EUCLID'S

ELEMENTS OF GEOMETRY,

CHIEFLY FROM THE TEXT OF Dr SIMSON,
WITH EXPLANATORY NOTES;

TOGETHER WITH A SELECTION OF GEOMETRICAL EXERCISES
FROM THE SENATE-HOUSE AND COLLEGE
EXAMINATION PAPERS;

TO WHICH IS PREFIXED AN INTRODUCTION, CONTAINING
A BRIEF OUTLINE OF THE HISTORY OF
GEOMETRY.

DESIGNED FOR THE USE OF THE HIGHER FORMS IN PUBLIC SCHOOLS
AND STUDENTS IN THE UNIVERSITIES.

BY

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PREFACE.

THIS new edition of Euclid's Elements of Geometry will be found to differ considerably from those at present in general use in Academical Education. The text is taken from Dr Simson's approved edition, with occasional alterations; but so arranged as to exhibit to the eye of the student the successive steps of the demonstrations, and to facilitate his apprehension of the reasoning. No abbreviations or symbols of any kind are employed in the text. The ancient Geometry had no symbols, nor any notation beyond ordinary language and the specific terms of the science. We may question the propriety of allowing a learner, at the commencement of his Geometrical studies, to exhibit Geometrical demonstrations in Algebraical symbols. Surely it is not too much to apprehend that such a practice may occasion serious confusion of thought. It may be remarked that the practice of exhibiting the demonstrations of Elementary Geometry in an Algebraical form, is now generally discouraged in this University. To each book are appended explanatory notes, in which especial care has been taken to guard the student against the common mistake of confounding ideas of number with those of magnitude. The work contains a selection of problems and theorems from the Senate-house and College Examination Papers, for the last forty-five years. These are arranged as Geometrical exercises to the several books of the Elements, and to a few only in each book the solutions are given. An Introduction is prefixed, giving a brief outline of the history and progress of Geometry.

The analysis of language, together with the sciences of number and magnitude, have been long employed as the chief elements of intellectual education. At a very early period, the study of Geometry was regarded as a very important mental discipline, as may be shewn from the seventh book of the Republic of Plato. To his testimony may be added that of the celebrated Pascal, (Œuvres, Tom. I. p. 66,) which Mr Hallam has quoted in his History of the Literature of the Middle Ages. "Geometry," Pascal observes, "is almost the only subject as to which we find truths wherein all men agree; and one cause of this is, that geometers alone regard the true laws of demonstration." These

are enumerated by him as eight in number. 1. To define nothing which cannot be expressed in clearer terms than those in which it is already expressed. 2. To leave no obscure or equivocal terms undefined. 3. To employ in the definition no terms not already known. 4. To omit nothing in the principles from which we argue, unless we are sure it is granted. 5. To lay down no axiom which is not perfectly evident. 6. To demonstrate nothing which is as clear already as we can make it. 7. To prove every thing in the least doubtful, by means of self-evident axioms, or of propositions already demonstrated. 8. To substitute mentally the definition instead of the thing defined. Of these rules he says, "the first, fourth, and sixth are not absolutely necessary to avoid error, but the other five are indispensable; and though they may be found in books of logic, none but the geometers have paid any regard to them."

If we consider the nature of Geometrical and Algebraical reasoning, it will be evident that there is a marked distinction between them. To comprehend the one, the whole process must be kept in view from the commencement to the conclusion; while in Algebraical reasonings, on the contrary, the mind loses the distinct perception of the particular Geometrical magnitudes compared; the attention is altogether withdrawn from the things signified, and confined to the symbols, with the performance of certain mechanical operations, according to rules of which the rationale may or may not be comprehended by the student. It must be obvious that greater fixedness of attention is required in the former of these cases, and that habits of close and patient observation, of careful and accurate discrimination will be formed by it, and the purposes of mental discipline more fully answered. In these remarks it is by no means intended to undervalue the methods of reasoning by means of symbolical language, which are no less important than Geometry. It appears, however, highly desirable that the provinces of Geometrical and Algebraical reasoning were more definitely settled than they are at present, at least in those branches of science which are employed as a means of mental discipline. The boundaries of Science have been extended by means of the higher analysis; but it must not be forgotten that this has been effected by men well skilled in Geometry and fully able to give a geometrical interpretation of the results of their operations; and though it may be admitted that the higher analysis is the more powerful instrument for that purpose, it may still be questioned whether it be well suited to

form the chief discipline of ordinary intellects without a previous knowledge of the principles of Geometry, and some skill in their application. Though the method of Geometrical analysis is very greatly inferior in power to the Algebraical, yet as supplementary to the Elements of Euclid, it is of great importance. It may be added, that a sound knowledge of the ancient geometry is the best introduction to the pursuits of the higher analysis and its extensive applications. On this subject the judgment of Sir Isaac Newton has been recorded by Dr Pemberton, in the preface to his view of Sir Isaac Newton's Discoveries. He says: "Newton censured the handling of geometrical subjects by algebraical calculations. He used to commend the laudable attempt of Hugo d'Omerique (in his 'Analysis Geometrica Nova et Vera,') to restore the ancient analysis, and very much esteemed the tract of 'Apollonius De Sectione Rationis,' for giving us a clearer notion of that analysis than we had before. The taste and mode of geometrical demonstration of the ancients he professed to admire, and even censured himself for not having more closely followed them than he did: and spoke with regret of his mistake, at the beginning of his mathematical studies, in applying himself to the works of Descartes and other algebraical writers, before he had considered the Elements of Euclid with the attention they deserve."

Regarding the study of Geometry as a means of mental discipline, it is obviously desirable that the student should be accustomed to the use of accurate and distinct expressions, and even to formal syllogisms. In most sciences our definitions of things are in reality only the results of the analysis of our own imperfect conceptions of the things; and in no science, except that of number, do the conceptions of the things coincide so exactly (if we may use the expression) with the things themselves, as in Geometry. Hence, in geometrical reasonings, the comparison made between the ideas of the things, becomes almost a comparison of the things themselves. The language of pure Geometry is always precise and definite. The demonstrations are effected by the comparison of magnitudes which remain unaltered, and the constant use of terms whose meaning does not on any occasion vary from the sense in which they were defined. It is this peculiarity which renders the study so valuable as a mental discipline: for we are not to suppose that the habits of thought thus acquired, will be necessarily confined to the consideration of lines, angles, surfaces and solids. The process of deduction pursued in Geometry from certain admitted principles and possible

constructions to their consequences, and the rigidly exact comparison of those consequences with known and established truths, can scarcely fail of producing such habits of mind as will influence most beneficially our reasonings on all subjects that may come before us.

In support of the views here maintained, that Geometrical studies form one of the most suitable and proper introductory elements of a scientific education, we may add the judgment of a distinguished living writer, the author of "The History and Philosophy of the Inductive Sciences," who has shewn, in his "Thoughts on the Study of Mathematics," that mathematical studies judiciously pursued, form one of the most effective means of developing and cultivating the reason: and that "the object of a *liberal education* is to develope the whole mental system of man;—to make his speculative inferences coincide with his practical convictions;—to enable him to render a reason for the belief that is in him, and not to leave him in the condition of Solomon's sluggard, who is wiser in his own conceit than seven men that *can* render a reason." To this we may subjoin that of Mr John Stuart Mill, which he has recorded in his invaluable System of Logic, (Vol. II. p. 180) in the following terms. "The value of Mathematical instruction as a preparation for those more difficult investigations (physiology, society, government, &c.) consists in the applicability not of its doctrines, but of its method. Mathematics will ever remain the most perfect type of the Deductive Method in general; and the applications of Mathematics to the simpler branches of physics, furnish the only school in which philosophers can effectually learn the most difficult and important portion of their art, the employment of the laws of simpler phenomena for explaining and predicting those of the more complex. These grounds are quite sufficient for deeming mathematical training an indispensable basis of real scientific education, and regarding, with Plato, one who is ἀγεωμέτρητος, as wanting in one of the most essential qualifications for the successful cultivation of the higher branches of philosophy."

R. P.

TRINITY COLLEGE,
October 1, 1845.

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ERRATA.

GREAT care has been taken in the correction of the proofs, and it is believed that the Student will not meet with any errata of importance in the text to impede his progress: the following, however, have been discovered on revising the sheets:

PAGE	LINE	FOR	READ
10	20	*CBG*	*CGB*.
12	14	*angles*	*angle*.
22	In the diagram, Prop. XXIII. *D* is misplaced.		
59	20	*BE*	*DE*.
80	17	equal less	less equal.
111	5	*ABC*	*ACB*.
125	23	$\frac{3-2}{3}$	$\frac{3-2}{3}.\pi$.
172	In the diagram, Prop. III. *A* is misplaced.		
177	In the diagram, Prop. IX. *D* and *E* are interchanged.		
192	In the diagram, Prop. XXVI. *for D read C.*		
242	13	*AE*	*AQ*.
323	46	arc	circumference.
328	1	centre	circumference.
341	4	of	at.

In the Geometrical Exercises, a few repetitions have occurred, and a few of the problems, perhaps, with more propriety, might have been arranged under a different book.

In the Index a few references are omitted, as the Examination Papers, from which the questions have been taken, were without dates; besides some four or five others, which were lost.

INTRODUCTION.

The aim of the following brief notices, is to give some account of the origin and progress of the science of Geometry, together with the names of the men by whom it has been successively advanced. Respecting the history of science it has been remarked that it serves, at least, to commemorate the benefactors of mankind; an object which can scarcely be considered as unworthy or unimportant. It is probable that this science had its origin, like all others, in the necessities of men. The word Geometry (γεωμετρία), formed from two Greek words, γῆ and μετρέω, seems to have been originally applied to the measuring of land. The earliest information on this subject is derived from Herodotus (Book II. c. 109), where he describes the customs of the Egyptians in the age of Sesostris, who reigned in Egypt from about 1416 to 1357, B.C. The account of Herodotus is to this effect. "I was informed by the priests at Thebes, that king Sesostris made a distribution of the territory of Egypt among all his subjects, assigning to each an equal portion of land in the form of a quadrangle, and that from these allotments he used to derive his revenue by exacting every year a certain tax. In cases however where a part of the land was washed away by the annual inundations of the Nile, the proprietor was permitted to present himself before the king, and signify what had happened. The king then used to send proper officers to examine and ascertain, by admeasurement, how much of the land had been washed away, in order that the amount of tax to be paid for the future, might be proportional to the land which remained. From this circumstance I am of opinion, that Geometry derived its origin; and from hence it was transmitted into Greece." The natural features and character of the land of Egypt, where rain is unknown, are such as to give credibility, at least, to the tradition recorded by Herodotus. In the earliest records of history, the population of Egypt is represented as numerous, and in the valley of the Nile, the extent of cultivated land is comparatively small Its extreme fertility is also placed in close contrast with the barrenness of the districts beyond the limits of the inundations of the Nile, by which the boundaries of the land on its margin are annually liable to alteration. There appear therefore some grounds for the belief that the geometrical allotment of land had its origin on the banks of the Nile. But independently of the tradition of Herodotus, it seems reasonable to suppose, that the science of Plane Geometry may have originated in the necessity of measuring and dividing lands, which must have arisen as soon as property in land came to be recognized among men. This recognition is found in the oldest historical records known in any language. The narrative in the 23rd chapter of the book of Genesis refers to Palestine, and belongs to a period 1860 years B.C. In Egypt

also, about 160 years later, as we learn from Gen. xlvii. not only was property in land recognized, but taxes were raised from the possessors and cultivators of the soil. This necessarily implies that there existed some method of estimating and dividing land, rude, probably, and inaccurate at first, but as society advanced and its wants increased, gradually becoming more exact.

The existence of the pyramids, the ruins of temples, and other architectural remains, supply evidence of some knowledge of Geometry; although it is possible that the geometrical properties of figures, necessary for such works, might have been known only in the form of practical rules, without any scientific arrangement of geometrical truths.

The word Geometry is used in a more extensive sense, as the science of Space; or that science which discusses and investigates the properties and relations existing between definite portions of space, under the fourfold division of lines, angles, surfaces, and volumes, without regard to any properties they may have of a physical nature. Of the origin and progress of Geometry, in this sense, it is proposed here to give some short account, as far as can be ascertained.

Whatever geometrical or astronomical science may have been possessed by the earlier Chaldeans and Egyptians, there are not known to be any historical records, which supply definite views of its limits or extent. In the most ancient Jewish writings, there is not the least allusion from which to infer that scientific Geometry was known and cultivated by that people. The traditions recorded by Josephus on this subject (Book I. c. 3, 9) can scarcely be considered worthy of being received as historical truth, since the subsequent history of the Jews does not inform us that they were, at any period, a scientific people.

Other ancient writers also confirm the tradition of Herodotus, that from Egypt the knowledge of Geometry passed over into Greece, where it attained a high degree of cultivation. Proclus, in his Commentary on Euclid's Elements of Geometry (Book II. c. 4), attributes to Thales the merit of having first conveyed the knowledge of Geometry from Egypt to Greece. Thales was a native of Miletus, at that time, the most flourishing of the Greek colonies of Ionia in Asia Minor. He was born about 640 B.C., and was descended from one of the most distinguished families, originally of Phœnicia. (Herod. I. 170. Diog. Laert. I. 22.) Thales, from a desire of knowledge, is reported by Diogenes Laertius to have travelled into Egypt, and to have held a friendly intercourse with the priests of that country; thus obtaining an acquaintance with the science of the Egyptians. The same writer also adds that he learned the art of Geometry among the Egyptians, and suggested a method of ascertaining the altitude of the pyramids by the length of their shadows. Plutarch relates this story, and adds, that Amasis, who was then king of Egypt, was astonished at the sagacity of Thales. If this tradition, recorded both by Diogenes Laertius and by Plutarch, be deemed worthy of credit, it would appear that the idea suggested by Thales was, to the Egyptians, a new application of a geometrical truth. Whatever mathematical knowledge the Egyptians might possess in the age of Thales, there are no writings, either cotemporary or of later times, which exhibit its extent. Thales is also said to have been the discoverer of some geometrical theorems, and to have left to his successors the principles of many others. The theorems which stand as the 5th, 15th, and 26th Propo-

sitions of the first Book of Euclid's Elements of Geometry are attributed to him by Proclus, and Prop. 31. Book III, also Props. 2, 3, 4, 5, of Book IV. He is reported by Herodotus (Book I. c. 74) to have foretold the year in which an eclipse of the sun would happen. He also designated the seasons, and found the year to consist of 365 days. All this implied not only some acquaintance with Geometry, but a considerable knowledge of the motions and periodical revolutions of the heavenly bodies. Anaximander of Miletus and Anaximenes are mentioned as disciples of Thales. The opinions of the latter are discussed by Aristotle immediately after those of Thales. Ameristus, the brother of Stesichorus the poet, is named by Hippias of Elis as a celebrated geometer.

Nearly at the same time with the commencement of speculative philosophy in Ionia, a spirit of enquiry began to shew itself in some of the Achæan and Dorian colonies in Magna Græcia. The most distinguished man of these times was Pythagoras. He was born at Samos, about 568 B.C.; and his descent is referred by Diogenes Laertius to the Tyrrhenian Pelasgi. After having been a disciple of Thales, he is reported to have visited Egypt, where he became the pupil of Oinuphis at Heliopolis, (which in the book of Genesis is called On), once a famous city of Lower Egypt. (Plutarch de Iside et Osiride, s. 10.) After his return from Egypt, he established a school at Crotona, an Achæan colony in Magna Græcia, which became very celebrated, and continued for nineteen generations. According to the account of Proclus (Book II. c. 4), Pythagoras was the first who gave to Geometry the form of a deductive science, by shewing the connexion of the geometrical truths then known, and their dependence on certain first principles. There are not known to be extant any particular accounts, or even fragments, of the earliest attempts to reduce geometrical truths to a system. It is, however, scarcely possible that any arrangement could have been attempted before a considerable number of geometrical truths had been discovered, and their connexion observed. The traditionary account, that Pythagoras was the founder of scientific mathematics, is, in some degree, supported by the statement of Diogenes Laertius, that he was chiefly occupied with the consideration of the properties of number, weight, and extension, besides music and astronomy. The passage of Cicero (De Nat. Deor. III. 36) may be referred to as evidence that later writers were unable to give any precise account of the mathematical discoveries of Pythagoras. To Pythagoras, however, is attributed the discovery of some of the most important elementary properties contained in the first book of Euclid's Elements. The very important truth contained in Prop. 47, Book I. is also ascribed to Pythagoras. Whether his discovery of this truth resulted from geometrical, or from numerical considerations is not certainly known: Proclus attributes to him the discovery of that right-angled triangle, the three sides of which are respectively 3, 4, and 5 units. To Pythagoras also belongs the discovery, that there are only three kinds of regular polygons which can be placed so as to fill up the space round a point; namely, six equilateral triangles, four squares, and three regular hexagons. Proclus attributes to him the doctrine of incommensurables, and the discovery of the five regular solids, which, if not due to Pythagoras, originated in his school. In Astronomy, he is reputed to have held, that the Sun is the centre of the system, and that the planets

revolve round it. This has been called, from his name, the Pythagorean System, which was revived by Copernicus, A.D. 1541, and proved by Newton. As a moral philosopher, many of his precepts relating to the conduct of life will be found in the verses which bear the name of the Golden Verses of Pythagoras. It is probable they were composed by some one of his school, and contain the substance of his moral teaching. The speculations of the early philosophers did not end in the investigation of the properties of number and space. The Pythagoreans attempted to find, and dreamed they had found, in the forms of geometrical figures and in certain numbers, the principles of all science and knowledge, whether physical or moral. The figures of Geometry were regarded as having reference to other truths besides the mere abstract properties of space. They regarded the unit, as the point; the duad, as the line; the triad, as the surface; and the tetractys, as the geometrical volume. They assumed the pentad as the physical body with its physical qualities. They seem to have been the first who reckoned the elements to be five in number, on the supposition of their derivation from the five regular solids. They made the cube, earth; the pyramid, fire; the octohedron, air; the icosahedron, water; and the dodecahedron, æther. The analogy of the five senses and the five elements was another favourite notion of the Pythagoreans.

Pythagoras was followed by Anaxagoras of Clazomene. Aristotle states that he wrote on Geometry, and Diogenes Laertius reports that he maintained the sun to be larger than the Peloponnesus. Ænopides of Chios was somewhat junior to Anaxagoras, both of whom are mentioned by Plato in his Rivals. Proclus ascribes to Ænopides the discovery of the truths which form the 12th and 23d Props. of the first Book of Euclid. Briso and Antipho are mentioned by Aristotle, as distinguished geometers, but no records are known to be extant of their writings or their discoveries.

About 450 B.C., Hippocrates of Chios, their cotemporary, was the most eminent geometer of his time, and is reported to have written a treatise on the Elements of Geometry; no fragments of which, however, are known to be in existence. He discovered the quadrature of the lunes which bear his name, by describing, in the same direction, semicircles on the three sides of an isosceles right-angled triangle, and observing that the sum of the two lunes, between the two quadrants of the larger semicircle and the two smaller semicircles, was equal to the area of the triangle. By means of the lunes he attempted the quadrature of the circle, but without success. The account of his attempts will be found in the Commentary of Simplicius, on the first book of Aristotle's Physics. Hippocrates also solved the problem of the duplication of the cube, which is, to find the length of the edge of a cube which shall be twice as great as a given cube. This problem, at that time, and for ages afterwards, excited very great attention among philosophers. He shewed that the solution depended on finding two geometric mean proportionals between two given lines.

About 100 years after Pythagoras, Zenodorus wrote a tract, preserved in Theon's commentary on the Almagest of Ptolemy, in which he shews that plane figures having equal perimeters have not equal areas.

Democritus, a native of Abdera, about the 80th Olympiad was cele-

brated for his knowledge both of Philosophy and the Mathematics. He is stated to have spent his large patrimony in travelling in distant countries.

Theodorus of Cyrene was eminent for his knowledge of Geometry, and is reported to have been one of the instructors of Plato.

We now come to the time of Plato, one of the most distinguished philosophers that ever lived; whose writings are still read, and regarded as of inestimable value. Plato visited Egypt, and on his return, founded his School at Athens, about 390 B.C. Over the entrance he placed the inscription, Οὐδεὶς ἀγεωμέτρητος εἰσίτω: "Let no one ignorant of Geometry enter here." This is a plain declaration of Plato's opinion respecting Geometry. He considered Geometry as the first of the sciences, and as introductory and preparatory to the pursuit of the higher subjects of human knowledge. Plato both cultivated and advanced the science of Geometry, as we learn from the testimony of Proclus and Pappus. The character of his writings, though not confined to discourses on Mathematics, affords incontestable evidence how great an admirer he was of Geometry, and how zealously he cultivated that science. To Plato is attributed the discovery of the method of the Geometrical Analysis; but by what means he was led to it, or to the invention of Geometrical loci, is not known. Hippocrates had, before his time, reduced the problem of the duplication of the cube to that of finding two mean proportionals between the edge of the given cube, and a line double that length; Plato attempted a solution of the problem in this form, by means of the right line and circle only. In this attempt, however, he failed, but effected a solution by means of two rulers, which could not be admitted as purely geometrical, since it involved other considerations besides those of the straight line and circle. It is uncertain whether the restriction of constructions in Geometry to the right line and circle, originated in the School of Plato, or at an earlier period. Plato is said to have discovered the Conic Sections,—curves which result from planes intersecting the surface of a cone,—and some of their more remarkable properties. Great attention was given to this subject, both during his lifetime, and after his death, by his cotemporaries and their successors. The Conic Sections and their properties were considered a distinct branch of the science, and called "the Higher Geometry." The trisection of an angle, or an arc of a circle, was another famous problem, which engaged the attention of the School of Plato. This, as well as the duplication of the cube, was, for many ages, believed possible, by means of the right line and circle, and has been repeatedly attempted from the earliest times, without success. It baffled the genius of Archimedes and others, who were not aware of the impossibility of its solution by the method they applied. The problem of the trisection of an angle or arc is reducible to the following problem: To draw a right line from a given point, cutting the semi-circumference of a circle in two points, and the diameter produced, so that the chord intercepted between the two points of intersection may be equal to the radius. This condition leads to the algebraical equation of one of the conic sections, whose properties are not the same as those of the right line and circle; and hence its impossibility is inferred.

From the Academy of Plato, proceeded many who successfully cultivated, and very considerably extended, the bounds of Geometrical science. Proclus names thirteen of the disciples and friends of Plato

who improved and made additions to the science. Among his cotemporaries may be named Leodamas of Thasus, Archytas of Tarentum, and Theætetus of Athens, by whom some theorems were discovered, and some improvements made in the methods of demonstration. Diogenes Laertius reports of Archytas, that he was the first who brought Mechanics into method by the use of mechanical principles, and the first who applied organic motions to Geometrical figures, and found out the duplication of the cube in Geometry. (Book VIII.) His solution of the duplication of the cube is given by Eutocius, in his commentary on the sphere and cylinder of Archimedes. Archytas was also the writer of a work on the elements of Geometry, which is not extant. The same writer relates, that Leodamas, by means of the Geometrical Analysis, which he had learned from Plato, discovered the solution of many problems, and made many discoveries. Theætetus was celebrated for a treatise on Geometry, which is lost. Plato honoured this disciple by giving to one of his dialogues the title of Theætetus; and in another, entitled Phædrus, he ascribes the origin of Geometry to Thoth, an Egyptian divinity. Aristeas, a disciple and friend of Plato, composed five books on the Conic Sections, which were highly esteemed. Another of the scholars of Plato was Neoclides, whose name is celebrated by Proclus. Leon was a scholar of Neoclides (B.C. 368), and was the author of several discoveries in Geometry. To Leon is attributed the invention of a method for discriminating the possibility or impossibility of a problem. He is also mentioned as the author of a work on the elements of Geometry. None of his discoveries or writings have descended to posterity. Amyclas was another friend of Plato; and the brothers Menæchmus and Dinostratus are both celebrated; the former for his application of the Conic Sections to solve the problem of the duplication of the cube, and the latter, for the discovery of a curve known by the name of the quadratrix. They are also said to have made some other additions to the science of Geometry, and to have rendered the whole more perfect. Theudias appears to have excelled both in mathematics and philosophy, and is said to have composed a work on Geometry, and to have generalized some theorems. At the same period, Cyzicenus of Athens, besides other branches of the mathematics, successfully cultivated Geometry. These friends and disciples of Plato used to resort to the Academy, and employ themselves in proposing, by turns, questions for solution. Eudoxus, a native of Cnidus, a town of Caria in Asia Minor, was one of the most intimate of the friends of Plato. He is reported to have written on the Elements, and to have generalized many results which had originated in the school of Plato, and to have advanced the science of Geometry by many important discoveries. To him is attributed the invention of the doctrine of proportion as treated in the 5th book of Euclid's Elements. He is said to have been the first who discovered that the volume of a cone or pyramid is equal to one third of its circumscribing cylinder or prism; that is, of a cylinder or prism having the same base and altitude. He is also reported to have advanced the knowledge of the higher Geometry, by the discovery of several important properties of the Conic Sections. He died B.C. 368, at 53 years of age. Diogenes Laertius, in his short memoir of Eudoxus, describes him as an astrologer, a physician, a legislator, and a geometer. Though none of his writings have descended to modern times, his name has been celebrated by the

eminent men, both of Greece and of Rome. Hermotimus wrote on Loci, and is said to have extended the results of Eudoxus and Theætetus. Philippus the Mendean, another disciple of Plato, is reported to have discovered problems, and to have proposed questions, being a great lover of the mathematical sciences. All these were either disciples of Plato, or attached themselves to the school he founded at Athens, and are mentioned by Proclus as having advanced or improved the mathematical sciences. Xenocrates also was a hearer of Plato, and is said to have been one of the instructors of Aristotle; he was renowned for his knowledge of the mathematical sciences. Aristotle, though originally a disciple of Plato, and attached to his school for a period of twenty years, became the founder of a new sect of philosophers—the Peripatetics, and opened a school, B.C. 341, at the Lyceum on the banks of the Ilissus. There he continued twelve years, till the false accusation of Eurymedon obliged him to flee to Chalcis, where he died at the age of sixty-three.

On this division of the Platonic school, the two sects—both the Academics and the Peripatetics—continued to hold the same opinion on the utility of Geometry, as the necessary introductory knowledge for all who were desirous of proceeding with the study of philosophy. Thus the science of Geometry continued to be cultivated, and to make advancement. Among the numerous writings of Aristotle, there is a treatise on Mechanics, and a collection of Problems in 38 divisions; the 15th consists of mathematical questions. There is also a tract on indivisible lines, which has been ascribed by some ancient commentators to Theophrastus, as we learn from Simplicius. (Rev. J. W. Blakesley's *Life of Aristotle.*)

Two of the Peripatetic school are especially celebrated, Theophrastus and Eudemus, who devoted themselves chiefly to mathematical studies. Theophrastus was the author of the first history of the mathematical sciences, from the earliest times to his own. The work consisted of eleven books, of which there were four on Geometry, six on Astronomy, and one on Arithmetic. Eudemus also wrote a history of Astronomy in six books, and a history of Geometry in six books, from which Proclus acknowledges that most of his facts were taken. None of these writings have been preserved.

Autolycus, of Pitane in Æolis, lived about 300 B.C. He was preceptor in mathematics to Arcesilaus, a disciple of Theophrastus the successor of Aristotle, as we learn from Diogenes Laertius. His treatise on the moveable sphere is the earliest written on that subject. The original Greek, with a Latin translation, was published in 1572. He also was the author of another treatise "On the Rising and Setting of the Stars," which is still extant, and has been translated and printed.

Aristeas is said to have composed five books on the Conic Sections, and five books on Solid Loci. He is also said to have been the friend of Euclid, and his instructor in Geometry.

We come next to the time of Euclid. The birth-place and even the country of Euclid are unknown, and he has been very frequently confounded with another philosopher of the same name, who was a native of Megara. He studied at Athens, and became a disciple of the Platonic school. He flourished in the time of Ptolemy Lagus (B.C. 323 to 284), to whom he made the celebrated reply, that "there is no royal road to Geometry." He is said to have successfully cultivated and

taught Geometry and the mathematical sciences at Alexandria, shortly after the school of Philosophy was founded in that city. The school at Alexandria became most distinguished for the eminent mathematicians it produced, both in the lifetime of Euclid and afterwards, until the destruction of the great library at Alexandria, and the subjugation of Egypt by the Arabians. Euclid has become celebrated chiefly by his work on the Elements of Geometry, for which his name has become a synonym. It consists of thirteen books, nine of which are devoted to the subject of Geometry, and four to the properties of numbers, as discussed by the Greek Arithmetic, and applied to Geometry. There are two other books on the five regular solids, usually found appended to the thirteen books of Euclid. These, however, were subsequently added to the Elements by Hypsicles of Alexandria. In some editions a sixteenth book is found, which was added by Flussas. Another geometrical work of Euclid is the Data, which consists of 100 propositions. This is the oldest specimen of the principles auxiliary to the Geometrical Analysis. The object of the several propositions of this book is to shew that, in cases where certain properties or ratios are given, other properties or ratios are also given, or may be found geometrically; and thus pointing out what data are essential in order that Geometrical Problems may be determinate and free from all ambiguity. Three books on Porisms are attributed to Euclid by Pappus and Proclus; the former, in the seventh book of his Mathematical Collections, has given some account of them and the general enunciations of some propositions. This contains all that is known to exist of these three books. Attempts have been made with some success in later times for their restoration. Euclid also wrote a Treatise on Fallacies in geometrical reasoning, and another on Divisions. Pappus makes mention of another work on Geometry attributed to Euclid under the title of *τόπων πρὸς ἐπιφάνειαν* (Coll. Math. Lib. VII. Introd.), which his Latin translator Commandine has rendered by "Locorum ad Superficiem." Pappus also states that Euclid composed four books on the Conic Sections, which were afterwards augmented by Apollonius Pergæus. Proclus has made no allusion to this work in his account of the writings of Euclid. Besides these writings on Geometry, Euclid is reported by Proclus to have been the author of a work on Optics and Catoptrics, and a work on Harmonics; but it is very questionable whether the treatise on Harmony which is extant, and attributed to Euclid, was really composed by him. Pappus mentions a work on Astronomy entitled "The Phænomena." This treatise contains some geometrical properties of the sphere.

It has been a question whether Euclid was the *author* or the *compiler* of the Elements of Geometry, which bear his name. If Euclid were the discoverer of the propositions contained in the thirteen books of the Elements, and the author of the demonstrations, he would be a phenomenon in the history of science. It is by far more probable that he collected and arranged the books on Geometry in the order in which they have come down to us, and made a more scientific classification of the geometrical truths which were known in his time. Euclid may also have been the discoverer of some new propositions, and may have amended and rendered more conclusive the demonstrations of others. From the slow advances of the human mind in making discoveries, and the general history of the progress of the sciences, it would seem

unreasonable to assign to Euclid a higher place than that of the *compiler* and *improver* of the Elements of Geometry. This is in complete accordance with the statement of Proclus, who relates that "Euclid composed Elements of Geometry, and improved and arranged many things of Eudoxus, and perfected many things which had been discovered by Theætetus, and gave invincible demonstrations of many things which had been left loosely or unsatisfactorily demonstrated before him." The Elements of Geometry, thus arranged and improved by Euclid, were acknowledged so far superior in completeness and accuracy to the elementary treatises then existing, that they entirely superseded them, and in course of time all have disappeared. The book of Euclid's Elements is therefore the most ancient work on Geometry known to be extant. The Greek Arithmetical Notation, employed in the arithmetical portion, has yielded to the more perfect system of the Indian; but the geometrical portion, from the time it was first put forth till the present day, a period of upwards of 2000 years, has maintained its high character as an elementary treatise, in all nations wherever the sciences have been cultivated.

At this part of our subject we cannot forbear making a few remarks on the comparative claims of the Egyptians and Greeks to the merit of being the authors of Scientific Geometry. The learned Sir Gardiner Wilkinson, in his profound work on the Ancient Egyptians, unhesitatingly assigns this honour to the Egyptians. In page 342 of Volume I. he expresses his judgment in the following terms: "Anticlides pretends that Mœris was the first to lay down the elements of that science, which, he says, was perfected by Pythagoras; but the latter observation is merely the result of the vanity of the Greeks, who claimed for their countrymen the credit of enlightening a people on the very subjects which they had visited Egypt for the purpose of studying." The vanity both of the later and the earlier Greeks may be readily admitted, without allowing that it suggests the true answer to this question. Diogenes Laertius, (Book VIII.) in his life of Pythagoras, writes thus: "Anticlides reports that *Mœris* was the first who invented Geometry, and Pythagoras brought his imperfect notions to perfection." Mœris was an early king of Egypt (Herod. II. 13), the son of Amenophis, and lived before the age of Sesostris. Whatever we learn from Herodotus respecting Geometry in Egypt, is referred to the age of Sesostris, and does not carry us beyond such processes as might exist without any attempts at science. The Geometry which was brought from Egypt to Greece, appears to have been in its infancy; and all that Pythagoras and others borrowed from the Egyptians could not have exceeded some practical rules and their applications. In the learned work referred to, any positive information as to the existence of Scientific Geometry among the early Egyptians, has been sought in vain; nor do ancient writers exhibit any remains of the Scientific Geometry of the Egyptians, or even make any claims in their favor to the merit of its invention. The general tenor of all the traditional and probable evidence tends to shew, that the scientific form of the Elements of Geometry is due to the acute intellect of the Greeks. And this presumption is reduced almost to historical certainty by the existing remains of the Greek Geometry. We may further observe, that in the very brief review we have given of the earliest commencement of the Greek Geometry and Mathematics, their simplicity is such as might be expected to characterize the original

attempts of an acute people. The advances were so gradual that any supposition of the sudden introduction of more perfect science from foreign sources is completely removed. Moreover, at the time that the Greek Geometrical science was rapidly advancing towards perfection in the school of Plato, there is a total absence of any accounts that the Geometry of the Egyptians or of any other nation had proceeded so far in its development. Plato truly ascribes to the Egyptians and Phœnicians a certain commercial activity, but distinguishes their native character from that of the Greeks, which he represents as remarkable for its curiosity and desire of knowledge. (De Repub. IV.) Five centuries afterwards the same characteristic of that people is remarked by St Paul in his first epistle to the Corinthians.

Archimedes was born at Syracuse, B.C. 287, about the period of the death of Euclid, and became the most eminent of all the Greek mathematicians. His discoveries in Geometry, Mechanics, and Hydrostatics, form a distinguished epoch in the history of mathematical science; and his remaining writings on the pure Mathematics are the most valuable portion of the ancient Geometry. He was the first who discovered that the volume of a sphere is two thirds of its circumscribing cylinder, or of a cylinder having the same diameter and altitude as the sphere, and that the curved surface of each is equal to four great circles of the sphere. He also found the relation of the volumes and surfaces of a hemisphere and cone upon the same base. These and other properties are investigated in two books, entitled, "On the Sphere and Cylinder," which have descended to our times in the original Doric Greek, together with the Commentary of Eutocius. Another work is still extant on the measurement of the circle, in which he shews that the area of a circle is equal to that of a right-angled triangle whose altitude is equal to the radius and whose base is equal to the circumference. Though he failed in his attempts to discover the exact proportion of the circumference to the diameter of a circle, he discovered an useful approximation to that ratio. He found, by numerical calculation, that the perimeter of a regular polygon of 192 sides, circumscribing a circle, is to the diameter in a less ratio than $3\frac{10}{70}$ to 1; and that the perimeter of the inscribed polygon of 96 sides is to the diameter in a greater ratio than $3\frac{10}{71}$ to 1: whence he concluded that the ratio of the circumference to the diameter of the circle must lie between these two ratios. The book of Lemmas, a collection of Problems and Theorems on Plane Geometry, attributed to Archimedes, is not known to be extant in the original Greek. It comes to us from the Arabic, of which a translation in Latin was published, for the first time, in 1659. On the Higher Geometry, three tracts of Archimedes are still extant. 1. On the Quadrature of the Parabola, in which he proves that the area included between the curve and two ordinates is equal to two thirds of the circumscribing parallelogram. This is the first instance known of the discovery of the quadrature of a figure bounded partly by a curved line, if we except that of the lunes of Hippocrates. 2. His Treatise on Conoids and Spheroids, which contains many discoveries: among them may be named, the ratio of the area of an ellipse to a circle having the same diameter as the axis major of the ellipse; and that the sections of conoids and spheroids are conic sections. He also first proved that the volume of the cone and parabolic conoid of the same base and altitude are in the proportion of 2 to 3. This tract also contains the demonstrations of several discoveries

respecting hyperbolic conoids and spheroids. 3. A Tract on Spirals. The curve known by the name of the Spiral of Archimedes was originally discovered by his friend Conon, whose premature death prevented his completing the investigation of the properties of the curve. Archimedes completed the investigations, and put them forth as they appear in this tract. Besides his writings on Geometry, we have a tract on Arithmetic, entitled Ψαμμιτης, or Arenarius. The object was to prove the possibility of expressing the number of grains of sand which would fill the whole space of the universe considered as a sphere extending to the stars. In this tract he alludes to a system of numeration which he had discovered, and which he had described in a work addressed to Zeuxippus. Two books are also extant on the Equilibrium of Planes, and on their centres of gravity, in which he has proved the fundamental property of the Lever, and shewn how to find the centre of gravity of a triangle, and other figures. Two books, "On Bodies which are carried in a fluid," in which the general conditions of a body floating on a fluid are investigated and applied to different forms of bodies. Archimedes was also the discoverer of the method of finding the specific gravity of bodies. The story of the crown of king Hiero, to whom Archimedes was related, is briefly this. Hiero had delivered to a goldsmith a certain weight of gold to be converted into a votive crown. The king suspected that the crown he received from the smith was not of pure gold, though of the proper weight, but that it was alloyed with silver. He applied to Archimedes to ascertain, without melting the crown, whether it contained alloy. It was observed by Archimedes, on going into a bath full of water, that when his body was immersed in the bath, a quantity of water equal to the bulk of his body flowed over the edge of the bath. It occurred at once to him, that if a weight of pure gold equal to the weight of the crown were immersed in a vessel full of water, and the quantity of water left in the vessel measured, on the gold being taken out; by doing the same with the crown in the same vessel, he would be able to ascertain whether the bulk of the crown were greater than the bulk of an equal weight of pure gold. For any weight of silver is larger in bulk than an equal weight of pure gold. According to Vitruvius, as soon as he had discovered the method of solution, he leaped out of the bath, and ran hastily through the streets to his own house, shouting εὕρηκα, εὕρηκα! He was also the inventor of a machine for raising water from lower to higher levels, which was called the screw of Archimedes. An important application of the principle of this screw has lately been made in the propulsion of ships by means of steam power. When Syracuse was besieged by a land and naval armament under Appius and Marcellus, the besieged held out a successful resistance for three years chiefly by the aid of the machines invented by Archimedes. The city was at length surprised and taken B.C. 212. Archimedes, when seventy-five years of age, was slain by a soldier, while intent on the solution of a problem. He is reported to have expressed a desire that a sphere inscribed in a cylinder might be engraved on his tomb, to record his discovery of the relation between the volumes and the surfaces respectively of these two solids. About 200 years after his death, his tomb was discovered near Syracuse by Cicero, while quæstor in Sicily. It was nearly overgrown with bushes and brambles, which he caused to be cleared away. The tomb was identified as the tomb of Archimedes by

the Sphere and Cylinder engraved upon the stone with the inscription; the latter part of which was completely effaced. (Cic. Tusc. Quæst. lib. v.)

Conon was the friend and cotemporary of Archimedes, and is celebrated by Virgil in his third Eclogue. In the treatise on the quadrature of the parabola, speaking of his genius, Archimedes exclaims—"How many theorems in geometry, which to others have appeared impossible, would Conon have brought to perfection, if he had lived!"

Cotemporary with Archimedes was Eratosthenes, a distinguished geometrician and astronomer. He is celebrated for his construction in solving the problem of the duplication of the cube. He was the first who attempted to measure the circumference of the earth by means of observations of the Sun at two different places, near the same meridian, at the time of the solstice. Though he did not completely succeed, on account of the inaccuracy of his data, he pointed out the method. None of his works have descended to modern times except a few fragments, and a list of the names of forty-four constellations, and the principal stars in each constellation.

Apollonius of Perga, a city of Pamphylia, lived about the same time, and stands next in fame to Archimedes. He was born at the time when Ptolemy Euergetes was king of Egypt: he studied the mathematical sciences at Alexandria in the school which Euclid's disciples had founded, and passed there the greater part of his life. He was the author of several works on Geometry, and became so eminent in that science that he was called by his cotemporaries the Great Geometer. His principal work is a treatise on the Conic Sections. From the author's dedicatory epistle to Eudemus, a geometer of Pergamus, it appears that the treatise consisted of eight books. The first four books still exist in the original Greek. An Arabic version of seven books, made about the middle of the thirteenth century, was found in the East about four centuries later by Golius, a professor of the oriental languages at Leyden; and was translated into Latin. It has been said that Apollonius appropriated the discoveries of others in the Conic Sections. It is well known that although Archimedes discovered many important properties of the curves which bear that name, it cannot be pretended that he was the original discoverer. Long before the time of Archimedes they had been studied in the school of Plato and at Alexandria, and many properties of them were well known. It is highly probable that Apollonius, in his treatise on the Conic Sections, availed himself of the writings of Archimedes as well as of others who had gone before him. His treatise on Conics was most highly esteemed; and to him is justly accorded the honour of having composed a better treatise on that difficult subject than any who had written before him. He made important improvements in the problems both of Euclid and of Archimedes. Before Apollonius, we are informed, by his commentator Eutocius, that writers on the Conic Sections required three different sorts of cones from which to cut the three different sections. They used to cut the parabola from a right-angled cone, the ellipse from an acute-angled cone, and the hyperbola from an obtuse-angled cone: because they always supposed the sections made by the cutting planes to be at right angles to the side of the cone. But Apollonius cut his sections from any cone by only varying the inclination and position of the cutting plane. It may be remarked that Apollonius first gave the names of *ellipse* and *hyperbola* to two of the

curves: the name of *parabola* had been already applied to the third by Archimedes. He also first made the distinction between the *diameters* of the sections, and the *axes*, giving the latter name to the two diameters which are at right angles to each other in the ellipse and hyperbola, and restricting the term axis in the parabola, to the line which passes through the focus and vertex of that curve. The following is a very brief account of the subject of each book of this treatise.

Book I. treats of the generation of the Conic Sections and their distinguishing properties.

Book II. treats of the properties of diameters and axes of these three curves, and of the asymptotes of the hyperbola.

Book III. consists of Theorems useful in the solution of solid loci.

Book IV. explains his new method of the intersection of the sections of cones with each other and with the circumferences of circles.

Book V. treats of maxima and minima in the Conic Sections.

Book VI. treats of equal and similar sections of the cone.

Book VII. contains a collection of Theorems useful in the solution of Problems.

Book VIII. was a collection of Problems with their solutions by means of the Theorems in Book VII.

Besides the treatise on the Conic Sections, Apollonius was the author of several treatises relating to the Geometrical Analysis. They bear the following titles in the translation of Pappus's Collections, and each treatise consisted of two Books.

1. De Rationis Sectione.
2. De Spatii Sectione.
3. De Sectione Determinatâ.
4. De Tactionibus.
5. De Inclinationibus.
6. De Planis Locis.

An Arabic Version of the Treatise De Sectionis Ratione was translated into Latin by Dr Halley and published at Oxford in 1708. The rest are not known to be extant either in the original Greek or in Arabic. Attempts however have been made since the revival of learning in Europe to restore these lost treatises; notices of which will be found under the names of the respective authors in their proper places. Proclus, in his commentary on Euclid, informs us that Apollonius attempted to demonstrate the axioms of Euclid, and cites his method of proving the first axiom, that things which are equal to the same thing are equal to one another. Proclus examines his so-called proof, and shews that properties are assumed not more self-evident than the axiom itself.

Nicomedes lived during the second century before the Christian era, and is known for the invention of a curve called the conchoid, and for the application he made of it in finding two mean proportionals between two given lines. He was also celebrated for the invention of several useful machines.

About the same time also lived Nitocles, Thrasideus, and Dositheus, whose discoveries in mathematics, and whose writings, if they left any, have not descended to our times.

Geminus, a native of Rhodes, and a mathematician of some repute, lived about 100 years before the Christian era, and is reported to have been the author of a work entitled 'Enarrationes Geometricæ,' which is not known to be extant.

Hipparchus, a native of Nice, though not a writer on the Elements of Geometry, is regarded as the first who reduced Astronomy to a science, and either devised or greatly improved the methods of calculation in Trigonometry, which form the basis of the science of Astronomy. His work on the calculation of chords originally consisted of 12 books, of which a few fragments only are known to be extant. He divided the circumference of the circle into 360 equal parts, and also the radius into 60 equal parts, which he likewise called degrees, each degree into 60 parts, and so on. His rules of calculation were derived from the properties of chords, and were estimated in sexagesimal parts of the radius, and the lengths of chords were calculated to every half degree of the semicircumference. It is, however, as an astronomer that his name is most celebrated. He was the first who discovered the precession of the equinoxes, and taught how to foretell eclipses, and form tables of them. The catalogue of stars which he observed and registered between the years B.C. 160 and 135, is preserved in Ptolemy's Almagest. They are arranged according to their longitudes and their apparent magnitudes. He was also the first who suggested the idea of fixing the position of places on the earth, as he did in the heavens by means of their latitude and longitude. He pursued his astronomical studies at Rhodes, whence he obtained the name of Rhodius, and afterwards in Bithynia, and at Alexandria. His writings on Astronomy are highly spoken of by ancient authors, but are not now extant: his commentary, however, on the Phænomena of Aratus, still exists. A full but rather exaggerated account of the discoveries of Hipparchus will be found in the work of Delambre, on the ancient Astronomy of the Greeks.

There is some uncertainty with respect to the exact period when Hypsicles flourished. He was a native of Alexandria, and, it is said, a disciple of Isidorus. To him are attributed the 14th and 15th books of the Elements, which were added to the 13 books of Euclid. In the introduction, he makes mention of Apollonius, who flourished in the reign of Ptolemy Euergetes, and probably Hypsicles came after him.

Theodosius of Tripoli flourished about the time of Cicero, and was the author of a treatise on the sphere in three books, which has come down to our time, in the original Greek. In this treatise, he investigates the properties of circles, which are made by sections of the surface of the sphere. It was translated and published by Dr Barrow in 1675: another edition was published at Oxford, in 1707.

For a period of some hundreds of years after the time of Theodosius, we shall find that few additional discoveries were made in geometrical and mathematical science. There were, however, some instances of individuals during that period, not entirely unskilled in Grecian science. Sosigenes the peripatetic, was a mathematician and astronomer. He was an Egyptian, and was brought by Julius Cæsar to Rome, for the purpose of assisting in the reformation of the Roman Calendar. This philosopher had discovered, by astronomical observation, that the year consists of 365 days and 6 hours; and to make allowance for the accumulation of the hours which were above 365 whole days, he invented the intercalation of one day in four years. The duplication every fourth year of the sixth day before the Calends of March was the intercalary day; and hence the year in which this took place consisted of 366 days, and was named bissextile. This was called

the Julian correction of the Calendar, and the reckoning by Julian years commenced B.C. 45, and continued till the more accurate correction was made under Pope Gregory XIII. Marcus Vitruvius was also greatly esteemed by Julius Cæsar, and he was subsequently employed by Augustus in constructing public buildings and warlike machines. Vitruvius was the author of a work on Architecture, in ten books, which was addressed to Augustus. This work is still extant, and is the only one on the Architecture of the ancients, which has descended to modern times. It affords evidence in the ninth book, that the writer was well skilled in Geometry.* Menelaus was born at Alexandria, in the time of the Emperor Trajan, and was of Grecian origin. He composed a Treatise on Trigonometry in six books, and another on the Sphere in three books, both of which have come down to us through the medium of translations in Arabic. A Latin translation of the Spherics was published at Paris in 1664.

Claudius Ptolemæus, one of the most eminent mathematicians and

* The Romans were a nation of warriors, and at no period of their history distinguished for their cultivation or advancement of the sciences. Though it must be admitted that their history exhibits many noble instances of patriotism and the sterner virtues, the leading principle of the Roman policy was nothing less than universal dominion. In the life of Agricola, Tacitus has recorded the substance of the address of Calgacus to the Northern Britons when invaded by the Romans, in which are the following lines: "Sed nulla jam ultra gens, nihil nisi fluctus et saxa, et infestiores Romani: quorum superbiam frustra per obsequium et modestiam effugeris: raptores orbis, postquam cuncta vastantibus defuere terræ, et mare scrutantur: si locuples hostis est, avari: si pauper, ambitiosi: quos non Oriens, non Occidens, satiaverit: soli omnium, opes atque inopiam pari affectu concupiscunt: auferre, trucidare, rapere falsis nominibus, imperium; atque, ubi solitudinem faciunt, pacem appellant." To this may be added the following passage from the popular work of M. Aimé-Martin; "La règne de Rome fut celui d'un brigand; elle s'aggrandit par la guerre et la pillage; et aussi elle perit par ses richesses et par la guerre." This will scarcely be deemed too highly coloured a description of Roman policy and practice, when it is compared with the following lines which Virgil puts into the mouth of Anchises.

"Excudent alii spirantia mollius æra,
Credo equidem; vivos ducent de marmore vultus;
Orabunt causas melius, cœlique meatus
Describent radio, et surgentia sidera dicent:
Tu regere imperio populos, Romane, memento:
Hæ tibi erunt artes; pacisque imponere morem,
Parcere subjectis, et debellare superbos." (*Æn.* VI. 847.)

These lines were written when nearly the whole world was subjected to the Roman arms, and when the future prospect of undisputed sway appeared to threaten that the dominion of Rome would be universal and perpetual. It may seem surprising that even in the golden age of Roman literature and magnificence, there does not appear to have existed one Roman of original genius, who successfully cultivated and advanced the mathematical sciences. The slender amount of scientific knowledge attained at that period, was acquired at Alexandria and at Athens, which seem to have been the chief places of resort for the philosophers and their hearers. Horace humorously concludes his description of the course of his education by declaring, that the object for which he was sent to Athens by his father was merely

"Scilicet ut possem curvo dignoscere rectum."

From this, perhaps, we may infer the opinion of that age, that a little mathematical knowledge was not deemed wholly useless, or incompatible with the pursuit of literature. With regard to the knowledge of astronomy which existed among the Romans, it may be observed, that it was rather cultivated for its supposed utility in relation to astrology and the prognostication of future events, than for its real value as science.

astronomers of antiquity, was born at Pelusium, in Egypt, about A.D. 70, and died A.D. 147. Though he is not known to have left any writings on Geometry, his great works still extant on Astronomy and Geography, for which he is justly celebrated, supply evidence of their author's having been deeply skilled in the applications of that science. His work on Geography consists of seven books, and his great work on Astronomy of thirteen, entitled "ἡ μεγάλη σύνταξις." In the early part of the 9th century it was translated into Arabic under the title of Almagest, a word formed from the Arabic article, and a Greek superlative. Both the original Greek and the Arabic version are still extant. Ptolemy adopted that system of the universe which placed the earth in the centre, and from him it acquired the name of the Ptolemaic system.

We may pass over the period which intervened between Ptolemy and Pappus, as no mathematicians of eminence appeared in the school of Alexandria, which at that period was the chief place where philosophy and the sciences were still cultivated. Pappus lived in the time of the Emperor Theodosius, who reigned between 379 and 395 A.D. He was the author of a work entitled "Mathematical Collections," which consists, as its title implies, of Problems and Theorems collected from the works of different mathematicians, with commentaries and historical notices. This work and the Commentary of Proclus are the chief repositories of information respecting the ancient Geometry, and especially the Geometrical Analysis. The work of Pappus originally consisted of eight books, the whole of which are extant in the original Greek except the first book, and the first half of the second book. A translation of the last six books into Latin was made by Commandine, who also wrote a Commentary on the work. These, after his death, were published by the Duke of Urbino, at Pisaurum, in 1588. The first two books of Pappus were not then known to be extant; however, there were found in a MS. of Pappus, in the Savilian Library at Oxford, the last twelve Propositions of the second book. They were translated into Latin by Dr Wallis, and printed at the end of his Aristarchus Samius, in 1688. This fragment being on Arithmetic, it was conjectured that the first two books were on the same subject. The next five books are on Geometry, and the last is chiefly on Mechanics. The following is a very brief account of the contents of the remaining six books of Pappus. The third book discusses four general problems. 1. The solution of the Delian problem, or duplication of the cube by means of the Conic Sections: besides three other solutions, one by Eratosthenes, another by Hero, and a third by himself. There is also a fourth by Nicomedes, by means of the conchoid. 2. A problem respecting the Medietates, a name given to three lines when they were in arithmetical, geometrical, or harmonical proportion. 3. To draw two straight lines from two points in one side of a triangle to a point within it so that they may be greater than the other two sides of the triangle. 4. To inscribe the five regular solids in spheres. The fourth book consists of Theorems. Prop. I. contains an extension of Euclid, I. 47. Prop. X. is one of the tangencies of Apollonius, to which the three preceding Props. are preliminary. Prop. XIII. On the property of the Arbelon. Prop. XIX. On the Spiral of Conon; also a solution of the Delian problem by means of the conchoid, and the trisection of an arc of a circle, with the properties of the quadratrix, and some problems. The fifth book commences

with a preface, in which Pappus remarks the instinct of bees whereby they construct their cells on geometrical principles, employing the figure whose base is a regular hexagon, which supplies, with the smallest labour, the greatest possible accommodation. The object of the book is to prove that, of these plane figures which are equilateral and equiangular, and have equal perimeters, the greatest area is contained by the figure with the greatest number of sides; and that of all plane figures of equal perimeters, the circle is the greatest. The subject of isoperimetrical figures is treated in 57 Propositions. It is proved that of plane figures with equal perimeters, the greatest is that which is equilateral and equiangular. This principle is extended to solids. The regular solids are then compared, and it is proved that of those with equal surfaces, the greatest is that with the greatest number of faces. These are introductory to the proposition, that of all solids with equal surfaces, the greatest is the sphere. At the end of the book, it is shewn that there can be only five regular solids, or that only equilateral triangles, squares, and pentagons, can form the boundaries of regular solid bodies. The sixth book is employed chiefly in explaining and connecting some propositions of Theodosius and others in treatises on the sphere, &c. The object of the book is stated in a short preface with reference to the three selected propositions: namely, Prop. 6, Book III. of Theodosius on the Sphere; Prop. 6 of Euclid's Phenomena, and Prop. 4 of Theodosius on Days and Nights. Eight different treatises are quoted or alluded to by Pappus in this book, and it may be considered as peculiarly worthy of notice, that in Props. 31, 32, 33, 34, which are preliminary to some on the sphere, there are stated, as examples, some distinctions of magnitudes which may either be increased or diminished without limit, or may be decreased while there is a limit to the decrease, and conversely. The seventh book is employed on the ancient geometrical Analysis. The preface contains an exposition of the method employed by the ancients in the discovery both of the solution of problems, and the demonstration of theorems. After that follows a particular description of the object and contents of some of the most important treatises of the analytical Geometry of the ancients, the whole of which are said to have consisted of thirty-three books. The seventh book itself consists of Lemmas or subsidiary propositions assumed or employed in the treatises described in the preface. The whole of these thirty-three books existed in the time of Pappus, but the greater part of them have since perished, or at least, are not known to be in existence, either in the original Greek or in any translation. The Data of Euclid, in Greek, two books, de Rationis Sectione in an Arabic version, and seven books of the Conics of Apollonius, four in Greek and three in Arabic, are all that are preserved. The descriptions however of eleven of these books are so particular and entire, that some eminent mathematicians have attempted the restoration of them. Of these attempts some short notices will be found under the names of their respective authors. The treatises of Apollonius, entitled, De Rationis Sectione, De Spatii Sectione, De Sectione Determinatâ, De Tactionibus, and De Inclinationibus, contained the discussion of general problems of frequent occurrence in geometrical investigations, completely solved, and all the possible cases distinguished; also of each case a separate analysis and synthesis were given with determinations in

all cases which required them. The use of these general problems was, the more immediate solution of any proposed geometrical problems, which could be easily reduced to a particular case of some one of them The other treatises in the list were useful for the same purpose. The seventh book contains 238 propositions, some of which exhibit complete examples of the ancient analysis and synthesis. The eighth book gives some account of the science of Mechanics, and exhibits the progress it had made in the age of Pappus: it contains many references to the mechanical inventions of Archimedes. A considerable part of the book is employed in describing what are called the five mechanical powers, and the most obvious combinations of them for raising or drawing large weights. Pappus acknowledges that the substance of the book is chiefly borrowed from the works of Hero the Elder, who lived about fifty years after Archimedes. His long preface contains some statements of the mechanical notions and of the arts of that period, as well as some observations on the utility of Mechanics, and on the connexion of Mechanics with Geometry. The branches of the science are distinguished, and some notices are given of treatises which are lost.

Serenus lived about the same time, and is chiefly known for his treatise, in three books, on the Cone and Cylinder.

Theon, of the same period, was a native of Smyrna: he was a mathematician of the Platonic school at Alexandria, of which he subsequently became president. He wrote a commentary on Euclid's Elements; the earliest of which we have any notice, and another on the first eleven books of the Almagest of Ptolemy. The commentary was translated into Latin by Commandine, and published with his Latin translation of the Elements of Euclid from the Arabic.

Hypatia, the daughter of Theon, became so well skilled in the mathematical Sciences as to be chosen to succeed her father in the school at Alexandria. Her commentaries on the Conic Sections of Apollonius, and on the Arithmetic of Diophantus, are not known to be in existence. Though of blameless life, she was assassinated, A.D. 415, and there are some grounds for the opinion, that Cyril the patriarch of Alexandria, was not quite exempt from blame in that horrid deed.

Hero the younger was the instructor of Proclus in the mathematical sciences at Alexandria. He was the author of a work on Mensuration, entitled Geodæsia, and another on Mechanics, both of which were published in Latin, at Venice, in 1572. In the Geodæsia, there is given the method of finding the area of a plane triangle in terms of the sides of the triangle.

Diocles, his cotemporary, discovered the generation of the curve called the cissoid, which still bears his name, and was applied by him in finding two mean proportionals between two given lines. Another solution of this problem was given by Sporus, who lived about the same time.

We now come down to the latter times of the Greek school of science and philosophy at Alexandria, which city seems to have been the chief place of refuge for the Grecian sciences. Proclus was born, A.D. 412, at Byzantium, and died at Athens at the age of 75 years. His parents, Patricius and Marcella, were both of Lycian origin, and are spoken of by Marinus as excelling in virtue. Proclus studied first at Alexandria under the most eminent Platonic philosophers. He fre-

quented the discourses of Olympiodorus, for the purpose of learning the doctrines of Aristotle; and in mathematical science he gave himself up to Hero, whose constant companion he became. He next studied at Athens, where he was the pupil of the celebrated Syrianus, and at length became the chief of the Platonic school established in that city. Of his numerous writings on the Mathematical sciences, his commentary on the first book of Euclid's Elements of Geometry is still extant in the original Greek. It was translated into English by Thomas Taylor, in 1788, and inscribed "To the Sacred Majesty of Truth." The commentary of Proclus, though tinged with much of the mysticism of the later Platonic school, contains some interesting facts relating to the history of Geometry, and many judicious remarks on the definitions, postulates, axioms, and propositions of the first book of the Elements. From a remark at the end, it appears to have been the intention of Proclus, if he had lived, to write commentaries on the other books of Euclid, in a similar style. There are extant two other mathematical works ascribed to him: one is a small treatise on the sphere, which was published in 1620 by Bainbridge, the professor of Astronomy at Oxford: the other is a compendium of Ptolemy's Almagest, entitled Hypotyposis. The original Greek was published in 1540, and a Latin translation by Valla in the following year. Proclus also wrote on many other subjects; commentaries on several dialogues of Plato, of which some are still extant; lectures on Aristotle, and a commentary on the writings of Homer and Hesiod. Four Hymns, one to the Sun, one to the Muses, and two to Venus, are attributed to him. He also wrote on Providence and Fate, and concerning the existence of Evil; besides numerous other pieces, of which we may mention his eighteen arguments against Christianity. These arguments, except the first, are all preserved in the answer of Philoponus. The Greek was published at Venice in 1535, and a Latin version at Lyons in 1557.

Marinus of Naples was a disciple of Proclus, and his successor in the school at Athens. There is a commentary still extant, on Euclid's Data, of which he was the author: his other writings on mathematical subjects, though not numerous, have not survived. His most celebrated work, however, is a life of his master Proclus: it was published in 1700, with a Latin version by Fabricius. Of the mathematicians who lived about the middle of the sixth century, (the latest period of the decline of Grecian science,) Eutocius may be regarded as the most distinguished. He was a native of Ascalon in Palestine, and a disciple of Isidorus, one of the architects who designed and built the church at Constantinople, which is now called the Mosque of St Sophia. The only works of Eutocius which have descended to modern times, are two commentaries; one on the Sphere and Cylinder of Archimedes, and the other on the first four books of the Conics of Apollonius Pergæus. The latter was published with the Oxford edition of that author by Dr Halley, in 1710; and the former with the works of Archimedes at Oxford, in 1792. The commentary on the Sphere and Cylinder contains ten various methods of solving the celebrated Delian problem, which are of little importance in the present state of mathematical science. Besides elucidations of difficult passages of the two works, these commentaries contain many useful observations on the historical progress of the mathematical sciences.

We must not pass over in silence one writer who lived at the end of the fifth century, and the early part of the sixth, and who was almost the latest author of any eminence that wrote in the Latin language. Boëthius was the most distinguished of the Romans for his scientific writings; which, however, consisted chiefly of translations and commentaries. He was a senator and consul in the reigns of Odoacer and Theodoric, and was put to death by order of Theodoric, A.D. 526. He was educated at Athens; and his writings were numerous on almost every branch of literature and science. He was the author of a treatise on Arithmetic, and another on Geometry: of the latter there is an ancient MS. copy preserved in the Library of Trinity College, Cambridge. He also translated Euclid's Elements, and some of the writings of Archimedes and Ptolemy. Boëthius, however, is chiefly celebrated for his work entitled "De Consolatione Philosophiæ," which was much read in the middle ages, and has been translated into almost all the European languages. An Anglo-Saxon version was made by King Alfred; and ancient MS. copies exist in several public libraries: it was printed at Oxford, in 1698.

The rise of the Mahommedan power in the seventh century, and the rapid and desolating conquests which followed, hastened the extinction of the Grecian sciences. In A. D. 640, the Mahommedans invaded and conquered Egypt. The great Library of Alexandria, which is said to have contained at that time many thousand volumes, the writings of geometers, astronomers, and philosophers, was committed to the flames. As a justification of the act, the Khalif Omar declared, that, "if they agreed with the Koran, they were useless, and if they did not, they ought to be destroyed." The learned men who were congregated at Alexandria for the cultivation of science and philosophy, either fell by the swords of the conquerors, or escaped by flight, and these carried with them some remains of the sciences. In somewhat more than a century after this event, the Arabians became the most zealous patrons and cultivators of the science and philosophy of the Greeks and Hindus. The rapid progress of the Mahommedan power, both in the East and in the West, led to the foundation of a powerful empire. The second Abbaside Khalif Almansur ascended the throne A.D. 753, and shortly after, transferred the seat of his government from Damascus to the newly-founded city of Bagdad. Haroun Alrashid, the grandson of Almansur, before his accession to the Khalifat had overrun the Greek provinces of Asia Minor, and penetrated as far as the Hellespont. The reigns of Alrashid and his successor Almamun displayed at Bagdad the highest degree of luxury and splendour, which are depicted in many scenes of the famous tales of the Arabian Nights' Entertainment. The Arabians became acquainted with the astronomical and arithmetical science of the Hindus before they had any knowledge of the writings of the Greek astronomers and mathematicians. It is related in the preface to the Astronomical Tables of Ebn Aladami, that in the reign of Almansur, in the 156th year of the Hegira, A.D. 773, an Indian astronomer visited the court of the Khalif of Bagdad, bringing with him astronomical tables, which, he affirmed, had been computed by an Indian prince whose name was Phighar. The Khalif, embracing the opportunity thus presented to him, commanded the book to be translated into Arabic, and to be published for a guide to the

Arabians in matters pertaining to the stars: this task was committed to Alfazari. An abridgment of these tables was made in the succeeding age by Mohammed Ben Musa, under the patronage of Almamun before his accession to the Khalifat. (Colebrooke). The same Mohammed Ben Musa is recognized by the Arabians as the first who made known the Indian Arithmetic and Algebra: his Treatise on Algebra, the earliest written in Arabic, is still extant: a MS. copy is preserved in the Bodleian Library at Oxford, which was translated into English by the late Dr Rosen, and published at the expense of the Oriental Translation Fund. In this treatise the rules of the science are given in prose, and their accuracy is established by geometrical illustrations. In the Sanscrit treatises on Arithmetic and Algebra, the rules are given in verse. Almansur ascended the throne of the Khalifs A.D. 813, and it was the glory of his reign, that he invited learned men from various countries for the introduction of science and literature into his dominions. He founded a college at Bagdad, and appointed Mesuc of Damascus, a famous Christian physician, its president. It was here, under the auspices and encouragement of Almamun that Arabic translations of Indian and Greek science commenced, which were continued in succeeding reigns. The few manuscripts of the mathematical and philosophical writings of the Greeks which had escaped the general ruin, were diligently sought for and translated into Arabic. Translations were made of the writings of Euclid, Archimedes, Apollonius, and others: besides which, Arabic commentaries were written to elucidate and explain these writings. To the Arabians of Bagdad is due the merit of preserving many writings, which, at present, are not known to be extant in the original Greek. Almamun died at about forty-eight years of age, after a reign of more than twenty years, A. D. 833. He is reported to have uttered the following ejaculation just before his death, "O Thou who never diest, have mercy on me, a dying man!"

The celebrated Alkindi is mentioned among the mathematicians and astronomers of the age of Almamun: his medical writings, which are still extant, prove that he sustained a very honourable rank among Arabian physicians.

Alfragan was a celebrated astronomer, who flourished at the latter part of the eighth century; he was a native of Fargan, in Samarcand. In his Elements of Astronomy, a work which consists of thirty chapters, he adopts Ptolemy's hypothesis, and frequently quotes from his writings. Professor Golius, of Leyden, translated this treatise into Latin, with notes on the first nine chapters, which were published with the original Arabic, in 1669, after his death.

Albategni was an Arabian astronomer of the ninth century: Dr Halley describes him as a man of great acuteness. His work, entitled "The Science of the Stars," was founded on his own observations, combined with those of Ptolemy. On his observations were founded the famous Alphonsine Tables. He died A.D. 888; his work was first printed in 1537.

Albumazar was a physician and astronomer of the ninth century. His Introductio ad Astronomiam was printed in 1489, and his work "De magnis conjunctionibus, annorum revolutionibus ac eorum perfectionibus," in 1526, at Venice.

Rhazes was a most distinguished physician, and received the appel-

lation of the experimenter. He is also said to have been profoundly skilled in astronomy and other sciences: he had the reputation of being a skilful alchemist, and of having been the first to use chemical preparations in medicine. Many of his numerous works have come down to us, and some of them have been translated and printed.

Honain was an Arabian physician and a Christian, who lived in the ninth century. He is reported to have travelled into Greece and Persia, and afterwards to have settled at Bagdad, where he translated the Elements of Euclid, and the writings of Hippocrates, into Arabic.

Thabet Ben Korrah lived during the latter part of the ninth and the early part of the tenth century. Ebn Khallikan relates, in his Biographical Dictionary, which has been translated into English, that Thabet Ben Korrah left Harran, and established himself at Kafratutha, where he remained, till Abu Jafar Mohammed Ben Musa arrived there, on his return from the Greek dominions, to Bagdad. He became acquainted with Thabet, and on seeing his skill and sagacity, invited him to Bagdad, made him lodge at his own house there, introduced him to the Khalif Almotaded, and procured him an appointment amongst the astronomers. Thabet became secretary to the Khalif, and was distinguished for his skill in the mathematics and astronomy. To him is attributed a translation of the Conics of Apollonius, and the Almagest of Ptolemy.

In the tenth century several treatises on Geometry are ascribed to Mohammed Bagdedin, among which is one "On the Division of Surfaces;" it was translated into Latin, by Commandine, and published in 1570; it was also translated into English, by John Dee. This treatise on the Division of Surfaces is by some attributed to Euclid.

Avicenna, who lived in the tenth century, has been accounted the prince of Arabian physicians and philosophers. He was a most voluminous writer; his greatest work was an Encyclopædia, in twenty volumes, with the title of "Utility of Utilities." He is reported to have written on almost every subject of physics, metaphysics, and the mathematical sciences. Among his mathematical works was an abridgment of Euclid's Elements of Geometry. Many of his writings are extant, and some of them have been printed at Venice.

About the middle of the eleventh century, Diophantus was translated into Arabic, by Abulwafa Buzjani. In the twelfth century, Nassir Eddin, a Mahommedan, acquired the highest reputation in all branches of literature and science. He translated the Elements of Euclid into Arabic, which, with his commentary, was printed at Rome, in 1594, under the patronage of the Medici. He is also reported to have written commentaries on the Spherics of Theodosius and Menelaus.

The Kholasat al Hisab, is a compendium of Arithmetic and Geometry, composed by Baha Eddin, who died A.D. 1575. The Arabic text, and a Persian commentary written by Roshan Ali, were printed at Calcutta, in 1812. In this treatise, Baha Eddin remarks, that "Learned Hindus have invented the well-known nine figures for them;" meaning the Arabians.

The Arabian mathematicians are not noted for any very important discoveries or improvements in Geometry. There is, however, one improvement in Trigonometry of considerable importance, to which they have an undoubted claim. The Greeks in their astronomical cal-

culations employed the chords of arcs; but the Arabians introduced the sines, or half the chord of the whole arc, which led to greater simplicity in calculation.

During the middle ages but few names have come down to us of men who were skilled in the mathematical sciences. Beda, commonly called the Venerable Bede, was born A.D. 672, near Wearmouth, in the bishopric of Durham; he was the first of the Anglo-Saxon historians. His Ecclesiastical History was completed in A.D. 731: his writings are numerous, and the first collection of them was printed at Paris, in 1544.

Athelard or Adelard, was a monk of Bath, and the first who made known in England, Euclid's Elements of Geometry. This he did by making a translation from the Arabic into Latin, long before any Greek copies of Euclid were known. He is said to have travelled into France, Germany, Italy, Spain, Egypt and Arabia, to increase his knowledge. He also wrote numerous works; some MS. copies of which are said by Vossius to be in some of the College Libraries, at Oxford. The works of Ptolemy, the astronomer of Alexandria, became known to the learned of these times through the same language as the works of Euclid.

A Treatise on the Sphere, by John de Sacro Bosco, or John of Holywood, was first put forth in the early part of the thirteenth century. It is asserted by Montucla (I. p. 506), to be only an abridgment of Ptolemy. MS. copies of it exist, and it has been printed several times; there is also a commentary on this treatise, written by Clavius.

Roger Bacon, an English monk of the Franciscan order, was born near Ilchester, in Somersetshire, in 1214. He studied first at Oxford, and afterwards at Paris, and became the most celebrated philosopher of his age. Of his writings, which are numerous, some only have been printed; his tract on Chronology has not been printed. By his great skill in astronomy, he discovered the error which gave occasion for the reformation of the Calendar, and his plan was afterwards followed by Pope Gregory XIII, with this variation, that Bacon would have had the correction to begin from the birth of our Saviour; whereas Gregory's amendment reached no higher than the Nicene Council. His great knowledge of the sciences in an ignorant age, no doubt gave rise to the story of his having dealings with the devil.

John de Basingstoke, who died A.D. 1252, is reported to have introduced into England the knowledge of the Greek numerals. His merits and learning recommended him to the favour of Robert Grossetete, then bishop of Lincoln, who is reported by Roger Bacon to have excelled in Geometry and the other mathematical sciences, and to have spent many years in the study of them.

Grossetete wrote on many subjects, and some of his pieces are still extant. He is reported to have studied first at Cambridge, afterwards at Oxford, and lastly at Paris; he was made Bishop of Lincoln in 1235.

About the year 1261 another translation of Euclid's Elements of Geometry from the Arabic was made into Latin, by Campanus of Novara, who also wrote a Commentary on Euclid. It was printed at Venice, in 1482, without a title-page, by Erhardus Ratdolt, and was the first printed edition of the Elements in Latin. The translation of Campanus has been supposed by some to be only a revision of the translation which had been made by Adelard. It is possible he might

have revised and improved Adelard's translation, and annexed his Commentary; but this could only be determined by a collation of early MS. copies of the two versions. Campanus was the author of a treatise on the Quadrature of the Circle, which has been printed, in the Appendix to the Margarita Filosofica.

In the thirteenth century, Vitello, a native of Poland, displayed an intimate knowledge of the geometrical writings of the Arabians and Greeks. He is chiefly celebrated for the treatise on Optics, which he composed from the writings of the Greeks and Arabians on that science. It was printed in 1572: he also translated the Spherics of Theodosius.

As geometers in the fourteenth century, Chaucer and Wallingfort may be mentioned. Chaucer, the father of English poetry, was born in London, A.D. 1328, and died at the age of seventy-two. He is said to have been learned in all the sciences. He was the author of a treatise on the Astrolabe, in which he describes the methods of astronomical observation known in his day: it has been printed.

Richard Suisset or Swineshead wrote a work called "the Calculator." It is highly commended by Cardan, and has been printed.

Thomas Bradwardine, Archbishop of Canterbury, in the time of Edward III. was the author of a work entitled "Geometria Speculativa cum Arithmetica Speculativa." He also wrote "de Proportionibus," and "de Quadratura Circuli." Bradwardine was the most distinguished geometrician of his time. He died A.D. 1348: his work on Geometry was printed in 1495.

George of Purbach, commonly called Purbach, was born in 1423, at a town of that name on the confines of Bavaria and Austria, and became the most eminent astronomer and mathematician of his time. He amended the Latin translation of Ptolemy's Almagest, which had been made, not from the Greek original, but from the Arabic translation. He also corrected, by means of the Greek text, the translation of Archimedes, made by Gerrard of Cremona, and wrote commentaries on those books of Archimedes which Eutocius had omitted. He translated the Conics of Apollonius, and made a Latin version of the Spherics of Theodosius and Menelaus; and of the book of Serenus on Cylinders. He composed an introduction to Arithmetic, and a treatise on Dialling and Gnomonics. He also made very great improvements in Trigonometry by introducing the table of Sines, and a new division of the radius into 600,000 instead of 60 equal parts. He thus completely changed the appearance of that science, so important in Astronomy.

John Muller, or, as he was called, Regiomontanus, from the Latinized name of his native town, Konigsberg, died at Rome, in 1476, at the early age of forty years: he had been invited thither by Pope Sixtus IV. to assist in rectifying the Calendar. Muller was the disciple and friend of Purbach; after whose premature death, he revised and completed, at Rome, the Latin version of Ptolemy, which Purbach had left unfinished; he also further improved Purbach's division of the radius. He rejected the sexagesimal subdivision of the radius, and made it to consist of 1,000,000 equal parts. With this new division, he computed the sines of arcs, to every minute of the quadrant, to seven places of figures; he also annexed a table of secants. His celebrated treatise on Triangles, in five books, was not published in his lifetime, and did not appear till 1533, when it was edited and

published by Schener. The solutions of the more difficult cases of plane and spherical triangles are to be found in his work; and, with the exception of what Spherical Trigonometry owes to Napier, that science may be said to have made but small advances for more than two centuries after the age of Muller. In his work on Triangles he gave a table for finding the angle of a right-angled triangle, from the base and perpendicular, without knowing the hypothenuse. This table, which he styled "Canon Fœcundus," was calculated for every degree of the quadrant, and was, in reality, a table of tangents, which he was the first to introduce into the science of Trigonometry. He added also many new theorems to that science; and after him few improvements were made in it till the time of Euler. His fifth book contains numerous problems concerning rectilinear triangles, of which some are solved by Algebra. The tables of Regiomontanus were printed in 1490.

The revival of ancient literature in Europe, about the middle of the fifteenth century, contributed to bring the mathematical writings of the Greeks into notice; and the discovery of the art of printing, about the same time, was the commencement of a new era in literature and science. The writings of the ancients were now no longer confined in MS. to the religious houses, and to the few who had the means of purchasing copies of the manuscripts. The fall of the Eastern empire, and the capture of Constantinople by the Turks, in 1458, drove many Greeks to seek their safety and subsistence in Italy and other parts of Europe. These became living instructors in the Greek language, and very much facilitated its revival in Italy. As the literature of the Greeks became known, their mathematical writings also attracted notice: they were printed and translated, and published with commentaries. Restorations also of lost treatises were attempted by mathematicians of that and following centuries.

Mr Hallam in a note, p. 157, Vol. I. of his History of the Middle Ages, remarks, "It may be considered a proof of the attention paid to Geometry in England, that two books of Euclid were read at Oxford, about the middle of the fifteenth century." With respect to the mathematical science of the middle ages, though we do not find any original writers who made additions to it by their discoveries; there must, in justice, be conceded to the men of those times, the possession of no mean or scanty knowledge of Geometry. The splendid ecclesiastical buildings of the middle ages, which are still standing both in Great Britain and in the south and west of Europe, evince in their structure a practical knowledge, at least, of some of the most difficult problems of Geometry and the science of Equilibrium.

Lucas de Burgo's work, entitled "Summa de Arithmetica, Geometriâ, &c." was published in 1494. Fifteen years later he published the Latin translation of Euclid, which had been made from the Arabic by Campanus.

Copernik, or (as he is commonly called) Copernicus, was born at Thorn in Prussia, in 1473, and his name is immortalized by his discovery of the true Solar system. The motion of the sphere, of the fixed stars with the sun and moon round the earth, in twenty-four hours, appeared to him too complex; and he felt persuaded that such could not be the system of nature. The Pythagoreans and some later philosophers had held the rotation of the earth round its own axis, and its revolution round the sun. Copernicus collected the writings of pre-

ceding astronomers, and examined all the hypotheses they had devised, for the explanation of the phænomena of the heavens: the result of his labours was his great work, "De Revolutionibus Orbium Cœlestium," Libri VI. It was completed in 1530, but was not published till a very few days before his death in 1543. In the title-page is quoted the admonition of Plato, *ἀγεωμέτρητος οὐδεὶς εἰσίτω*. Copernicus was the author of a tract on Plane and Spherical Trigonometry, which contained a Table of Sines. It was first printed at Nuremberg, and afterwards at the end of the first book of his great work, De Revolutionibus, &c. The publication of this work was superintended by George Joachim Rhæticus, the disciple and latterly the assistant of Copernicus, in his astronomical labours. He was born in 1514, and died in 1576 at Feldkirk in the Tyrol. With the view of making astronomical calculations more accurate, he commenced a table of sines, tangents, and secants for every ten seconds of the quadrant, to fifteen places of figures; which he did not live to complete. This work was completed and published by his disciple Valentine Otho in 1596. The table of sines for every ten seconds, and for every second in the first and last degrees of the quadrant, which he had completed, was published in 1613 by Pitiscus, who extended the value of some of the latter sines to twenty-two places of figures.

Nicholas Tartaglia was a celebrated mathematician, born at Brescia in 1479. He was the original discoverer of the solution of Cubic Equations, which he first effected in 1530, and which Cardan, who has generally had the merit of the discovery, surreptitiously obtained from him. The treatise of Tartaglia on the Theory and Practice of Gunnery, is the earliest which treated of the motion of projectiles. He also published an edition of Euclid's Elements, in Italian, with a Commentary. The last part of his great work, "Trattato de Numeri et Misure," was published in 1558.

John Werner of Nuremberg, one of the most distinguished astronomers and geometers of his day, was born in 1468 and died in 1528: he was the first who attempted to restore the geometrical analysis of the ancient Greeks. In his "Opera Mathematica," which he published in 1522, will be found what he effected, both in the Conic Sections and in some solid problems. He also wrote a work on Triangles.

Zamberti made the first translation of Euclid's Elements, from the original Greek into Latin, which he published at Venice in 1505. The original Greek was first published at Basle in 1533, edited by Simon Grynæus. This edition of Euclid was the foundation of Commandine's translation in 1572, as it has been, in great measure, that of later editions.

During the sixteenth century Euclid was held in so high estimation among mathematicians, that no attempts appear to have been made to advance the science of Geometry, beyond the point at which he left it. Commentaries and translations seem to have been almost all they attempted.

Erasmus Reinhold wrote commentaries on Euclid's Elements and the great work of Copernicus. He was born at Salfeldt in Thuringia in 1511, and died in 1553, and was considered one of the most eminent mathematicians of his time. He wrote on Plane and Spherical Triangles, improved Muller's Tables of Sines and Tangents, and was the author of numerous other works on the sciences.

F. Maurolyco, of Messina, was born in 1494, and died in 1575; he made some improvements in Trigonometry, and edited the Spherics of Theodosius and Autolycus; he also published his "Emendatio et Restitutio Conicorum Apollonii Pergæi," in 1575.

Frederick Commandine was born at Urbino in 1509, and died in 1575: he is justly accounted one of the first geometers of his age. He composed several original works on the sciences; but is chiefly known for his translations of several of the Greek geometers, of whose works some had not previously been translated into Latin. He translated the geometrical writings of Archimedes, and wrote a commentary upon them; also four books of the Conics of Apollonius Pergæus, with the Lemmas of Pappus and the Commentaries of Eutocius, together with Serenus on Sections of the Cone and Cylinder. He was the first who translated the last six books of the Mathematical Collections of Pappus into Latin, which were published after his death, in 1588, by the munificence of the duke of Urbino. The edition most commonly met with is that of Manolessius, printed in 1660. This translation of Pappus first directed the attention of mathematicians to the subjects of the lost works of the ancient geometers, and gave rise to various attempts for the restoration of several, which are described in the preface to the seventh book. Commandine's translation of Euclid with a commentary, was put forth in 1572. An Italian version was published under his direction in 1575. An English translation of the Latin version was published in 1715, by Dr John Kiel, at that time Savilian Professor of Astronomy at Oxford. Commandine also translated into Latin, an Arabic version of Euclid's tract on the Division of Surfaces.

To this period belongs John Dee, a man distinguished for his mathematical and astrological knowledge. He was educated at St John's College, Cambridge, where he chiefly devoted his attention to mathematical studies. He was made one of the fellows of Trinity College, at its foundation by Henry VIII., and afterwards, as we learn from Lilly's Memoirs, read lectures on Euclid at Rheims. He wrote a learned preface, of fifty folio pages, to Billingsley's Euclid: it bears the date of February 9, 1570, and was written at Mortlake. He translated into English the tract on Division of Surfaces, from the Latin version of Commandine.

To Henry Billingsley, a citizen of London, is due the merit of making the first English translation of Euclid's Elements of Geometry. It comprises the whole of the 13 books, with the 14th and 15th which were added by Hypsicles, and a 16th by Flussas. It was chiefly made from the Latin of Campanus, and contains a commentary, besides the preface, by John Dee, above-mentioned. It was first published in 1570 in a large folio volume; and a second edition was edited by Leeke and Serle in 1661.

Francis Vieta was born at Fontenoy in Lower Poitou in 1540, and was a man of original genius, as is manifest from his discoveries and improvements in different branches of the mathematical sciences. He introduced the use of letters into Algebra, and invented many theorems. He effected great improvements in Geometry, made considerable additions to the science of Trigonometry, and reduced it to a system. He wrote a treatise on Angular Sections, and restored the tract of Apollonius on Tangencies, which he published with the title of

Apollonius Gallus. His collected works were published at Leyden by Schooten.

Galileo Galilei, the cotemporary of Milton and friend of Kepler, was born at Pisa in Tuscany in 1564, and at a very early age, gave evidence of great genius for geometrical and philosophical pursuits. This became more evident while he was under the direction of Guido Ubaldi at the University of Pisa. From the time of Archimedes, a period of nearly 2000 years, little or nothing had been done in Mechanical Geometry, till Galileo first extended the bounds of that science by the application of Geometry to Motion. He first taught the true theory of uniformly accelerated and retarded motions, and of their composition, and proved that the spaces described by heavy bodies, *falling freely* from the beginning of their motion, are as the squares of the times. Contrary to the general belief, he maintained that all bodies, whether light or heavy, fall to the earth, through the same space in equal times; and attempted to verify the truth of his proposition by experiment. The two bodies, however, which he let fall from the top of the hanging tower of Pisa, did not reach the ground exactly at the same instant. The reason Galileo assigned was that the resistance of the air retarded the lighter body more than the heavier. He invented the cycloid, and the simple pendulum which he used in his astronomical experiments. The application of the pendulum to clocks was made by his son; and subsequently brought to perfection by Huygens. Galileo first proved that a body projected in any direction not perpendicular to the horizon describes a parabola; and it may be remarked that the Geometry of Galileo was wholly applied to explain and advance the science of Motion. The invention, in 1609, of the refracting telescope which still bears his name, disclosed new views of the solar system. By the aid of this he discovered that the moon is an opaque body with mountains and vallies, and that she receives her light by reflection from the sun. He also put forth the conjecture that the moon might be an inhabited world like the earth. He observed the different phases of the planet Venus, proving her motion round the sun. He discovered four of Jupiter's satellites, and caught an imperfect view of the ring of Saturn, which at the time of his observation appeared like two small stars, one on each side of the planet's disk. He was the first who discovered spots on the sun's disk, and from their varying position he inferred the motion of the sun on its axis. By his telescope he also discovered that the whiteness of the Milky Way is caused by innumerable stars apparently more close together than in the other parts of the heavens. His celebrated work, the "Dialogues on the Ptolemaic and Copernican Systems," was published at Florence, in 1632; and though dedicated to Ferdinand II., brought Galileo under the hatred of the Jesuits, and the power of the Inquisition. In June 1632, that court condemned him of heresy, for teaching that the sun is the centre of the solar system, and that the earth revolves on its own axis, and moves round the sun. He was obliged, on his knees, to abjure his belief of all he had advanced in his Dialogues, and to swear, that for the future, he would never assert or write any thing in favour of such heretical opinions. He was sentenced to imprisonment during the pleasure of the court; and for a certain time to recite daily the seven penitential psalms. Galileo is reported to have whispered in the ears of a friend, as he rose from his knees, "E pur se muove."

The influence of his powerful friends no doubt moderated the sentence of the Inquisition, whose proceedings in what they called heresy, were always of the most cruel, frequently of the most horrid description. Galileo was nevertheless kept strictly confined in the prison of the Inquisition for two years. Even when upwards of 70 years of age, new rigours were exercised against him, on account of some fresh suspicions of pope Urban VIII., which were inflamed by the philosopher's inveterate foes, the Jesuits. His health greatly suffered, and he was afterwards released from confinement; but became blind some years before his death. During this period he finished his dialogues on Motion. His death took place in 1642, at the age of 78. Most of the works of Galileo were collected and printed in 1656; they were translated into English by T. Salusbury, and published in 1661, in his Mathematical Collections. A more complete collection of his works was published at Milan, in 1811. At Florence, in 1674, was published a work of Galileo, under the title of "Quinto Libro de gli Elementi d'Euclidi, &c.," by Vincenzio Viviani, one of his distinguished pupils.

Christopher Clavius was born at Bamberg in Germany, in 1537, and died at Rome, in 1612: his writings on the mathematical sciences were collected and printed at Mayence, in five folio volume, in 1612: he superintended the reformation of the Calendar under the direction of pope Gregory XIII. He was skilled in the ancient Geometry, and edited several of the Greek mathematical writings, and on some of them he wrote commentaries, among which may be mentioned the Elements of Euclid. He was the author of a work on Practical Geometry, and of a commentary on Sacro Bosco's treatise on the Sphere, which works were first published in 1570. He was a zealous cultivator of the sciences; but no discoveries or improvements are attributed to him.

Willebrod Snell was a man of original genius: he was born at Leyden, in 1591, and died in 1626. To him is attributed by Huygens the discovery of the law of the refraction of light, before it was made known by Descartes. He was a skilful geometer, and published in 1608, with the title of Apollonius Batavus, his attempted restoration of three tracts of Apollonius "De Sectione Determinata," "De Sectione Rationis," and "De Sectione Spatii." The tract "De Sectione Determinata" was translated into English by the Rev. J. Lawson, and published in 1772. This tract is imperfectly restored, as there is omitted the distinction of the situation of the points; and there is no complete exposition of the determinations. He also wrote a tract on the Circle; in which are given various approximations, both geometrical and arithmetical. Ludolph Van Ceulen is also noted for having calculated the ratio of the diameter to the circumference of the circle, to thirty-five places of decimals.

Sir Henry Savile, an accomplished scholar, founded two professorships at Oxford; one of Geometry, and the other of Astronomy, in 1619. As the first Savilian professor of Geometry, he delivered thirteen lectures on the first book of Euclid's Elements of Geometry, in 1620, which he published in Latin during the following year. In 1585 he was appointed Warden of Merton College, and Provost of Eton in 1596; the former appointment he held for thirty-six years. In 1613 he published the works of Chrysostom in Greek, and was the author and editor of several other works.

Leonard Digges was a mathematician of some note, and the author of a treatise on Geometry, in three books, which he called Pantometria. It was published by his son in 1591, with a supplementary book on the five regular solids.

The ancient Geometry of the Greeks was considered perfect, as indeed within the bounds of its legitimate province it is; and no attempts were made to improve or extend the methods handed down from the ancients till the time of Kepler. He was born at Wiel, in 1571, and died in 1630: he introduced the new principle of infinity into Geometry. He conceived a circle to be made up of an infinite number of infinitely small triangles with their vertices in the centre, and their bases coinciding with the circumference of the circle. A cone, in the same manner, was supposed to consist of an infinite number of indefinitely small pyramids. This idea of Kepler lies at the foundation of the higher analysis. He published his views in a work entitled "Nova Stereometria," in 1615, which have been discussed under the names of "Infinitesimals;" "Fluxions and Fluents;" "Differential and Integral Calculus."

Pierre de Fermat was born in 1595, and died in 1665; though a person of extraordinary vanity, he was a mathematician of original genius. He attempted the restoration of the two books of Apollonius on Plane Loci: he has given the synthesis, but omitted the analysis of the propositions; he has also omitted the distinction of cases of each proposition, and has not ascertained the determinations. He wrote a treatise on Spherical Tangencies, in which he has demonstrated, in the case of spheres, properties analogous to those which Vieta had before demonstrated in his restoration of Apollonius on Tangencies. He was the author of a treatise on Geometric Loci, both plane and solid. Fermat had acquired some general notion of the Porisms of Euclid. He was the author of a method of maxima and minima, and of the quadrature of parabolas of all orders, besides several discoveries in the properties of numbers, one of which still bears the name of Fermat's theorem. His collected works were, after his death, published at Toulouse, in 1679.

Descartes was born in 1596, and died at Stockholm in 1650. He was the cotemporary of Galileo, Fermat, Roberval, and many other celebrated mathematicians. He has been cited as the inventor of the New Geometry, or, as it is called, Analytical Geometry; the foundation of which, however, had certainly been laid before his time. Algebra had been applied to Geometry by Vieta, and, to some extent, by other mathematicians. But though he did not originate, he certainly extended the limits and powers of Analytical Geometry, by the discovery of a new principle. The use of co-ordinates appears for the first time, though under a different name, in the second book of his Geometry, which was published in 1637. He first taught the method of expressing curves by equations. The simple conception of expressing curve lines by means of equations between the two variable co-ordinates of a point in the curve; and curve surfaces by means of equations which contain the three co-ordinates of any variable point in the surface, has led to a new science, and entitles the discoverer to be classed amongst men of profound genius. This discovery and its applications by means of the higher analysis, have given a power to Geometry before unknown. In the Epistles of Descartes, which were printed in 1683, he

remarks (Part III. Ep. 72), "In searching out the solution of geometrical questions, I always make use of lines parallel and perpendicular as much as possible: and I consider no other theorems than the two following; the sides of similar triangles are proportionals; and in right-angled triangles the square of the hypothenuse is equal to the squares of the two sides. And I am not afraid to suppose several unknown quantities, that I may reduce the proposed equation to such terms as that it may depend on no other theorems than these two."

Buonaventura Cavalieri, better known by the Latinized appellation, Cavalerius, was born at Milan in 1598. He was a pupil of the celebrated Galileo, and became Professor of Mathematics at Bologna; he died in 1647. He was the author of several works on the mathematical sciences, the most important of which is a treatise on Indivisibles in seven books, which he put forth in 1635: it is important as being one of the first attempts to extend the powers of the ancient Geometry. He conceived a line to be made up of an infinite number of points; a surface to be formed of an infinite number of such lines; and a solid to be composed of an infinite number of such surfaces. These elements of geometrical magnitudes he named Indivisibles: and the principle he assumed in the application of these assumptions was, that the ratio of the infinite sums of lines, or of planes as compared with the unit of surface or volume, was the same as that of the surface or volume of which they were the measures. He shewed that his new principle was, in effect, the same as the method of Exhaustions, but a more convenient mode of reasoning, being less tedious and more direct. In the first six books he explains and applies his theory of Indivisibles, and in the seventh book he proves the same results by methods independent of Indivisibles; with the view of shewing the agreement of the results, and the consequent truth of his new principle. Guldin controverted and wrote against the doctrine of Indivisibles, and was answered by Cavalerius in the third of his "Exercitationes Geometricæ sex," which were published in 1647. This work consists of exercises in the Method of Indivisibles, with answers to Guldin's objections. The method of Cavalerius is not free from error, as he applies the process of simplification at too early a stage of the investigation, by which means the strict logic of the reasoning is violated, while the correctness of the result is not affected.

Roberval was born in 1602, and adopted the theory of Cavalerius, but improved his modes of expression. His Traité des Indivisibles was published in 1693, after his death, and contains his application of the new principles. His method of drawing tangents was an approximation to that applied afterwards by the principles of Fluxions and the Differential Calculus.

Albert Girard was a Fleming who displayed great genius in the Mathematics. He was the first who announced the restoration of the three books of the Porisms of Euclid, in a work on Trigonometry, which was printed at the Hague in 1629. To what extent he succeeded is not known, as the results of his labours have never been published. To him are due some general theorems for the measuring of solid angles.

Marinus Ghetaldus was a distinguished geometer at the beginning of the seventeenth century: he was the author of several works on the ancient Geometry: he attempted, from what could be gathered from

Pappus, a restoration of the lost book of Apollonius on Inclinations, and published it in 1607, with the title of Apollonius Redivivus. In the same year also he put forth a supplement to the Apollonius Gallus of Vieta, and a collection of problems. Among his other writings may be mentioned his Archimedes Promotus, which was first published in 1603.

Blaise Pascal was born at Clermont in 1623, and at an early age gave proofs of extraordinary ability. He was of an enquiring mind, desirous of knowing the reasons of every thing. It is reported of him that whenever he could not obtain from others sound reasons, he used to seek them out for himself; never giving his assent except on conviction. Pascal challenged the mathematicians of his day to prove some properties of the cycloid; but as the answers he received from Wallis and other eminent men were unsatisfactory, he himself gave the complete proofs of all the properties mentioned in his challenge. He invented the arithmetical machine which bears his name, and was the author of some pieces on other mathematical subjects. His intense application to study injured his health. He became a Jansenist, and retired to the Abbey of Port Royal, near Paris, where he composed his Provincial Letters, and wrote his thoughts on Religion and other subjects. These were published after his death, which happened when he was only thirty-nine years of age. His works were collected and published at Paris in 1779.

Schooten was Professor of Mathematics at Leyden. In 1649 he published an edition of Descartes' Geometry, and in 1657 an original work entitled Exercitationes Mathematicæ. He attempted the restoration of the Loci Plani of Apollonius. He has given the synthesis only, omitting the analysis, except in a few instances. He has also omitted the determinations and the distinction of cases; and in the preface he acknowledges that his attempted restoration was designed to be an illustration of the Geometry of Descartes, by furnishing appropriate examples to his method.

Christian Huygens was born at the Hague in 1629, where he died in 1696. He was one of the most ingenious mathematicians of his age. He was the author of two treatises, one entitled "Theoremata de Quadratura Hyperbolæ, Ellipsis et Circuli, ex dato portionum gravitatis centro;" and the other, "De Circuli magnitudine inventa;" which, at the time of their publication, were highly esteemed. He was also the author of some other pieces on Geometry, which were published at Paris in 1693, and he discovered the theory of Evolutes. His studies were not confined merely to the speculative portion of the mathematical sciences, but were extended to questions of practical utility. Among them may be named his improvements in telescopes, and his method of rendering the oscillations of pendulums isochronous. He was the discoverer of Saturn's ring, and a third satellite of that planet. When nearly sixty years of age he read and admitted the theory of centripetal forces, and the gravitation of the planets to the sun, which had been proved in Newton's Principia. His writings are numerous.

Dr Isaac Barrow was a distinguished scholar and geometer, and the tutor of Isaac Newton when an undergraduate at Trinity College. Dr Barrow was appointed Greek Professor at Cambridge in 1660, and Gresham Professor of Geometry in 1662: the latter office he resigned

on his appointment to the new Professorship, founded at Cambridge by Mr Lucas, in 1663. This also he resigned, in favour of Mr Newton, in 1669. He became Master of Trinity College in 1672, and died in 1677 at the age of 47 years. In 1655 he edited, in Latin, the thirteen books of Euclid's Elements of Geometry, which were translated into English and published in 1660. He also put forth Euclid's Data in 1657. His Lectiones Geometricæ were published in 1670, and translated into English, by Stone, in 1735. They contain his method of drawing tangents to curves, which is similar to that by the method of fluxions or the differential calculus: the difference being only in the notation. His Lectiones Opticæ were published in 1669. He also edited Archimedes, Apollonius and Theodosius. The Lectures which he delivered, as Lucasian Professor, were published after his death in 1683, with the title of Lectiones Mathematicæ, and were translated into English by Kirkby, in 1734: they are confined to Euclid's Elements of Geometry. He also applied the method of indivisibles to the propositions of Archimedes on the Sphere and Cylinder. His treatise was printed in 1678.

Sir Isaac Newton was one of the greatest philosophers that ever lived. The inscription on the pediment of the statue of Newton in the chapel of Trinity College, "Qui genus humanum ingenio superavit," records the unquestioned judgment of posterity. He was the original inventor of the method of fluxions and fluents; in 1665, his attention was first directed to the subject, and his method was completed before 1669. The merit of the invention was claimed for Leibnitz, who had put forth, in 1684, his view of the principles of infinitesimals or differentials. The terms employed and the notations adopted by these two great men were different; they agreed in the substance and object of the theory, but there was a considerable difference in their conception of the principles. An angry controversy arose respecting these claims, and an appeal was at length made to the Royal Society, by which a committee of enquiry was appointed. The result of their enquiries and deliberations was a decision in favour of Newton. The papers relating to this enquiry were printed in 1712, with the title of Commercium Epistolicum de Analysi promota. His discoveries in optics, and his new theory of light and colours for the explanation of optical phenomena, he did not publish till 1704: nearly thirty years after his chief discovery in that science.

His greatest discovery, however, was that of universal gravitation. His thoughts were first led to the subject in 1666, just after he had left Cambridge for the country, on account of the Plague. As he was sitting alone in a garden, some apples fell from a tree, and his thoughts were led to the subject of gravity. He considered that as this power is not found to be sensibly diminished at the remotest distance from the surface of the earth to which we can rise, it seemed reasonable to conclude that it must extend much further than is commonly believed. He enquired: "Why not as high as the moon? and if so, her motion must be influenced by the force of gravity: perhaps she is retained in her orbit by it. Though the power of gravity is not sensibly lessened, in the little change of distance at which we can place ourselves from the surface of the earth, yet it is possible that at the height of the moon this power may differ much in degree from what it is here.' To make an estimate of the amount of this diminution, he considered,

that if the moon were retained in her orbit by the force of gravity, no doubt the primary planets are carried round the sun by the like power: and by comparing the periods of the several planets with their mean distances from the sun, he found, that if any power like gravity held them in their courses, its intensity must decrease inversely as the square of their distances from the sun. He arrived at this conclusion, from considering them to move in circles concentric with the sun; from which form the orbits of the greater part of them do not greatly differ. By supposing therefore the force of gravity, when extended to the moon, to decrease in the same manner, he computed whether that force would be sufficient to keep the moon in her orbit. In this computation, taking sixty miles to a degree, the result at which he arrived did not shew the power of gravity to decrease as the inverse square of the moon's distance from the earth: whence he concluded that some other cause must, at least, combine with the action of gravity on the moon. Being unable to satisfy himself respecting this, he laid aside, for that time, all thought upon the subject. In the winter of 1676, he discovered the two grand propositions, that by a centripetal force varying inversely as the square of the distance, a planet must revolve in an ellipse about a centre of force, placed in one of the foci; and by a radius vector drawn to that focus, describe areas proportional to the times. These are proved in the second and third sections of the first book of the Principia. After the year 1679 his thoughts were again turned to the moon; and, by using in his computation the more accurate length of a degree, he arrived at the conclusion, that that planet appeared, agreeably to his former conjecture, to be retained in her orbit by the force of gravity, varying as the inverse square of the distance. On this principle he proved that the primary planets really moved in such orbits as Kepler had supposed. He afterwards drew up about twelve propositions relating to the motion of the primary planets round the sun, and sent them, at the end of the year 1683, to the Royal Society. Soon after this communication, Dr Halley became known to Newton, and having learned from him that the proof of the propositions respecting the primary planets was completed, earnestly solicited him to finish the work. Accordingly the first edition was printed under the care of Dr Halley, with the title of "Philosophiæ Naturalis Principia Mathematica," and published in 1687.

The reader of the Principia cannot fail of perceiving that Sir Isaac Newton was a most profound geometer. Newton and his cotemporary Maclaurin were the first who applied the consideration of the degrees of equations to the discovery of the general properties and characteristics of curved lines, and to them and Cotes are due the discovery of their most important general properties. Newton has given the results of his investigations on this subject, in his "Enumeratio Linearum tertii ordinis," which, with his tract on the quadrature of curves by the method of fluxions and fluents, were first printed at the end of the first edition of his Optics, in 1704. His principal object was the enumeration of lines contained in an equation of the third degree between two variables. He discovered seventy-two different species, and four more were added by Stirling. In his "Arithmetica Universalis," he has applied the method of Descartes to the solution of geometrical problems and the construction of the roots of equations. The whole works of Newton were edited by Dr Horsley in five quarto volumes, in 1779.

Edmund Halley was one of the most eminent geometers and astronomers of his age. In 1703 he succeeded Dr Wallis as Savilian Professor of Geometry at Oxford. Soon after his appointment, he commenced a translation from the Arabic, of the treatise of Apollonius De Sectione Rationis, and attempted the restoration of his two books De Sectione Spatii, from the account given by Pappus in the seventh book of his Mathematical Collections: the whole of these he published in 1706. The tract de Sectione Rationis, recovered from the Arabic, is a complete specimen of the ancient method of analysis. He also published at Oxford in 1710, an amended translation of seven books of the Conics of Apollonius, and attempted a restoration of the eighth, which was lost. This magnificent folio edition contains the first four books in the original Greek, together with a Latin translation; the next three in Latin from the Arabic version; and the last book in Latin as restored by Dr Halley himself, together with the Lemmas of Pappus and the Commentaries of Eutocius. Halley united a profound knowledge of the ancient geometry with the new geometry of Descartes, and applied the latter to the construction of equations of the third and fourth degrees by means of a parabola and a circle. His memoir on this subject was published in the Philosophical Transactions for 1687. At 63 years of age, he succeeded Flamsteed as Astronomer Royal at Greenwich; and for eighteen years discharged the duties of that office without an assistant. He died at the age of 86, in the year 1742.

James Gregory was an eminent mathematician, the cotemporary and correspondent of Newton, Huygens, Wallis, and others of that time. He was the inventor of a reflecting telescope which bears his name. He discovered a method of drawing tangents to curves, geometrically. His "Geometriæ Pars Universalis" was first published at Padua, in 1668, and his "Exercitationes Geometricæ" in the same year.

Dr David Gregory, the nephew of James, was chosen Savilian Professor of Astronomy, in 1691, in preference to Dr Halley. In 1702 he published his chief work, entitled "Astronomiæ Physicæ et Geometricæ Elementa," founded on the Newtonian hypothesis; and in the following year, the works of Euclid, in Greek and Latin.

Abraham Sharp, a skilful mathematician and expert mechanic, became the amanuensis of Flamsteed, in 1688, and assisted him in his "Historia Cælestis." He had the chief hand in constructing the mural arc at the Greenwich Observatory. He published in 1717, "Geometry Improved, by A. S. Philomath;" an elaborate treatise, and containing solutions of many difficult problems: he died in 1742, at the age of 91.

Alexis Claude Clairaut was born in 1713, and died in 1765. To Clairaut is due the merit of having exhibited methodically the doctrine of three co-ordinates in space, applied to curve surfaces and the lines which originate in their intersections. His celebrated treatise on this subject was published in 1731, entitled "Traité des Courbes à double Courbure." Before Clairaut, however, M. Pacent, in a memoir read before the French Academy of Sciences, in 1700, had illustrated the extension of the principle of Descartes by a curve surface, expressed by an equation between three variables. Also John Bernouilli had expressed surfaces by an equation, involving three co-ordinates in his solution of the problem, "To find the shortest line which can be drawn on a surface between two given points."

The expression "curve of double curvature" arises from the consideration, that such a curve partakes of the curvature of two plane curves, which are in fact its projections on two co-ordinate planes. It was first suggested by M. Pitot, who employed it in a memoir on the spiral thread on a right cylinder, and which he read before the French Academy of Sciences, in 1724.

Euler, in his "Introductio in Analysin Infinitorum," published in 1748, explains the general principles of the analytical theory of curves, and in the extension of his investigations to geometry of three dimensions, discusses the equation between three variables, which includes surfaces of the second degree. The treatise of Cramer, first published in 1750, with the title "Introduction à l'Analyse des lignes courbes Algébraïques," is one of the most complete on this branch of geometry.

Edmund Stone published, in 1731, an edition of Euclid's Elements, with an account of his life and writings, and a defence of the Elements against modern objectors. In 1735 he translated and published Dr Barrow's Geometrical Lectures. He wrote an account of two new species of lines of the third order, not noticed by Newton, which was published in the forty-first volume of the Philosophical Transactions of the Royal Society, of which he was a fellow: he was also the author of a Mathematical Dictionary and some other works*.

Dr Robert Simson was a native of Ayrshire, and born in 1687. He was appointed Professor of Mathematics at Glasgow in 1711, where he continued to discharge his duties as an instructor for nearly fifty years. During this period his attention was principally directed to the writings of the ancient Greek geometers. His restoration of the Loci Plani and the Determinate Section of Apollonius, and his treatise on the Porisms of Euclid, entitle him to the high reputation he still holds as a geometer.

* The following account of Stone from Dr Hutton, may be cited as an example of true genius overcoming all the disadvantages of birth, fortune, and education. Edmund Stone was the son of a gardener of the Duke of Argyle. At eight years of age he was taught to read; and at eighteen, without further assistance, he had made such advances in mathematical knowledge as to be able to read the Principia of Newton. As the Duke was one day walking in his garden, he saw a copy of Newton's Principia lying on the grass, and called some one near him to take it back to the library. Young Stone, the gardener, modestly observed, that the book belonged to him. To you! replied the Duke; do you understand Geometry, Latin, Newton? I know a little of them, replied the young man, with an air of simplicity. The Duke was surprised, and having himself a taste for the sciences, he entered into conversation with the young mathematician. He asked him several questions, and was astonished at the force, the accuracy, and the candour of his answers. But how, said the Duke, came you by the knowledge of all these things? Stone replied, a servant taught me, ten years since, to read. Does any one need to know more than the twenty-four letters of the alphabet, in order to learn any thing else that one wishes? The Duke's curiosity was redoubled: he sat down on a bank, and requested a detail of all his proceedings. I first learned to read, said Stone; the masons were then at work upon your house; I went near them one day, and saw that the architect used a rule and compasses, and that he made calculations. I enquired what might be the meaning and use of these things, and was informed that there is a science called Arithmetic. I purchased a book of Arithmetic, and learned it. I was told there was another science called Geometry; I bought the books, and learned Geometry. By reading, I found that there were good books in these two sciences in Latin: I bought a dictionary, and learned Latin. I understood, also, that there were good books of the same kind in French; I bought a dictionary, and learned French; and this, my Lord, is what I have done. It seems to me that we may learn every thing when we know the twenty-four letters of the alphabet. The Duke, highly pleased with the account, brought this wonderful genius out of obscurity, and provided him with an employment which left him leisure to apply himself to the Sciences.

Dr Simson's first endeavours were directed to improve the defective restorations of the books on the Geometrical Analysis by preceding geometers. His restoration of Apollonius is entirely according to the ancient method; and is more complete than any preceding attempt of the kind. In his preface to the Sectio Determinata, which he restored, he points out the defects in Snell's restorations, and notices the solutions of some of the problems by Alexander Anderson, in his supplement to Apollonius Redivivus, published at Paris in 1612. He also remarks on some of the problems in the Treatise on Geometrical Analysis, by Hugo d'Omerique, and in his work adopts some propositions from these performances. But Dr Simson is more generally known at the present day for his translation of the first six and the eleventh and twelfth books of Euclid's Elements of Geometry. The first edition was published both in Latin and English in 1756. The English translation has almost superseded every other, and may be regarded as the standard text of Euclid in English, having maintained its character in this country for nearly a century. The Data of Euclid was added to the second edition of the Elements in 1762. Dr Simson's first publication, except his paper on Porisms in the Philosophical Transactions, was his Geometrical Treatise on the Conic Sections, which was published in 1735.

The description of the Porisms of Euclid by Pappus is so mutilated, that every attempt, before Dr Simson's, to restore them had failed. Dr Halley, though successful in the restoration of some portions of the ancient Geometry, gave up the Porisms in 1706 as a hopeless task, as is obvious from his remark in the preface to the seventh book of Pappus, "Hactenus Porismatum descriptio, nec mihi intellecta, nec lectori profutura." Dr Simson had been occupied on the subject of the Porisms in 1715, and perhaps earlier, for he observes that in that year he had demonstrated the first case of Fermat's fourth Porism, before he had acquired the knowledge of the true nature of that class of Propositions. The first object of his researches seems to have been, to discover the Porisms, from the general description given of them by Pappus: and when he had failed in this, he tried to discover some of the individual Porisms, from which he expected to ascertain the distinctive character of these propositions; but in this attempt he had no better success. For a considerable time, he informs us, his imagination was completely occupied by the subject: his mind was harassed by the constant, but unsuccessful exertion: he lost his sleep, and his health suffered; all his endeavours were ineffectual, and he finally determined to banish the subject from his thoughts. For some time he maintained this resolution, and applied himself to other pursuits; but afterwards, as he was walking on the banks of the river Clyde, he inadvertently fell into a reverie respecting the Porisms. Some new ideas struck his mind, and having drawn the diagram with chalk on an adjoining tree, at that moment, for the first time, he acquired a just notion of one of Euclid's Porisms. This account is given in his preface to the Porisms, p. 319, of his "Opera Reliqua," a volume of his writings on Geometry, published after his death, by the munificence of the late Earl Stanhope.

Matthew Stewart was a pupil of Dr Simson, and afterwards became Professor of Mathematics in the University of Edinburgh. Dr Stewart was a successful cultivator of the ancient geometry. His "General

Theorems of considerable use in the higher parts of the Mathematics," was published in 1746, and placed him among the first geometers of his time. In 1761 he published another volume, entitled "Tracts Physical and Mathematical," and two years after his celebrated work on geometry, "Propositiones Geometricæ more veterum demonstratæ ad Geometriam antiquam illustrandam et promovendam idoneæ." In this work he has given both the analysis and synthesis of a series of geometrical theorems, many of which were not known before.

Dr Waring extended the discoveries of Newton in the theory of curves much beyond his predecessors. His "Miscellanea Analytica de æquationibus Algebraicis et curvarum proprietatibus," was published in 1762, and his "Proprietates Algebraicarum Curvarum," in 1772.

Bishop Horsley was born in 1732, and died in 1806. Though educated at Cambridge he removed to Oxford, and there in 1769 published his edition of the Inclinations of Apollonius. Besides his edition of Newton's Works, he put forth in 1801 his Practical Mathematics, in three volumes, containing Euclid's Elements, the Data, &c.

William Wales, by his talents and application rose from obscurity to an eminent position among men of science. He was the person selected to observe at Hudson's Bay the transit of Venus over the Sun in 1769, and afterwards accompanied, as astronomer, the celebrated Captain Cook on his first voyage in the years 1772 to 1774; and again in his other voyage in the years 1776 to 1779. A short time before he set sail in 1772, his friend Mr Lawson put forth his restoration of the two books of Apollonius, De Sectione Determinata, together with an English translation of Snell's restoration of the same two books. Another restoration of these two books was made by Giannini, and published in his Opuscula Mathematica in 1773.

Dr Robertson, the late Savilian Professor of Astronomy, to his treatise entitled "Sectionum Conicarum, Libri VII. &c.," (1792, Oxon.) has annexed a learned history of the Conic Sections.

Mascheroni published in Italian, in 1797, a treatise on geometry, entitled "Geometria del Compassa," in which the solutions of geometrical problems are effected by means of the circle only, instead of the straight line and circle. It was translated into French, and published in 1798, and again reprinted in 1823. In the twelfth year of the Republic, a similar treatise by M. Servois, was put forth, in which the solutions of geometrical problems were effected by means of straight lines only.

During the period of the French Revolution, at the end of the eighteenth century, the celebrated Monge discovered and put forth a new kind of geometry, under the appellation of "Géométrie Descriptive." The discoveries of Monge mark a new era in the history of geometry, as did the discovery of Descartes. The new geometry of Monge has two objects in view; first, to represent geometrical solids on a plane surface; and secondly, to deduce from this method of representation, the mathematical properties of the figures. It is chiefly conversant with the determination of the curves in which two or more surfaces intersect each other, when they are supposed to penetrate one another. To Monge is also due the theory of projections; and he was the first who proved that the square of any surface is equal to the sum of the squares of its projections on the three co-ordinate planes. It ought to be remarked, that attention had been before directed to this

subject by several who made contributions to it. Among these may be named Courcier, a jesuit, who published, at Paris, in 1613, a work in which he investigated the nature, and shewed the description of the curves which result from the penetration of cylindrical, spherical and conical surfaces; but to Monge is due the merit of its greatest extension.

The French school of Monge has produced many eminent geometers. The "Géométrie de Position," and "L'Essai sur la Théorie des Transversals," by Carnot, are a continuation of the method of Monge. Also may be mentioned "Les Développemens et les Applications de Géométrie," by Dupin, and the "Traité des Propriétés projectives," of M. Poncelet, with others, who have amply and skilfully written on the geometry of correlative figures, and its application to the physical sciences. To these may be added the names of Leslie, Playfair, Le Gendre, La Croix, L'Huillier, Vincent, and many others, too numerous here to mention, who by their writings have furthered the progress and advancement of scientific geometry.

We must not, before closing this subject, omit naming F. Peyrard, who published, at Paris, in 1818, an edition of Euclid's Elements from an ancient Greek manuscript which had not been collated or printed before. This edition, besides the Greek text, contains a version in Latin, and a French translation. F. Peyrard is also known as the editor and translator of the writings of Archimedes and Apollonius.

The short and cursory notices here given of the modern Geometry, are intended rather to awaken the curiosity of the student than to afford an ample and satisfactory account. In the "Mémoires Couronnés de l'Académie de Bruxelles" for 1837, the student will find in the "Aperçu Historique" of M. Chasles, a full and particular account of the history of the methods and developements of the modern Geometry.

ON THE ABBREVIATIONS AND ALGEBRAICAL SYMBOLS EMPLOYED IN GEOMETRY.

It has been remarked that the ancient geometry of the Greeks admitted no symbols besides the diagrams and ordinary language. In later times, after symbols of operation had been devised by writers on algebra, they were very soon adopted and employed, on account of their brevity and convenience, in writings purely geometrical. Dr Barrow was one of the first who introduced algebraical symbols into the language of elementary geometry, and distinctly states, in the preface to his Euclid, that his object is "to content the desires of those who are delighted more with symbolical than verbal demonstrations." As algebraical symbols are employed in almost all works on the mathematics, whether geometrical or not, it seems proper in this place to give some brief account of the marks which may be regarded as the alphabet of symbolical language.

The mark = was first used by Robert Recorde, in his treatise on Algebra entitled, "The Whetstone of Witte, &c.", for the sign of equality; "because," as he remarks, "no two things can be more equal than a pair of parallels, or *gemowe* lines of one length." It was employed by him, in his algebra, simply in the sense of *æquatur*, or, *is equal to*, affirming the equality of two numerical or algebraical ex-

pressions. Geometrical equality is not exactly the same as numerical equality, and when this symbol is used in geometrical reasonings, it must be understood as having reference to pure geometrical equality.

The signs of relative magnitude, > meaning, *is greater than,* and <, *is less than,* were first introduced into algebra by Thomas Harriot, in his "Artis Analyticæ Praxis," which was published after his death in 1631.

The signs + and − were first employed by Michael Stifel, in his "Arithmetica Integra," which was published in 1544. The sign + was employed by him for the word *plus,* and the sign −, for the word *minus.* These signs were used by Stifel strictly as the arithmetical or algebraical signs of addition and subtraction.

The sign of multiplication × was first introduced by Oughtrede in his "Clavis Mathematica," which was published in 1631. In algebraical multiplication he either connects the letters which form the factors of a product by the sign ×, or writes them as words without any sign or mark between them, as had been done before by Harriot, who first introduced the small letters to designate known and unknown quantities. However concise and convenient the notation $AB \times BC$ may be in practice for "*the rectangle contained by the lines* AB *and* BC;" the student is cautioned against the use of it, in the early part, at least, of his geometrical studies, as the use of it is likely to occasion a misapprehension of Euclid's meaning. Dr Barrow sometimes expresses "*the rectangle contained by* AB *and* BC" by "*the rectangle* ABC."

Michael Stifel was the first who introduced integral exponents to denote the powers of algebraical symbols of quantity, for which he employed capital letters. Vieta afterwards used the vowels to denote known, and the consonants, unknown quantities, but used words to designate the powers. Simon Stevin, in his treatise on Algebra, which was published in 1605, improved the notation of Stifel, by placing the figures that indicated the powers within small circles. Peter Ramus adopted the initial letters *l, q, c, bq* of *latus, quadratus, cubus, biquadratus,* as the notation of the first four powers. Harriot exhibited the different powers of algebraical symbols by repeating the symbol, two, three, four, &c. times, according to the order of the power. Descartes restored the numerical exponents of powers, placing them at the right of the numbers, or symbols of quantity, as at the present time. Dr Barrow employed the notation ABq, for "*the square of the line* AB," in his edition of Euclid. The notations AB^2, AB^3, for "*the square and cube of the line whose extremities are* A *and* B," are found in almost all works on the Mathematics, though not wholly consistent with the algebraical notations a^2 and a^3.

The symbol $\surd$, being originally the initial letter of the word *radix,* was first used by Stifel to denote the square root of the number, or the symbol, before which it was placed.

The Hindus, in their treatises on Algebra, indicated the ratio of two numbers, or of two algebraical symbols, by placing one above the other, without any line of separation. The line was first introduced by the Arabians, from whom it passed to the Italians, and from them to the rest of Europe. This notation has been employed for the expression of geometrical ratios by almost all writers on the Mathematics, on account of its great convenience. Oughtrede first used points to indicate proportion; thus, $a \,.\, b :: c \,.\, d$, means, that a bears the same proportion to b, as c does to d.

EUCLID'S

ELEMENTS OF GEOMETRY.

BOOK I.

DEFINITIONS.

I.

A POINT is that which has no parts, or which has no magnitude.

II.

A line is length without breadth.

III.

The extremities of lines are points.

IV.

A right line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of superficies are lines.

VII.

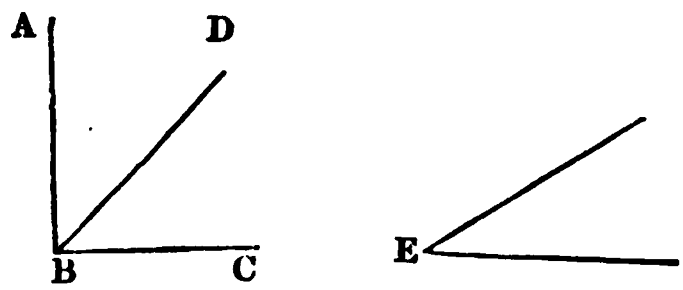
A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII.

A plane angle is the inclination of two lines to each other in a plane which meet together, but are not in the same straight line.

IX.

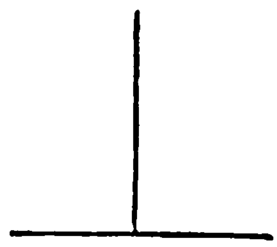
A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



N. B. When several angles are at one point *B*, either of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the right lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of these right lines, and the other upon the other line. Thus the angle which is contained by the right lines *AB*, *CB*, is named the angle *ABC*, or *CBA*; that which is contained by *AB*, *DB*, is named the angle *ABD*, or *DBA*; and that which is contained by *DB*, *CB*, is called the angle *DBC*, or *CBD*. But, if there be only one angle at a point, it may be expressed by the letter at that point; as the angle at *E*.

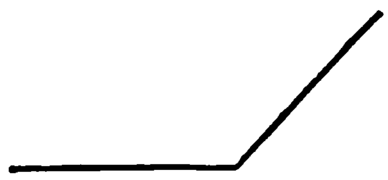
X.

When a straight line standing on another straight line makes the adjacent angles equal to each other, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



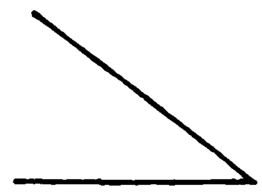
XI.

An obtuse angle is that which is greater than a right angle.



XII.

An acute angle is that which is less than a right angle.



XIII.

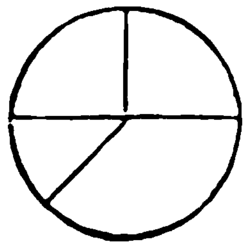
A term or boundary is the extremity of any thing.

XIV.

A figure is that which is inclosed by one or more boundaries.

XV.

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.

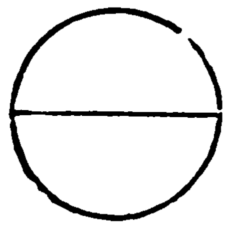


XVI.

And this point is called the centre of the circle.

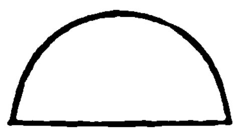
XVII.

A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.



XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.



XIX.

The centre of a semicircle is the same with that of the circle.

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or triangles, by three straight lines.

XXII.

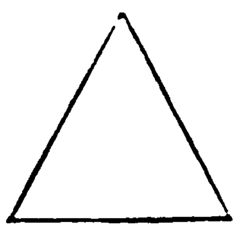
Quadrilateral, by four straight lines.

XXIII.

Multilateral figures, or polygons, by more than four straight lines.

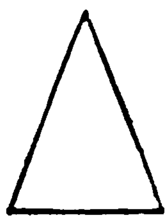
XXIV.

Of three-sided figures, an equilateral triangle is that which has three equal sides.



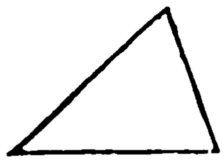
XXV.

An isosceles triangle is that which has two sides equal.



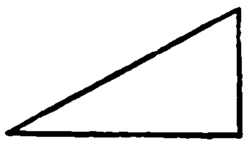
XXVI.

A. scalene triangle is that which has three unequal sides.



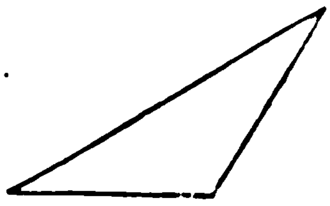
XXVII.

A right-angled triangle is that which has a right angle.



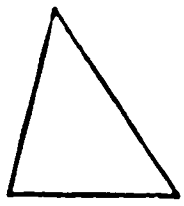
XXVIII.

An obtuse-angled triangle is that which has an obtuse angle.



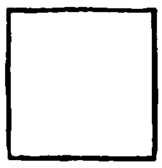
XXIX.

An acute-angled triangle is that which has three acute angles.



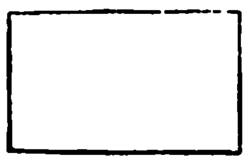
XXX.

Of quadrilateral or four-sided figures, a square has all its sides e and all its angles right angles.



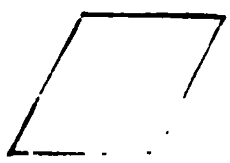
XXXI.

An oblong is that which has all its angles right angles, but has all its sides equal.



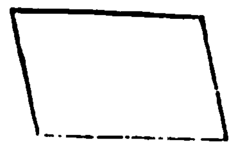
XXXII.

A rhombus has all its sides equal, but its angles are not right an



XXXIII.

A rhomboid has its opposite sides equal to each other, but all its sides are not equal, nor its angles right angles.



XXXIV.

All other four-sided figures besides these, are called Trapeziums.

XXXV.

Parallel straight lines are such as are in the same plane, and whic being produced ever so far both ways, do not meet.

A.

A parallelogram is a four-sided figure, of which the opposite sides are parallel: and the diameter or the diagonal is the straight line joining two of its opposite angles.

POSTULATES.

I.

Let it be granted that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any centre at any distance from that centre.

AXIOMS.

I.

Things which are equal to the same thing are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same, are equal to one another.

VII.

Things which are halves of the same, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot inclose a space.

XI.

All right angles are equal to one another.

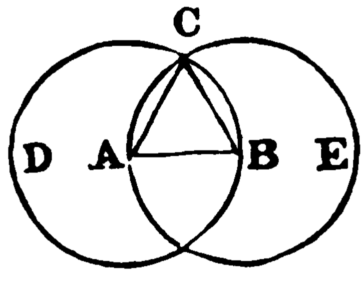
XII.

If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line.
It is required to describe an equilateral triangle upon AB.

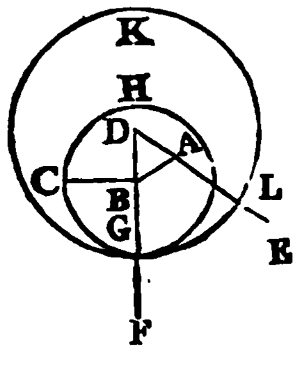


From the centre A, at the distance AB, describe the circle BCD; (post. 3.)
from the centre B, at the distance BA, describe the circle ACE;
and from the point C, in which the circles cut one another,
draw the straight lines CA, CB to the points A, B. (post. 1.)
Then ABC shall be an equilateral triangle.
Because the point A is the centre of the circle BCD,
therefore AC is equal to AB; (def. 15.)
and because the point B is the centre of the angle ACE,
therefore BC is equal to BA;
but it has been proved that CA is equal to AB;
therefore CA, CB are each of them equal to AB;
but things which are equal to the same thing are equal to one another; (ax. 1.)
therefore CA is equal to CB;
wherefore CA, AB, BC are equal to one another:
and the triangle ABC is therefore equilateral,
and it is described upon the given straight line AB.
Which was required to be done.

PROPOSITION II. PROBLEM.

From a given point, to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.
It is required to draw from the point A a straight line equal to BC.



From the point A to B draw the straight line AB; (post. 1.)
upon AB describe the equilateral triangle DAB, (I. 1.)
and produce the straight lines DA, DB to E and F; (post. 2.)
from the centre B, at the distance BC describe the circle CGH, (post. 3.)
and from the centre D, at the distance DG, describe the circle GKL.
Then the straight line AL shall be equal to BC.
Because the point B is the centre of the circle CGH,

therefore BC is equal to BG; (def. 15.)
and because D is the centre of the circle GKL,
therefore DL is equal to DG;
and DA, DB parts of them are equal; (I. 1.)
therefore the remainder AL is equal to the remainder BG; (ax. 3.)
but it has been shewn that BC is equal to BG,
wherefore AL and BC are each of them equal to BG;
and things which are equal to the same thing are equal to one another;
therefore the straight line AL is equal to BC. (ax. 1.)

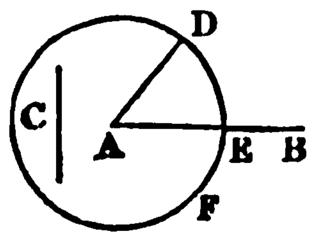
Wherefore from the given point A a straight line AL has been drawn equal to the given straight line BC. Which was to be done.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB a part equal to C, the less.



From the point A draw the straight line AD equal to C; (I. 2.)
and from the centre A, at the distance AD, describe the circle DEF. (post. 3.)
Then AE shall be equal to C.
Because A is the centre of the circle DEF,
therefore AE is equal to AD; (def. 15.)
but the straight line C is equal to AD; (constr.)
whence AE and C are each of them equal to AD;
wherefore the straight line AE is equal to C. (ax. 1.)

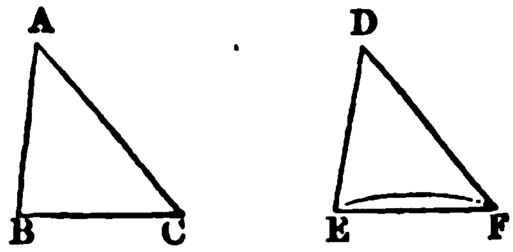
And therefore from AB the greater of two straight lines, a part AE has been cut off equal to C the less. Which was to be done.

PROPOSITION IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to each other; they shall likewise have their bases, or third sides, *equal; and the two triangles shall be equal; and their other angles shall be equal, each to cach, viz. those to which the equal sides are opposite.*

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, and the included angle BAC equal to the included angle EDF.

Then shall the base BC be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles to which the equal sides are opposite shall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

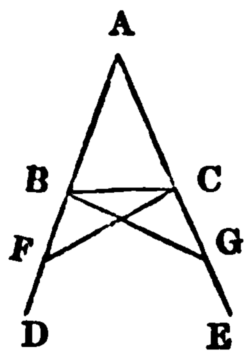


For, if the triangle *ABC* be applied to the triangle *DEF*, so that the point *A* may be on *D*, and the straight line *AB* on *DE*;
because *AB* is equal to *DE*,
therefore the point *B* shall coincide with the point *E*;
and *AB* coinciding with *DE*, because the angle *BAC* is equal to the angle *EDF*,
therefore the straight line *AC* shall fall on *DF*;
also because *AC* is equal to *DF*,
therefore the point *C* shall coincide with *F*;
but the point *B* coincides with the point *E*;
wherefore the base *BC* shall coincide with the base *EF*;
because the point *B* coinciding with *E*, and *C* with *F*,
if the base *BC* do not coincide with the base *EF*, the two straight lines *BC* and *EF* would inclose a space, which is impossible. (ax. 10.)
Therefore the base *BC* does coincide with *EF*, and is equal to it.
Wherefore the whole triangle *ABC* coincides with the whole triangle *DEF*, and is equal to it;
and the other angles of the one coincide with the remaining angles of the other, and are equal to them,
viz. the angle *ABC* to the angle *DEF*,
and the angle *ACB* to *DFE*.
Therefore, if two triangles have two sides of the one equal to two sides, &c.
Which was to be demonstrated.

PROPOSITION V. THEOREM.

The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base shall be equal.

Let *ABC* be an isosceles triangle of which the side *AB* is equal to *AC*,
and let the equal sides *AB*, *AC* be produced to *D* and *E*.
Then the angle *ABC* shall be equal to the angle *ACB*,
and the angle *DBC* to the angle *ECB*.



In *BD* take any point *F*;
from *AE* the greater, cut off *AG* equal to *AF* the less, (I. 3.) and join *FC*, *GB*.
Because *AF* is equal to *AG*, (constr.) and *AB* to *AC*; (hyp.)
the two sides *FA*, *AC*, are equal to the two *GA*, *AB*, each to each;

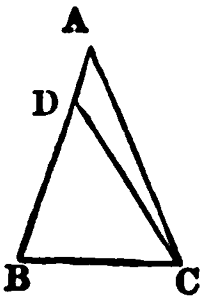
and they contain the angle *FAG* common to the two triangles *AFC, AGB*;
therefore the base *FC* is equal to the base *GB*, (I. 4.)
and the triangle *AFC* is equal to the triangle *AGB*,
also the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite;
viz. the angle *ACF* to the angle *ABG*,
and the angle *AFC* to the angle *AGB*.
And because the whole *AF* is equal to the whole *AG*,
of which the parts *AB, AC*, are equal;
therefore the remainder *BF* is equal to the remainder *CG*; (ax. 3.)
and *FC* was proved to be equal to *GB*;
hence because the two sides *BF, FC* are equal to the two *CG, GB*, each to each;
and the angle *BFC* has been proved to be equal to the angle *CGB*,
also the base *BC* is common to the two triangles *BFC, CGB*;
wherefore these triangles are equal, (I. 4.)
and their remaining angles, each to each, to which the equal sides are opposite;
therefore the angle *FBC* is equal to the angle *GCB*,
and the angle *BFC* to the angle *CBG*.
And, since it has been demonstrated,
that the whole angle *ABG* is equal to the whole *ACF*,
the parts of which, the angles *CBG, BCF* are also equal;
therefore the remaining angle *ABC* is equal to the remaining angle *ACB*,
which are the angles at the base of the triangle *ABC*;
and it has been proved,
that the angle *FBC* is equal to the angle *GCB*,
which are the angles upon the other side of the base.
Therefore the angles at the base, &c. Q.E.D.

COR. Hence every equilateral triangle is also equiangular.

PROPOSITION VI. THEOREM.

If two angles of a triangle be equal to each other, the sides also which subtend, or are opposite to, *the equal angles, shall be equal to one another.*

Let *ABC* be a triangle having the angle *ABC* equal to the angle *ACB*.
Then the side *AB* shall be equal to the side *AC*.



For, if *AB* be not equal to *AC*,
one of them is greater than the other.
Let *AB* be greater than *AC*;
and at the point *B*, from *BA* cut off *BD* equal to *CA* the less, (I. 3.) and join *DC*.
Then, in the triangles *DBC, ABC*,
because *DB* is equal to *AC*, and *BC* is common to both,
the two sides *DB, BC* are equal to the two sides *AC, CB*, each to each;

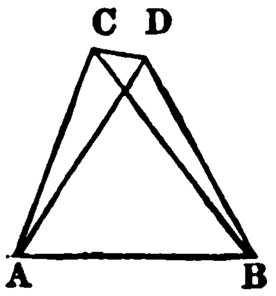
and the angle DBC is equal to the angle ACB; (hyp.)
therefore the base DC is equal to the base AB, (I. 4.)
and the triangle DBC is equal to the triangle ACB,
the less equal to the greater, which is absurd.
Therefore AB is not unequal to AC, that is, AB is equal to AC.
Wherefore, if two triangles, &c. Q.E.D.

COR. Hence every equiangular triangle is also equilateral.

PROPOSITION VII. THEOREM.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

If it be possible, on the same base AB, and upon the same side of it, let there be two triangles ACB, ADB, which have their sides CA, DA, terminated in the extremity A of the base, equal to one another, and likewise their sides, CB, DB, that are terminated in B.

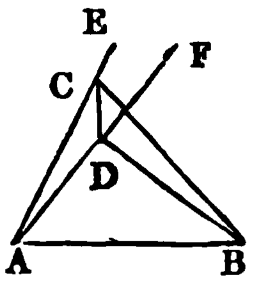


Join CD.

First. When the vertex of each of the triangles is without the other triangle.

Because AC is equal to AD in the triangle ACD,
therefore the angle ACD is equal to the angle ADC; (I. 5.)
but the angle ACD is greater than the angle BCD; (ax. 9.)
therefore also the angle ADC is greater than BCD;
much more therefore is the angle BDC greater than BCD.
Again, because the side BC is equal to BD in the triangle BCD, (hyp.)
therefore the angle BDC is equal to the angle BCD; (I. 5.)
but the angle BDC was proved greater than the angle BCD,
hence the angle BDC is both equal to, and greater than the angle BCD;
which is impossible.

Secondly. Let the vertex D of the triangle ADB fall within the triangle ACB.



Produce AC and AD to E and F.
Then because AC is equal to AD in the triangle ACD,
therefore the angles ECD, FDC upon the other side of the base CD are equal to one another; (I. 5.)
but the angle ECD is greater than the angle BCD; (ax. 10.)
therefore also the angle FDC is greater than the angle BCD;

much more then is the angle BDC greater than the angle BCD.
Again, because BC is equal to BD in the triangle BCD, therefore the angle BDC is equal to the angle BCD; (I. 5.)
but the angle BDC has been proved greater than BCD, wherefore the angle BDC is both equal to, and greater than the angle BCD;
which is impossible.

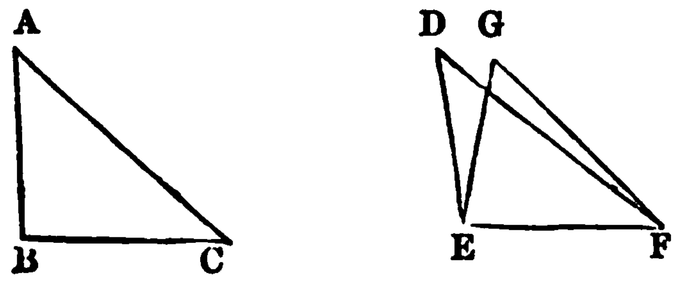
Thirdly. The case in which the vertex of one triangle is upon a side of the other needs no demonstration.

Therefore upon the same base and on the same side of it, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angles contained by the two sides equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC, equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF; and also the base BC equal to the base EF.
Then the angle BAC shall be equal to the angle EDF.

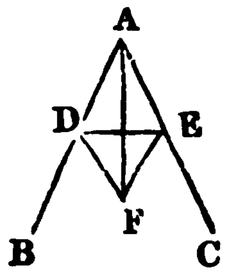


For, if the triangle ABC be applied to DEF, so that the point B be on E, and the straight line BC on EF;
then because BC is equal to EF, (hyp.)
therefore the point C shall coincide with the point F;
wherefore BC coinciding with EF,
BA and AC shall coincide with ED, DF;
for, if the base BC coincide with the base EF, but the sides BA, AC do not coincide with the sides ED, DF, but have a different situation as EG, FG:
Then, upon the same base, and upon the same side of it, there can be two triangles which have their sides which are terminated in one extremity of the base equal to one another, and likewise those sides which are terminated in the other extremity;
but this is impossible. (I. 7.)
Therefore, if the base BC coincide with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF;
wherefore likewise the angle BAC coincides with the angle EDF, and is equal to it, (ax. 8.)
Therefore if two triangles have two sides, &c. Q.E.D.

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle.
It is required to bisect it.

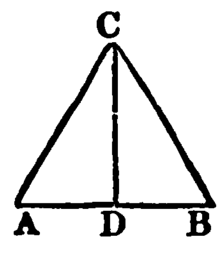


Take any point D in AB;
from AC cut off AE equal to AD; (I. 3.) and join DE,
describe an equilateral triangle DEF on the side of DE remote from A, (I. 1.) and join AF.
Then the straight line AF shall bisect the angle BAC.
Because AD is equal to AE, (constr.)
and AF is common to the two triangles DAF, EAF;
the two sides DA, AF, are equal to the two sides EA, AF, each to each;
and the base DF is equal to the base EF; (constr.)
therefore the angle DAF is equal to the angle EAF. (I. 8.)
Wherefore the angle BAC is bisected by the straight line AF. Q.E.F.

PROPOSITION X. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line.
It is required to divide AB into two equal parts.



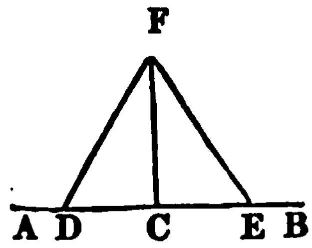
Upon AB describe the equilateral triangle ABC; (I. 1.)
and bisect the angle ACB by the straight line CD meeting AB in the point D. (I. 9.)
Then AB shall be cut into two equal parts in the point D.
Because AC is equal to CB, (constr.)
and CD is common to the two triangles ACD, BCD;
the two sides AC, CD are equal to BC, CD, each to each;
and the angle ACD is equal to BCD; (constr.)
therefore the base AD is equal to the base DB. (I. 4.)
Wherefore the straight line AB is divided into two equal parts in the point D. Q.E.F.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given straight line, and C a given point in it.

It is required to draw a straight line from the point C at right angles to AB.



In AC take any point D, and make CE equal to CD; (I. 3.)
upon DE describe the equilateral triangle DEF, (I. 1.), and join CF.
Then CF drawn from the point C shall be at right angles to AB.
Because DC is equal to CE, and FC is common to the two triangles DCF, ECF;
the two sides DC, CF are equal to the two sides EC, CF, each to each;
and the base DF is equal to the base EF; (constr.)
therefore the angle DCF is equal to the angle ECF: (I. 8.)
and these two angles are adjacent angles.

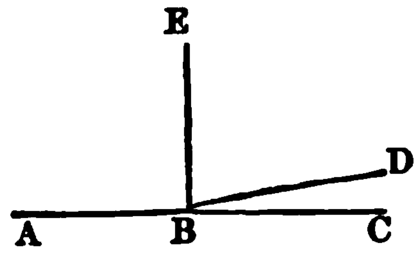
But when the two adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called a right angle: (def. 10.)

Therefore each of the angles DCF, ECF is a right angle.

Wherefore from the given point C, in the given straight line AB, FC has been drawn at right angles to AB. Q.E.F.

COR. By help of this problem, it may be demonstrated that two straight lines cannot have a common segment.

If it be possible, let the segment AB be common to the two straight lines ABC, ABD.



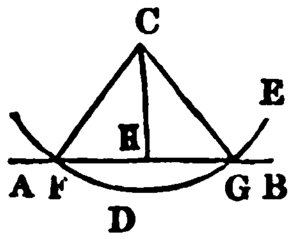
From the point B, draw BE at right angles to AB; (I. 11.)
then because ABC is a straight line,
therefore the angle ABE is equal to the angle EBC; (def. 10.)
Similarly, because ABD is a straight line,
therefore the angle ABE is equal to the angle EBD;
but the angle ABE is equal to the angle EBC,
wherefore the angle EBD is equal to the angle EBC, (ax. 1.)
the less equal to the greater angle, which is impossible.
Therefore two straight lines cannot have a common segment.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced any length both ways, and let C be a point without it.

It is required to draw a straight line perpendicular to *AB* from the point *C*.



Take any point *D* upon the other side of *AB*,
and from the centre *C*, at the distance *CD*, describe the circle *EGF*
meeting *AB* in *F* and *G*; (post. 3.)
bisect *FG* in *H* (I. 10.), and join *CH*.

Then the straight line *CH* drawn from the given point *C* shall be perpendicular to the given straight line *AB*.

Join *CF*, and *CG*.

And because *FH* is equal to *HG* (constr.), and *HC* is common to the triangles *FHC*, *GHC*;
the two sides *FH*, *HC*, are equal to the two *GH*, *HC*, each to each;
and the base *CF* is equal to the base *CG*; (def. 15.)
therefore the angle *FHC* is equal to the angle *GHC*; (I. 8.)
and these are adjacent angles.

But when a straight line standing on another straight line, makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it. (def. 10.)

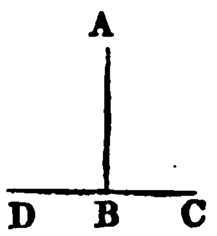
Therefore from the given point *C* a perpendicular *CH* has been drawn to the given straight line *AB*. Q.E.F.

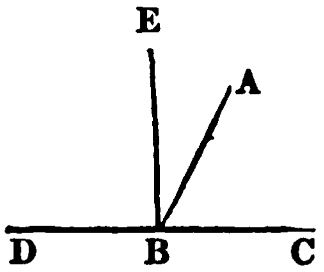
PROPOSITION XIII. THEOREM.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line *AB* make with *CD*, upon one side of it, the angles *CBA*, *ABD*.

Then these shall be either two right angles, or shall be together equal to two right angles.





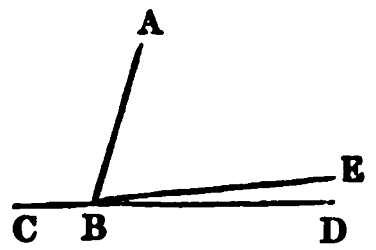
For if the angle *CBA* be equal to the angle *ABD*,
each of them is a right angle. (def. 10.)
But if the angle *CBA* be not equal to the angle *ABD*,
from the point *B* draw *BE* at right angles to *CD*. (I. 11.)
Then the angles *CBE*, *EBD* are two right angles. (def. 10.)
And because the angle *CBE* is equal to the angles *CBA*, *ABE*,
add the angle *EBD* to each of these equals;
therefore the angles *CBE*, *EBD* are equal to the three angles
CBA, *ABE*, *EBD*. (ax. 2.)

Again, because the angle DBA is equal to the two angles DBE, EBA,
add to each of these equals the angle ABC;
therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC.
But the angles CBE, EBD have been proved equal to the same three angles;
and things which are equal to the same thing are equal to one another;
therefore the angles CBE, EBD are equal to the angles DBA, ABC;
but the angles CBE, EBD are two right angles;
therefore the angles DBA, ABC are together equal to two right angles. (ax. 1.)
Wherefore when a straight line, &c. Q.E.D.

PROPOSITION XIV. THEOREM.

If at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.
Then BD shall be in the same straight line with CB.



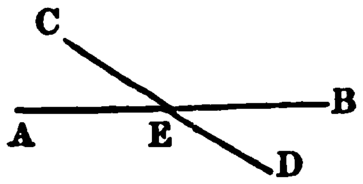
For, if BD be not in the same straight line with CB,
let BE be in the same straight line with it.
Then because AB meets the straight line CBE;
therefore the adjacent angles CBA, ABE are equal to two right angles; (I. 13.)
but the angles CBA, ABD are equal to two right angles; (hyp.)
therefore the angles CBA, ABE are equal to the angles CBA, ABD: (ax. 2.)
take away from these equals the common angle CBA,
therefore the remaining angle ABE is equal to the remaining angle ABD; (ax. 3.)
the less equal to the greater angle, which is impossible:
therefore BE is not in the same straight line with CB.
And in the same manner it may be demonstrated, that no other can be in the same straight line with it but BD, which therefore is in the same straight line with CB.
Wherefore, if at a point, &c. Q.E.D.

PROPOSITION XV. THEOREM.

If two straight lines cut one another, the vertical, or opposite *angles shall be equal.*

Let the two straight lines AB, CD cut one another in the point E.

Then the angle AEC shall be equal to the angle DEB, and the angle CEB to the angle AED.



Because the straight line AE makes with CD at the point E, the adjacent angles CEA, AED;

these angles are together equal to two right angles. (I. 13.)

Again, because the straight line DE makes with AB at the point E the adjacent angles AED, DEB;

these angles also are equal to two right angles;

but the angles CEA, AED have been shewn to be equal to two right angles;

wherefore the angles CEA, AED are equal to the angles AED, DEB;

take away from each the common angle AED,

and the remaining angle CEA is equal to the remaining angle DEB. (ax. 3.)

In the same manner it may be demonstrated, that the angle CEB is equal to the angle AED.

Therefore, if two straight lines cut one another, &c. Q.E.D.

COR. 1. From this it is manifest, that, if two straight lines cut each other, the angles which they make at the point where they cut, are together equal to four right angles.

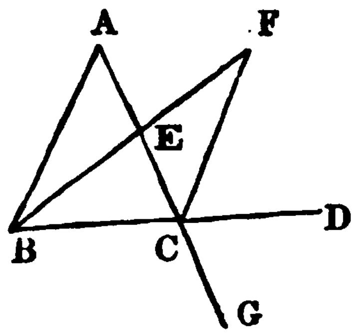
COR. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D.

Then the exterior angle ACD shall be greater than either of the interior opposite angles CBA or BAC.



Bisect AC in E, (I. 10.) and join BE;

produce BE to F, making EF equal to BE, (I. 3.) and join FC.

Because AE is equal to EC, and BE to EF;

the two sides AE, EB are equal to the two CE, EF, each to each, in the triangles ABE, CFE;

and the angle AEB is equal to the angle CEF,

because they are opposite vertical angles; (I. 15.)

therefore the base AB is equal to the base CF, (I. 4.)

and the triangle AEB to the triangle CEF,

and the remaining angles of one triangle to the remaining angles of the other, each to each, to which the equal sides are opposite;
wherefore the angle *BAE* is equal to the angle *ECF*;
but the angle *ECD* or *ACD* is greater than the angle *ECF*;
therefore the angle *ACD* is greater than the angle *BAE*.

In the same manner, if the side *BC* be bisected, and *AC* be produced to *G*; it may be demonstrated that the angle *BCG*, that is, the angle *ACD*, (I. 15.) is greater than the angle *ABC*.

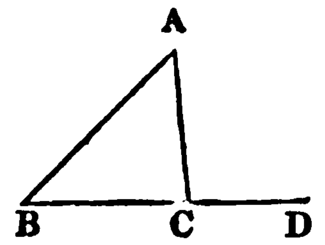
Therefore, if one side of a triangle, &c. Q.E.D.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let *ABC* be any triangle.

Then any two of its angles together shall be less than two right angles.



Produce any side *BC* to *D*.
Then because *ACD* is the exterior angle of the triangle *ABC*;
therefore the angle *ACD* is greater than the interior and opposite angle *ABC*; (I. 16.)
to each of these unequals add the angle *ACB*;
Therefore the angles *ACD*, *ACB* are greater than the angles *ABC*, *ACB*;
but the angles *ACD*, *ACB* are equal to two right angles; (I. 13.)
therefore the angles *ABC*, *BCA* are less than two right angles.
In like manner it may be demonstrated,
that the angles *BAC*, *ACB* are less than two right angles,
as also the angles *CAB*, *ABC*.

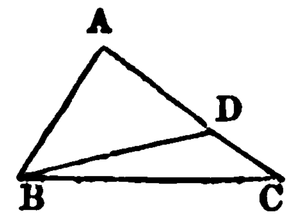
Therefore any two angles of a triangle, &c. Q.E.D.

PROPOSITION XVIII. THEOREM.

The greater side of every triangle is opposite to the greater angle.

Let *ABC* be a triangle, of which the side *AC* is greater than the side *AB*.

Then the angle *ABC* shall be greater than the angle *BCA*.



Since the side *AC* is greater than the side *AB*,
make *AD* equal to *AB*, (I. 3.) and join *BD*.
Then because *AD* is equal to *AB* in the triangle *ABD*,
therefore the angle *ADB* is equal to the angle *ABD*, (I. 5.)

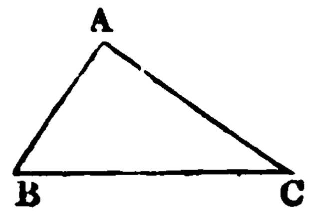
but because the side CD of the triangle DBC is produced to A,
therefore the exterior angle ADB is greater than the interior and opposite angle DCB; (I. 16.)
but the angle ADB has been proved equal to the angle ABD,
therefore the angle ABD is greater than the angle DCB;
wherefore much more is the angle ABC greater than the angle ACB.
Therefore the greater side, &c. Q.E.D.

PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

Let ABC be a triangle of which the angle ABC is greater than the angle BCA.

Then the side AC shall be greater than the side AB.

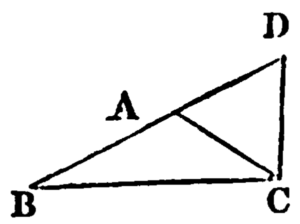


For, if AC be not greater than AB,
AC must either be equal to, or less than AB;
if AC were equal to AB,
then the angle ABC would be equal to the angle ACB; (I. 5.)
but it is not equal; (hyp.)
therefore the side AC is not equal to AB.
Again, if AC were less than AB,
then the angle ABC would be less than the angle ACB; (I. 18.)
but it is not less,
therefore the side AC is not less than AB;
and AC has been shewn to be not equal to AB;
therefore AC is greater than AB.
Wherefore the greater angle, &c. Q.E.D.

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle.
Then any two sides of it together shall be greater than the third side,
viz. the sides BA, AC greater than the side BC;
AB, BC greater than AC;
and BC, CA greater than AB.



Produce the side BA to the point D,
make AD equal to AC, (I. 3.) and join DC.

Then because AD is equal to AC,
therefore the angle ADC is equal to the angle ACD; (I. 5.)
but the angle BCD is greater than the angle ACD;
therefore also the angle BCD is greater than the angle ADC.
And because in the triangle DBC,
the angle BCD is greater than the angle BDC, (ax. 9.)
and that the greater angle is subtended by the greater side; (I. 19.)
therefore the side DB is greater than the side BC;
but DB is equal to BA and AC,
therefore the sides BA and AC are greater than BC.

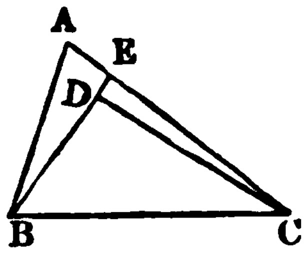
In the same manner it may be demonstrated, that the sides AB, BC are greater than CA;
also that BC, CA are greater than AB.
Therefore any two sides, &c. Q.E.D.

PROPOSITION XXI. THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to a point D within the triangle.

Then BD and DC shall be less than BA and AC the other two sides of the triangle,
but shall contain an angle BDC greater than the angle BAC.



Produce BD to meet the side AC in E.
Because two sides of a triangle are greater than the third side, (I. 20.)
therefore the two sides BA, AE of the triangle ABE are greater than BE;
to each of these unequals add EC;
therefore the sides BA, AC are greater than BE, EC. (ax. 4.)
Again, because the two sides CE, ED of the triangle CED are greater than DC; (I. 20.)
add DB to each of these unequals;
therefore the sides CE, EB are greater than CD, DB. (ax. 4.)
But it has been shewn that BA, AC are greater than BE, EC;
much more then are BA, AC greater than BD, DC.

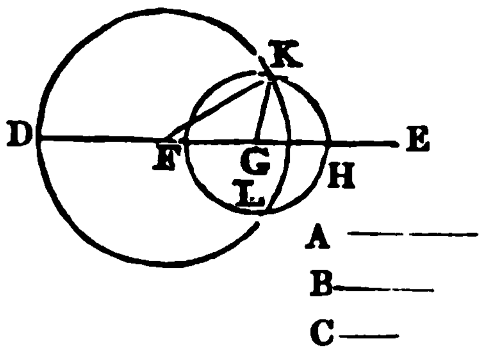
Again, because the exterior angle of a triangle is greater than the interior and opposite angle; (I. 16.)
therefore the exterior angle BDC of the triangle CDE is greater than the interior and opposite angle CED;
for the same reason, the exterior angle CED of the triangle ABE is greater than the interior and opposite angle BAC;
and it has been demonstrated,
that the angle BDC is greater than the angle CEB;
much more therefore is the angle BDC greater than the angle BAC.
Therefore, if from the ends of the side, &c. Q.E.D.

PROPOSITION XXII. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let A, B, C be the three given straight lines,
of which any two whatever are greater than the third, (I. 20.)
namely, A and B greater than C;
A and C greater than B;
and B and C greater than A.

It is required to make a triangle of which the sides shall be equal to A, B, C, each to each.



Take a straight line DE terminated at the point D, but unlimited towards E,
make DF equal to A, FG equal to B, and GH equal to C; (I. 3.)
from the centre F, at the distance FD, describe the circle DKL; (post. 3.)
and from the centre G, at the distance GH, describe the circle HLK; and join KF, KG.

Then the triangle KFG shall have its sides equal to the three straight lines A, B, C.

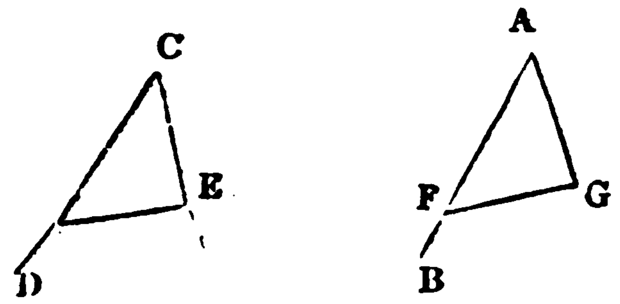
Because the point F is the centre of the circle DKL,
therefore FD is equal to FK; (def. 15.)
but FD is equal to the straight line A;
therefore FK is equal to A.
Again, because G is the centre of the circle HLK;
therefore GH is equal to GK, (def. 15.)
but GH is equal to C;
therefore also GK is equal to C; (ax. 1.)
and FG is equal to B;
therefore the three straight lines KF, FG, GK, are respectively equal to the three, A, B, C:
and therefore the triangle KFG has its three sides KF, FG, GK, equal to the three given straight lines A, B, C. Q.E.F.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle.

It is required to make an angle at the given point A in the given straight line AB, that shall be equal to the given rectilineal angle DCE.



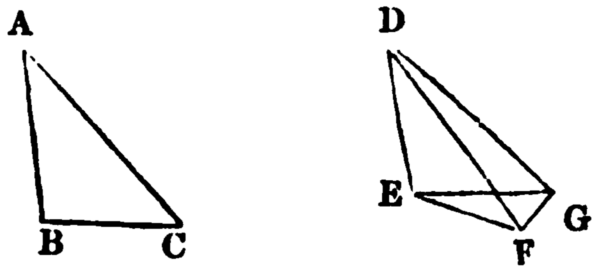
In CD, CE, take any points D, E, and join DE;
make the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC, so that AF be equal to CD, AG to CE, and FG to DE. (I. 22.)

Then the angle FAG shall be equal to the angle DCE.
Because FA, AG are equal to DC, CE, each to each,
and the base FG is equal to the base DE;
therefore the angle FAG is equal to the angle DCE. (I. 8.)
Wherefore at the given point A in the given straight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Q.E.F.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle, shall be greater than the base of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC, equal to the two DE, DF, each to each, namely AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF.
Then the base BC shall be greater than the base EF.



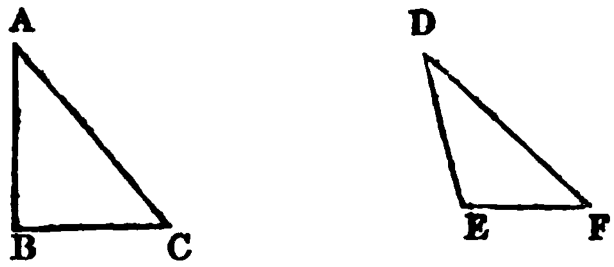
Of the two sides DE, DF, let DE be not greater than DF,
at the point D, in the straight line DE,
make the angle EDG equal to the angle BAC; (I. 23.)
make DG equal to DF or AC, (I. 3.) and join EG, GF.
Then, because DE is equal to AB, and DG to AC,
the two sides DE, DG are equal to the two AB, AC, each to each,
and the angle EDG is equal to the angle BAC;
therefore the base EG is equal to the base BC. (I. 4.)
And because DG is equal to DF in the triangle DFG,
therefore the angle DFG is equal to the angle DGF; (I. 5.)
but the angle DGF is greater than the angle EGF; (ax. 9.)
therefore the angle DFG is also greater than the angle EGF;
much more therefore is the angle EFG greater than the angle EGF.
And because in the triangle EFG, the angle EFG is greater than the angle EGF,
and that the greater angle is subtended by the greater side; (I. 19.)
therefore the side EG is greater than the side EF;
but EG was proved equal to BC;
therefore BC is greater than EF.
Wherefore if two triangles, &c. Q.E.D.

PROPOSITION XXV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.

Let *ABC, DEF* be two triangles which have the two sides *AB, AC* equal to the two sides *DE, DF*, each to each, namely, *AB* equal to *DE*, and *AC* to *DF*; but the base *BC* greater than the base *EF*.

Then the angle *BAC* shall be greater than the angle *EDF*.



For, if the angle *BAC* be not greater than the angle *EDF*, it must either be equal to it, or less than it.

If the angle *BAC* were equal to the angle *EDF*,
then the base *BC* would be equal to the base *EF*; (I. 4.)
but it is not equal,
therefore the angle *BAC* is not equal to the angle *EDF*.
Again, if the angle *BAC* were less than the angle *EDF*,
then the base *BC* would be less than the base *EF*; (I. 24.)
but it is not less,
therefore the angle *BAC* is not less than the angle *EDF*;
and it has been shewn, that the angle *BAC* is not equal to the angle *EDF*;
therefore the angle *BAC* is greater than the angle *EDF*.
Wherefore, if two triangles, &c. Q.E.D.

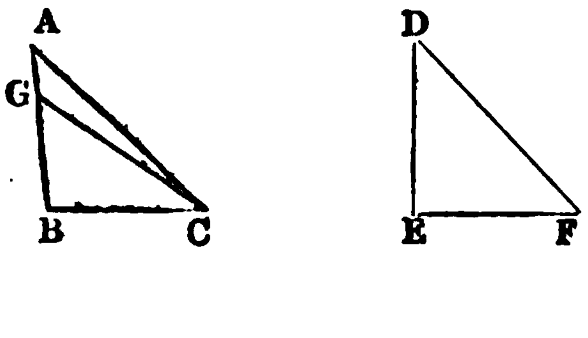
PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz. either the sides adjacent to the equal angles in each, or the sides opposite to them; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

Let *ABC, DEF* be two triangles which have the angles *ABC, BCA* equal to the angles *DEF, EFD*, each to each, namely *ABC* to *DEF*, and *BCA* to *EFD*; also one side equal to one side.

First, let those sides be equal which are adjacent to the angles that are equal in the two triangles, namely *BC* to *EF*.

Then the other sides shall be equal, each to each, namely *AB* to *DE*, and *AC* to *DF*, and the third angle *BAC* to the third angle *EDF*.

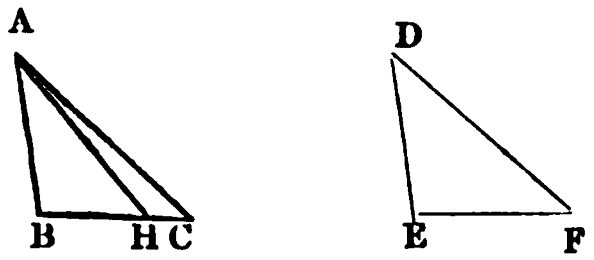


For, if AB be not equal to DE, one of them must be greater than the other.

Let AB be greater than DE,
make BG equal to ED, (I. 3.) and join GC.
Then in the two triangles GBC, DEF,
because GB is equal to DE, and BC to EF, (hyp.)
the two sides GB, BC are equal to the two DE, EF, each to each;
and the angle GBC is equal to the angle DEF;
therefore the base GC is equal to the base DF, (I. 4.)
and the triangle GBC to the triangle DEF,
and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle GCB is equal to the angle DFE;
but the angle DFE is, by the hypothesis, equal to the angle ACB;
wherefore also the angle GCB is equal to the angle ACB; (ax. 1.)
the less angle equal to the greater, which is impossible;
therefore AB is not unequal to DE,
that is, AB is equal to DE.
Hence, in the triangles ABC, DEF;
because AB is equal to DE, and BC to EF, (hyp.)
and the angle ABC is equal to the angle DEF; (hyp.)
therefore the base AC is equal to the base DF, (I. 4.)
and the third angle BAC to the third angle EDF.

Secondly, let the sides which are opposite to the equal angles in each triangle be equal to one another, namely, AB equal to DE.

Then in this case likewise the other sides shall be equal, AC to DF, and BC to EF, and also the third angle BAC to the third angle EDF.



For if BC be not equal to EF, one of them must be greater than the other.

Let BC be greater than EF; make BH equal to EF, (I. 3.) and join AH.
Then in the two triangles ABH, DEF,
because AB is equal DE, and BH to EF,
and the angle ABC to the angle DEF; (hyp.)
therefore the base AH is equal to the base DF, (I. 4.)
and the triangle ABH to the triangle DEF,
and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle BHA is equal to the angle EFD;
but the angle EFD is equal to the angle BCA; (hyp.)
therefore the angle BHA is equal to the angle BCA, (ax. 1.)
that is, the exterior angle BHA of the triangle AHC, is equal to its interior and opposite angle BCA; which is impossible; (I. 16.)
wherefore BC is not unequal to EF,
that is, BC is equal to EF.
Hence, in the triangles ABC, DEF;
because AB is equal to DE, and BC to EF, (hyp.)

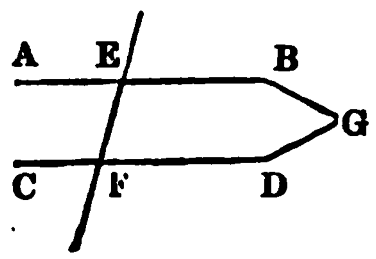
and the included angle ABC is equal to the included angle DEF; (hyp.)
therefore the base AC is equal to the base DF, (I. 4.)
and the third angle BAC to the third angle EDF.
Wherefore if two triangles, &c. Q.E.D.

PROPOSITION XXVII. THEOREM.

If a straight line, falling on two other straight lines, make the alternate angles equal to each other; these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD equal to one another.

Then AB shall be parallel to CD.



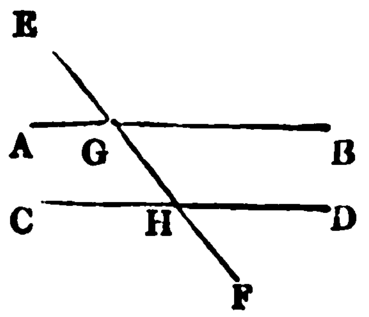
For, if AB be not parallel to CD,
AB and CD being produced will meet either towards A and C, or towards B and D.
Let AB, CD be produced and meet towards B and D, in the point G.
Then GEF is a triangle,
and its exterior angle AEF is greater than the interior and opposite angle EFG; (I. 16.)
but the angle AEF is equal to the angle EFG; (hyp.)
therefore the angle AEF is greater than and equal to the angle EFG; which is impossible.
Therefore AB, CD being produced do not meet towards B, D.
In like manner, it may be demonstrated, that they do not meet when produced towards A, C.
But those straight lines in the same plane which meet neither way, though produced ever so far, are parallel to one another; (def. 35.)
therefore AB is parallel to CD.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXVIII. THEOREM.

If a straight line falling upon two other straight lines, makes the exterior angle equal to the interior and opposite upon the same side of the line; or makes the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side; or make the two interior angles BGH, GHD on the same side together equal to two right angles.

Then AB shall be parallel to CD.



Because the angle EGB is equal to the angle GHD, (hyp.)
and the angle EGB equal to the angle AGH, (I. 15.)
therefore the angle AGH is equal to the angle GHD; (ax. 1.)
and they are alternate angles,
therefore AB is parallel to CD. (I. 27.)

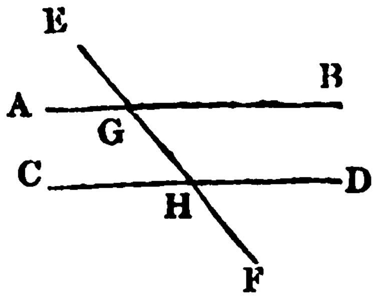
Again, because the angles BGH, GHD are together equal to two right angles, (hyp.)
and that the angles AGH, BGH are also together equal to two right angles; (I. 13.)
therefore the angles AGH, BGH are equal to the angles BGH, GHD; (ax. 1.)
take away the common angle BGH;
therefore the remaining angle AGH is equal to the remaining angle GHD; (ax. 3.)
and they are alternate angles;
therefore AB is parallel to CD. (I. 27.)

Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD.
Then the alternate angles AGH, GHD shall be equal to one another;
the exterior angle EGB shall be equal to the interior and opposite angle GHD upon the same side of the line EF;
and the two interior angles BGH, GHD upon the same side shall be together equal to two right angles.



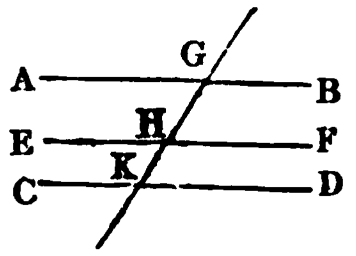
For, if the angle AGH be not equal to the alternate angle GHD,
let AGH be greater than GHD,
then because the angle AGH is greater than the angle GHD,
add to each of these unequals the angle BGH;
therefore the angles AGH, BGH are greater than the angles BGH, GHD; (ax. 4.)
but the angles AGH, BGH are equal to two right angles; (I. 13.)
therefore the angles BGH, GHD are less than two right angles;
but those straight lines which, with another straight line falling upon them, make the interior angles on the same side less than two right angles, will meet together if continually produced; (ax. 12.)
therefore the straight lines AB, CD, if produced far enough, will meet;
but they never meet, since they are parallel by the hypothesis;
therefore the angle AGH is not unequal to the angle GHD,
that is, the angle AGH is equal to the angle GHD:
but the angle AGH is equal to the angle EGB; (I. 15.)

therefore likewise the angle EGB is equal to the angle GHD: (ax. 1.)
add to each of them the angle BGH;
therefore the angles EGB, BGH are equal to the angles BGH, GHD; (ax. 2.)
but EGB, BGH are equal to two right angles; (I. 13.)
therefore also BGH, GHD are equal to two right angles. (ax. 1.)
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXX. THEOREM.

Straight lines which are parallel to the same straight line are parallel to each other.

Let the straight lines AB, CD be each of them parallel to EF.
Then shall AB be also parallel to CD.

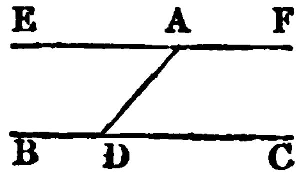


Let the straight line GHK cut AB, EF, CD.
Then because GHK cuts the parallel straight lines AB, EF,
therefore the angle AGH is equal to the alternate angle GHF. (I. 29.)
Again, because GHK cuts the parallel straight lines EF, CD,
therefore the exterior angle GHF is equal to the interior angle HKD; (I. 29.)
and it was shewn that the angle AGH is equal to the angle GHF;
therefore the angle AGH is equal to the angle GKD;
and these are alternate angles;
therefore AB is parallel to CD. (I. 27.)
Wherefore straight lines which are, &c. Q.E.D.

PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line.
It is required to draw through the point A a straight line parallel to the straight line BC.

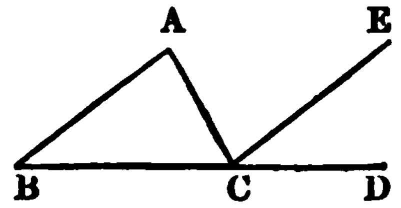


In the line BC take any point D, and join AD,
at the point A in the straight line AD, make the angle DAE equal to the angle ADC; (I. 23.)
and produce the straight line EA to F.
Then EF shall be parallel to BC.
Because the straight line AD meets the two straight lines EF, BC, and makes the alternate angles EAD, ADC, equal to one another,
therefore EF is parallel to BC. (I. 27.)
Wherefore through the given point A, has been drawn a straight line EAF parallel to the given straight line BC. Q.E.F.

PROPOSITION XXXII. THEOREM.

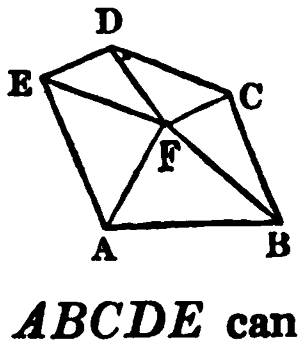
If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let *ABC* be a triangle, and let one of its sides *BC* be produced to *D*. Then the exterior angle *ACD* shall be equal to the two interior and opposite angles *CAB*, *ABC*:
and the three interior angles *ABC*, *BCA*, *CAB* shall be equal to two right angles.



Through the point *C* draw *CE* parallel to the side *BA*. (I. 31.)
Then because *CE* is parallel to *BA*, and *AC* meets them,
therefore the angle *ACE* is equal to the alternate angle *BAC*. (I. 29.)
Again, because *CE* is parallel to *AB*, and *BD* falls upon them,
therefore the exterior angle *ECD* is equal to the interior and opposite angle *ABC*; (I. 29.)
but the angle *ACE* was shewn to be equal to the angle *BAC*;
therefore the whole exterior angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*: (ax. 2.)
to each of these equals add the angle *ACB*,
therefore the angles *ACD* and *ACB* are equal to the three angles *CAB*, *ABC*, and *ACB*; (ax. 2.)
but the angles *ACD*, *ACB* are equal to two right angles, (I. 13.)
therefore also the angles *CAB*, *ABC*, *ACB* are equal to two right angles. (ax. 1.)
Wherefore, if a side of any triangle be produced, &c. Q.E.D.

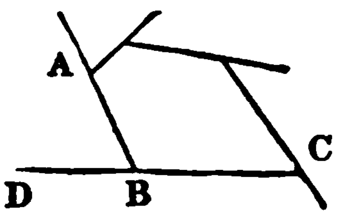
COR. 1. All the interior angles of any rectilineal figure together with four right angles, are equal to twice as many right angles as the figure has sides.



For any rectilineal figure *ABCDE* can be divided into as many triangles as the figure has sides, by drawing straight lines from a point *F* within the figure to each of its angles.
Then, because the three interior angles of a triangle are equal to two right angles, and there are as many triangles as the figure has sides,
therefore all the angles of these triangles are equal to twice as many right angles as the figure has sides;
but the same angles of these triangles are equal to the interior angles of the figure together with the angles at the point *F*:
and the angles at the point *F*, which is the common vertex of all the triangles, are equal to four right angles, (I. 15. Cor. 2.)

therefore the same angles of these triangles are equal to the angles of the figure together with four right angles;
but it has been proved that the angles of the triangles are equal to twice as many right angles as the figure has sides;
therefore all the angles of the figure together with four right angles are equal to twice as many right angles as the figure has sides.

Cor. 2. All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.

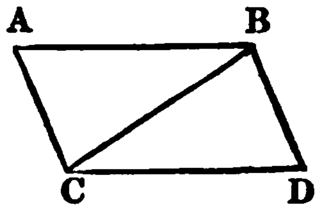


Since every interior angle *ABC*, together with its adjacent exterior angle *ABD*, are equal to two right angles, (I. 13.)
therefore all the interior angles, together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides;
but it has been proved by the foregoing corollary, that all the interior angles together with four right angles are equal to twice as many right angles as the figure has sides;
therefore all the interior angles together with all the exterior angles are equal to all the interior angles and four right angles, (ax. 1.)
take from these equals all the interior angles,
therefore all the exterior angles of the figure are equal to four right angles. (ax. 2.)

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let *AB*, *CD* be equal and parallel straight lines, and joined towards the same parts by the straight lines *AC*, *BD*.
Then *AC*, *BD* shall be equal and parallel.



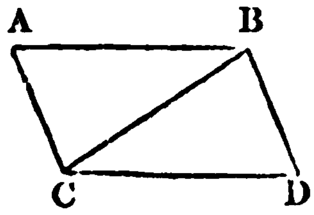
Join *BC*.
Then because *AB* is parallel to *CD*, and *BC* meets them,
therefore the angle *ABC* is equal to the alternate angle *BCD*; (I. 29.)
and because *AB* is equal to *CD*, and *BC* common to the two triangles *ABC*, *DCB*; the two sides *AB*, *BC*, are equal to the two *DC*, *CB*, each to each, and the angle *ABC* was proved to be equal to the angle *BCD*:
therefore the base *AC* is equal to the base *BD*, (I. 4.)
and the triangle *ABC* to the triangle *BCD*,

and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle ACB is equal to the angle CBD.
And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another;
therefore AC is parallel to BD; (I. 27.)
and AC was shewn to be equal to BD.
Therefore, straight lines which, &c. Q. E. D.

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, that is, divides them into two equal parts.

Let $ACDB$ be a parallelogram, of which BC is a diameter.
Then the opposite sides and angles of the figure shall be equal to one another; and the diameter BC shall bisect it.



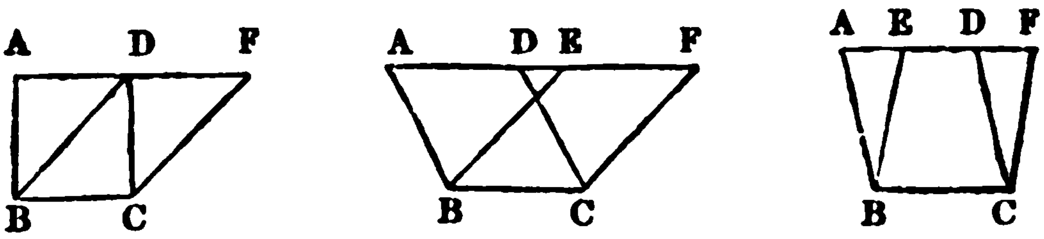
Because AB is parallel to CD, and BC meets them,
therefore the angle ABC is equal to the alternate angle BCD. (I. 29.)
And because AC is parallel to BD, and BC meets them,
therefore the angle ACB is equal to the alternate angle CBD. (I. 29.)
Hence in the two triangles ABC, CBD,
because the two angles ABC, BCA in the one, are equal to the two angles BCD, CBD in the other, each to each;
and one side BC, which is adjacent to their equal angles, common to the two triangles;
therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other, (I. 26.)
namely, the side AB to the side CD, and AC to BD, and the angle BAC to the angle BDC.
And because the angle ABC is equal to the angle BCD,
and the angle CBD to the angle ACB,
therefore the whole angle ABD is equal to the whole angle ACD; (ax. 2.)
and the angle BAC has been shewn to be equal to BDC;
therefore the opposite sides and angles of a parallelogram are equal to one another.
Also the diameter BC bisects it.
For since AB is equal to CD, and BC common, the two sides AB, BC are equal to the two DC, CB, each to each;
and the angle ABC has been proved to be equal to the angle BCD;
therefore the triangle ABC is equal to the triangle BCD; (I. 4.)
and the diameter BC divides the parallelogram $ACDB$ into two equal parts. Q. E. D.

PROPOSITION XXXV. THEOREM.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the parallelograms *ABCD, EBCF* be upon the same base *BC*, and between the same parallels *AF, BC*.

Then the parallelogram *ABCD* shall be equal to the parallelogram *EBCF*.



If the sides *AD, DF* of the parallelograms *ABCD, DBCF*, opposite to the base *BC*, be terminated in the same point *D*;

then it is plain that each of the parallelograms is double of the triangle *BDC*; (I. 34.)

and therefore the parallelogram *ABCD* is equal to the parallelogram *DBCF*. (ax. 6.)

But if the sides *AD, EF*, opposite to the base *BC*, be not terminated in the same point;

Then, because *ABCD* is parallelogram,
therefore *AD* is equal to *BC*; (I. 34.)
and for a similar reason, *EF* is equal to *BC*;
wherefore *AD* is equal to *EF*; (ax. 1.)
and *DE* is common;

therefore the whole, or the remainder, *AE* is equal to the whole, or remainder *DF*; (ax. 2 or 3.)

and *AB* is equal to *DC*; (I. 34.)
hence in the triangles *EAB, FDC*,
because *FD* is equal to *EA*, and *DC* to *AB*,

and the exterior angle *FDC* is equal to the interior and opposite angle *EAB*; (I. 29.)

therefore the base *FC* is equal to the base *EB*, (I. 4.)
and the triangle *FDC* equal to the triangle *EAB*.

From the trapezium *ABCF* take the triangle *FDC*, and from the same trapezium take the triangle *EAB*, and the remainders are equal, (ax. 3.)

therefore the parallelogram *ABCD* is equal to the parallelogram *EBCF*.

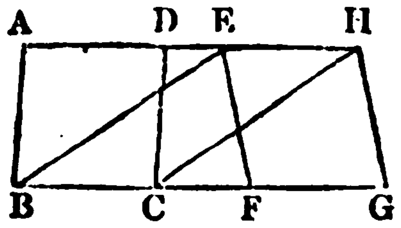
Therefore parallelograms upon the same, &c. Q.E.D.

PROPOSITION XXXVI. THEOREM.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let *ABCD, EFGH* be parallelograms upon equal bases *BC, FG*, and between the same parallels *AH, BG*.

Then the parallelogram *ABCD* shall be equal to the parallelogram *EFGH*.



Join BE, CH.

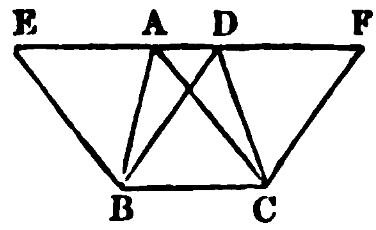
Then because BC is equal to FG, (hyp.) and FG to EH, (I. 34.)
therefore BC is equal to EH; (ax. 1.)
and these lines are parallels, and joined towards the same parts by the straight lines BE, CH;
but straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel; (I. 33.)
therefore BE, CH are both equal and parallel;
wherefore $EBCH$ is a parallelogram. (def. A.)
Then since the parallelograms $ABCD$, $EBCH$, are upon the same base BC, and between the same parallels BC, AH;
therefore the parallelogram $ABCD$ is equal to the parallelogram $EBCH$. (I. 35.)
For a similar reason, the parallelogram $EFGH$ is equal to the parallelogram $EBCH$;
therefore the parallelogram $ABCD$ is equal to the parallelogram $EFGH$. (ax. 1.)
Therefore parallelograms upon equal, &c. Q.E.D.

PROPOSITION XXXVII. THEOREM.

Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC.

Then the triangle ABC shall be equal to the triangle DBC.



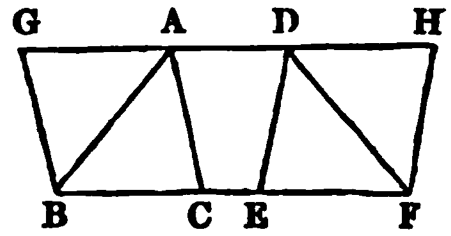
Produce AD both ways to the points E, F;
through B draw BE parallel to CA, (I. 31.)
and through C draw CF parallel to BD.
Then each of the figures $EBCA$, $DBCF$ is a parallelogram;
and $EBCA$ is equal to $DBCF$, (I. 35.) because they are upon the same base BC, and between the same parallels BC, EF.
And because the diameter AB bisects the parallelogram $EBCA$,
therefore the triangle ABC is half of the parallelogram $EBCA$; (I. 34.)
also because the diameter BC bisects the parallelogram $DBCF$,
therefore the triangle DBC is half of the parallelogram $DBCF$,
but the halves of equal things are equal; (ax. 7.)
therefore the triangle ABC is equal to the triangle DBC.
Wherefore triangles, &c. Q.E.D.

PROPOSITION XXXVIII. THEOREM.

Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels BF, AD.

Then the triangle ABC shall be equal to the triangle DEF.



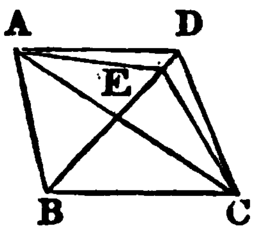
Produce AD both ways to the points G, H;
through B draw BG parallel to CA, (I. 31.)
and through F draw FH parallel to ED.
Then each of the figures $GBCA$, $DEFH$ is a parallelogram;
and they are equal to one another, (I. 36.)
because they are upon equal bases BC, EF,
and between the same parallels BF, GH.
And because the diameter AB bisects the parallelogram $GBCA$,
therefore the triangle ABC is the half of the parallelogram $GBCA$; (I. 34.)
also, because the diameter DF bisects the parallelogram $DEFH$,
therefore the triangle DEF is the half of the parallelogram $DEFH$;
but the halves of equal things are equal; (ax. 7.)
therefore the triangle ABC is equal to the triangle DEF.
Wherefore, triangles upon equal bases, &c. Q.E.D.

PROPOSITION XXXIX. THEOREM.

Equal triangles upon the same base and upon the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC, and upon the same side of it.

Then the triangles ABC, DBC shall be between the same parallels.

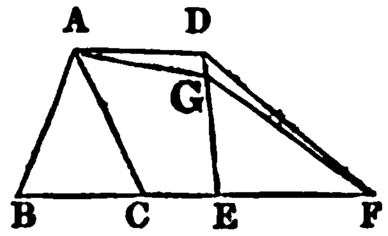


Join AD; AD shall be parallel to BC.
For, if it is not, through the point A draw AE parallel to BC, (I. 31.) meeting BD or BD produced in E, and join EC.
Then the triangle ABC is equal to the triangle EBC, (I. 37.)
because they are upon the same base BC,
and between the same parallels BC, AE;
but the triangle ABC is equal to the triangle DBC; (hyp.)
therefore the triangle DBC is equal to the triangle EBC,
the greater equal to the less triangle, which is impossible:
therefore AE is not parallel to BC.
In the same manner it can be demonstrated, that no other line but AD is parallel to BC;
AD is therefore parallel to BC.
Wherefore, equal triangles upon, &c. Q.E.D.

PROPOSITION XL. THEOREM.

Equal triangles, upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and towards the same parts.
Then they shall be between the same parallels.

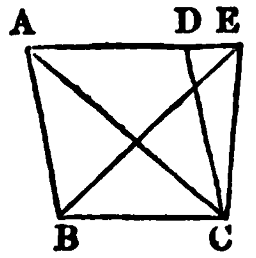


Join AD; AD shall be parallel to BF.
For, if it is not, through A draw AG parallel to BF, (I. 31.)
meeting ED, or ED produced in G, and join GF.
Then the triangle ABC is equal to the triangle GEF, (I. 38.)
because they are upon equal bases BC, EF,
and between the same parallels BF, AG;
but the triangle ABC is equal to the triangle DEF; (hyp.)
therefore the triangle DEF is equal to the triangle GEF, (ax. 1.)
the greater equal to the less triangle, which is impossible:
therefore AG is not parallel to BF.
And in the same manner it can be demonstrated, that there is no other parallel to it but AD;
AD is therefore parallel to BF.
Wherefore, equal triangles upon, &c. Q.E.D.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.

Let the parallelogram $ABCD$, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE.
Then the parallelogram $ABCD$ shall be double of the triangle EBC.

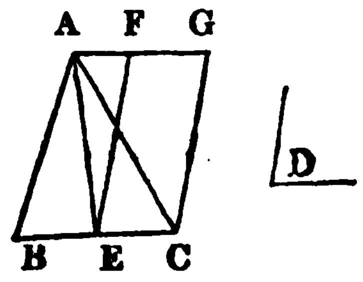


Join AC.
Then the triangle ABC is equal to the triangle EBC, (I. 37.)
because they are upon the same base BC, and between the same parallels BC, AE.
But the parallelogram $ABCD$ is double of the triangle ABC,
because the diameter AC bisects it; (I. 34.)
wherefore $ABCD$ is also double of the triangle EBC.
Therefore, if a parallelogram and a triangle, &c. Q.E.D.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle.
It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

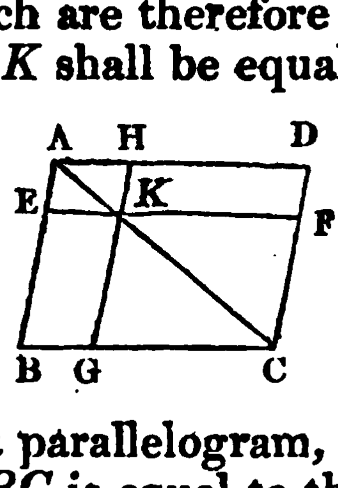


Bisect BC in E, (I. 10.) and join AE;
at the point E in the straight line EC,
make the angle CEF equal to the angle D; (I. 23.)
through A draw AFG parallel to BC, (I. 31.)
and through C draw CG parallel to EF.
Then the figure $CEFG$ is a parallelogram. (def. A.)
And because the triangles ABE, AEC are on the equal bases BE, EC, and between the same parallels BC, AG;
they are therefore equal to one another; (I. 38.)
and therefore the triangle ABC is double of the triangle AEC;
but the parallelogram $FECG$ is double of the triangle AEC, (I. 41.)
because they are upon the same base EC, and between the same parallels EC, AG;
therefore the parallelogram $FECG$ is equal to the triangle ABC, (ax. 6.)
and it has one of its angles CEF equal to the given angle D.
Wherefore a parallelogram $FECG$ has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D. Q.E.F.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let $ABCD$ be a parallelogram, of which the diameter is AC: and EH, GF the parallelograms about AC, *that is, through which AC passes:*
also BK, KD the other parallelograms which make up the whole figure $ABCD$, which are therefore called the complements.
Then the complement BK shall be equal to the complement KD.



Because $ABCD$ is a parallelogram, and AC its diameter,
therefore the triangle ABC is equal to the triangle ADC. (I. 34.)

Again, because $EKHA$ is a parallelogram, and AK its diameter, therefore the triangle AEK is equal to the triangle AHK; (I. 34.) and for the same reason, the triangle KGC is equal to the triangle KFC.

Wherefore the two triangles AEK, KGC are equal to the two triangles AHK, KFC, (ax. 2.)

but the whole triangle ABC is equal to the whole triangle ADC;

therefore the remaining complement BK is equal to the remaining complement KD. (ax. 3.)

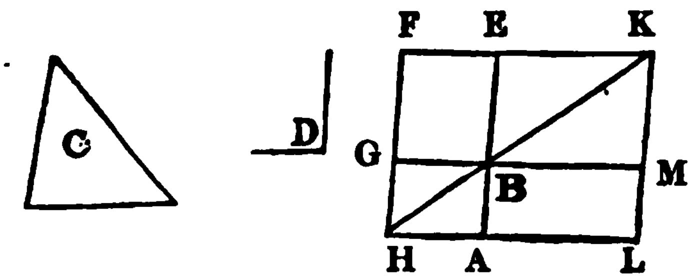
Wherefore the complements, &c. Q.E.D.

PROPOSITION XLIV. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle.

It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



Make the parallelogram $BEFG$ equal to the triangle C, (I. 42.)

and having the angle EBG equal to the angle D,

so that BE be in the same straight line with AB;

produce FG to H, through A draw AH parallel to BG or EF, (I. 31.) and join HB.

Then because the straight line HF falls upon the parallels AH, EF, therefore the angles AHF, HFE are together equal to two right angles; (I. 29.)

wherefore the angles BHF, HFE are less than two right angles:

but straight lines which with another straight line make the two interior angles upon the same side less than two right angles, do meet if produced far enough: (ax. 12.)

therefore HB, FE shall meet, if produced;

let them be produced and meet in K,

through K draw KL parallel to EA or FH,

and produce HA, GB to meet KL in the points L, M.

Then $HLKF$ is a parallelogram, of which the diameter is HK;

and AG, ME, are the parallelograms about HK;

also LB, BF are the complements;

therefore the complement LB is equal to the complement BF; (I. 43.)

but the complement BF is equal to the triangle C; (constr.)

wherefore LB is equal to the triangle C.

And because the angle GBE is equal to the angle ABM, (I. 15.) and likewise to the angle D; (constr.)

therefore the angle ABM is equal to the angle D. (ax. 1.)

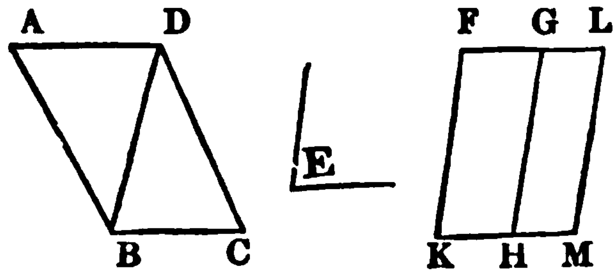
Therefore to the straight line AB, the parallelogram LB is applied, equal to the triangle C, and having the angle ABM equal to the angle D. Q.E.F.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let $ABCD$ be the given rectilineal figure, and E the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the figure $ABCD$, and having an angle equal to E.



Join DB.

Describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal to the angle E; (I. 42.)

to the straight line GH, apply the parallelogram GM equal to the triangle DBC, having the angle GHM equal to the angle E. (I. 44.)

Then the figure $FKML$ shall be the parallelogram required.

Because the angle E is equal to each of the angles FKH, GHM, therefore the angle FKH is equal to the angle GHM;

add to each of these equals the angle KHG;

therefore the angles FKH, KHG are equal to the angles KHG, GHM;

but FKH, KHG are equal to two right angles; (I. 29.)

therefore also KHG, GHM are equal to two right angles;

and because at the point H, in the straight line GH, the two straight lines KH, HM, upon the opposite sides of it, make the adjacent angles equal to two right angles,

therefore HK is in the same straight line with HM. (I. 14.)

And because the line HG meets the parallel KM, FG,

therefore the angle MHG is equal to the alternate angle HGF; (I. 29.)

add to each of these equals the angle HGL;

therefore the angles MHG, HGL are equal to the angles HGF, HGL;

but the angles MHG, HGL are equal to two right angles; (I. 29.)

therefore also the angles HGF, HGL are equal to two right angles,

and therefore FG is in the same straight line with GL. (I. 14.)

And because KF is parallel to HG, and HG to ML,

therefore KF is parallel to ML; (I. 30.)

and KM has been proved parallel to FL,

wherefore the figure $FKML$ is a parallelogram;

and since the triangle ABD is equal to the parallelogram HF,

and the triangle BDC to the parallelogram GM;

therefore the whole rectilineal figure *ABCD* is equal to the whole parallelogram *KFLM*.

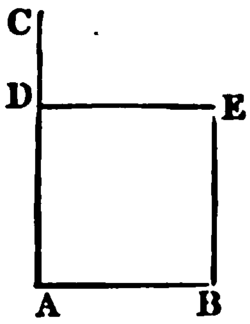
Therefore the parallelogram *KFLM* has been described equal to the given rectilineal figure *ABCD*, having the angle *FKM* equal to the given angle *E*. Q. E. F.

Cor. From this it is manifest how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; viz. by applying to the given straight line a parallelogram equal to the first triangle *ABD*, (I. 44.) and having an angle equal to the given angle.

PROPOSITION XLVI. PROBLEM.

To describe a square upon a given straight line.

Let *AB* be the given straight line.
It is required to describe a square upon *AB*.



From the point *A* draw *AC* at right angles to *AB*; (I. 11.)
make *AD* equal to *AB*, (I. 3.)
through the point *D* draw *DE* parallel to *AB*, (I. 31).
and through *B*, draw *BE* parallel to *AD*;
therefore *ABED* is a parallelogram;
whence *AB* is equal to *DE*, and *AD* to *BE*; (I. 34.)
but *BA* is equal to *AD*,
therefore the four lines *BA*, *AD*, *DE*, *EB* are equal to one another,
and the parallelogram *ADEB* is equilateral.
It has likewise all its angles right angles;
since *AD* meets the parallels *AB*, *DE*,
therefore the angles *BAD*, *ADE* are equal to two right angles; (I. 29.)
but *BAD* is a right angle; (constr.)
therefore also *ADE* is a right angle.
But the opposite angles of parallelograms are equal; (I. 34.)
therefore each of the opposite angles *ABE*, *BED* is a right angle;
wherefore the figure *ADEB* is rectangular, and it has been proved to be equilateral;
therefore the figure *ADED* is a square, (def. 30.) and it is described upon the given straight line *AB*. Q. E. F.

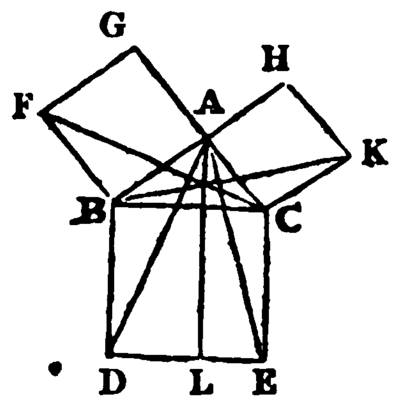
Cor. Hence, every parallelogram that has one right angle, has all its angles right angles.

PROPOSITION XLVII. THEOREM.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC.

Then the square described upon the side BC, shall be equal to the squares described upon BA, AC.

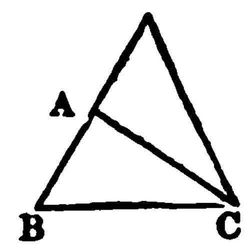


On BC describe the square $BDEC$, (I. 46.)
and on BA, AC the squares GB, HC;
through A draw AL parallel to BD or CE; (I, 31.)
and join AD, FC.
Then because the angle BAC is a right angle, (hyp.)
and that the angle BAG is a right angle, (def. 30.)
the two straight lines AC, AG upon the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles;
therefore CA is in the same straight line with AG. (I. 14.)
For the same reason, BA and AH are in the same straight line.
And because the angle DBC is equal to the angle FBA,
each of them being a right angle,
add to each of these equals the angle ABC,
therefore the whole angle DBA is equal to the whole angle FBC. (ax. 2.)
And because the two sides AB, BD, are equal to the two sides FB, BC, each to each, and the included angle ABD is equal to the included angle FBC,
therefore the base AD is equal to the base FC, (I. 4.)
and the triangle ABD to the triangle FBC.
Now the parallelogram BL is double of the triangle ABD, (I. 41.)
because they are upon the same base BD, and between the same parallels BD, AL;
also the square GB is double of the triangle FBC,
because these also are upon the same base FB, and between the same parallels FB, GC.
But the doubles of equals are equal to one another; (ax. 6.)
therefore the parallelogram BL is equal to the square GB.
Similarly, by joining AE, BK, it can be proved,
that the parallelogram CL is equal to the square HC.
Therefore the whole square $BDEC$ is equal to the two squares GB, HC; (ax. 2.)
and the square $BEDC$ is described upon the straight line BC, and the squares GB, HC, upon AB, AC:
therefore the square upon the side BC is equal to the squares upon the sides AB, AC.
Therefore, in any right-angled triangle, &c. Q. E. D.

PROPOSITION XLVIII. THEOREM.

If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other two sides AB, AC.
Then the angle BAC shall be a right angle.



From the point A draw AD at right angles to AC, (I. 11.)
make AD equal to AB, and join DC.
Then because AD is equal to AB,
therefore the square of AD is equal to the square of AB;
to each of these equals add the square of AC;
therefore the squares of AD, AC are equal to the squares of AB, AC:
but the squares of AD, AC are equal to the square of DC, (I. 47.)
because the angle DAC is a right angle;
and the square of BC, by hypothesis, is equal to the squares of BA, AC;
therefore the square of DC is equal to the square of BC;
and therefore the side DC is equal to the side BC.
And because the side AD is equal to the side AB,
and AC is common to the two triangles DAC, BAC;
the two sides DA, AC, are equal to the two BA, AC, each to each;
and the base DC has been proved to be equal to the base BC;
therefore the angle DAC is equal to the angle BAC; (I. 8.)
but DAC is a right angle;
therefore also BAC is a right angle.
Therefore, if the square described upon, &c. Q. E. D.

NOTES TO BOOK I.

ON THE DEFINITIONS.

GEOMETRY is one of the most perfect of the deductive Sciences, and seems to rest on the simplest inductions from experience and observation.

The first principles of Geometry are therefore in this view consistent hypotheses founded on facts cognizable by the senses, and it is a subject of primary importance to draw a distinction between the conception of things and the things themselves. These hypotheses do not involve any property contrary to the real nature of the things, and, consequently, cannot be regarded as arbitrary, but, in certain respects, agree with the conceptions which the things themselves suggest to the mind through the medium of the senses. The essential definitions of Geometry therefore being deductions from observation and experience, rest ultimately on the evidence of the senses.

It is by experience we become acquainted with the existence of individual forms of magnitudes; but by the mental process of abstraction, which begins with a particular instance, and proceeds to the general idea of all objects of the same kind, we attain to the general conception of those forms which come under the same general idea.

The essential definitions of Geometry express generalized conceptions of real existences in their most perfect ideal forms; the laws and appearances of nature, and the operations of the human intellect being supposed uniform and consistent.

But in cases where the subject falls under the class of simple ideas, the terms of the definition so called are no more than merely equivalent expressions. The simple idea described by a proper term or terms, does not in fact admit of definition properly so called. The definitions in Euclid's Elements may be divided into two classes, those which merely explain the meaning of the terms employed, and those, which, besides explaining the meaning of the terms, suppose the existence of the things described in the definitions.

Definitions in Geometry cannot be of such a form as to explain the nature and properties of the figures defined; it is sufficient that they give marks whereby the thing defined may be distinguished from every other of the same kind. It will at once be obvious, that the definitions of Geometry, one of the pure sciences, being abstractions of space, are not like the definitions in any one of the physical sciences. The discovery of any new physical facts may render necessary some alteration or modification in the definitions of the latter.

Def. I. Simson has adopted Theon's definition of a point. Euclid's definition is, *σημεῖον ἐστιν οὗ μέρος οὐδέν*, "A point is that, of which there is no part," or which cannot be parted or divided, as it is explained by Proclus. The Greek term *σημεῖον*, literally means, a visible *sign* or *mark* on a surface, in other words, *a physical point.* The English term *point*, means the sharp end of any thing, or a mark made by it. The word *point* comes from the Latin *punctum*, through the French word *point*. Neither of these terms, in its literal sense, appears to give a very exact notion of what is to be understood by a point in Geometry.

Euclid's definition of a point merely expresses a negative property, which excludes the proper and literal meaning of the Greek term, as applied to denote a physical point, or a mark which is visible to the senses.

Pythagoras defined a point to be μονὰς θέσιν ἔχουσα, "a monad having position." By uniting the positive idea of position, with the negative idea of defect of magnitude, the conception of a point in Geometry may be rendered perhaps more intelligible. A point may then be defined to be that which has no magnitude, but position only.

Def. II. Every visible line has both length and breadth, and it is impossible to draw any line whatever which shall have no breadth. The definition requires the conception of the length only of the line to be considered, abstracted from, and independently of, all idea of its breadth.

Def. III. This definition renders more intelligible the exact meaning of the definition of a point: and we may add, that, in the Elements, Euclid supposes that the intersection of two straight lines is a point, and that two straight lines can intersect each other in one point only.

Def. IV. The straight line or right line is a term so clear and intelligible as to be incapable of becoming more so by formal definition. Euclid's definition is Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται, wherein he states it to lie *evenly*, or *equally*, or *upon an equality*, (ἐξ ἴσου) between its extremities, and which Proclus explains as being stretched between its extremities, ἡ ἐπ' ἄκρων τεταμένη.

If the line be conceived to be drawn on a plane surface, the words ἐξ ἴσου may mean, that no part of the line which is called a straight line deviates either from one side or the other of the direction which is fixed by the extremities of the line; and thus it may be distinguished from a curved line, which does not lie, in this sense, evenly between its extreme points. If the line be conceived to be drawn in space, the words ἐξ ἴσου, must be understood to apply to every direction on every side of the line between its extremities.

Every straight line situated in a plane is considered to have two sides; and when the direction of a line is known, the line is said to be given in position; also, when the length is known or can be found, it is said to be given in magnitude.

From the definition of a straight line, it follows, that two points fix a straight line in position, which is the foundation of the first and second postulates. Hence straight lines which are proved to coincide in two or more points, are called "one and the same straight line," Prop. 14, Book I., or, which is the same thing, that, "Two straight lines cannot have a common segment," as Simson shews in his Corollary to Prop. 11, Book I.

Archimedes defined "a straight line to be the shortest distance between two points;" but this is a theorem considered by Euclid as requiring proof.

The following definition of straight lines has also been proposed. "Straight lines are those which, if they coincide in any two points, coincide as far as they are produced." But this is rather a criterion of straight lines, and analogous to the eleventh axiom, which states that, "all right angles are equal to one another," and suggests that all straight lines may be made to coincide wholly, if the lines be equal; or partially if the lines be of unequal lengths. A definition should properly be restricted to the description of the thing defined, as it exists, independently of any comparison of its properties or of tacitly assuming the existence of axioms.

Def. VII. Euclid's definition of a plane surface is, Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται, "A plane surface is that which lies evenly or equally with the straight lines in it;" instead of which Simson has given the definition which was originally proposed by Hero the Elder. A plane superficies may be supposed to be situated in any position, and to be continued in every direction to any extent.

Def. VIII. Simson remarks that this definition seems to include the angles formed by two curved lines, or a curve and a straight line, as well as that formed by two straight lines.

Angles made by straight lines only, are treated of in Elementary Geometry.

Def. IX. It is of the highest importance to attain a clear conception of an angle.

The literal meaning of the term *angulus* suggests the Geometrical conception of an angle, which may be regarded as formed by the divergence of two straight lines from a point. In the definition of an angle, the magnitude of the angle is independent of the lengths of the two lines by which it is included; their mutual divergence from the point at which they meet, is the criterion of the magnitude of an angle, as it is pointed out in the succeeding definitions. The point at which the two lines meet is called the vertex of the angle, and must not be confounded with the magnitude of the angle itself. The right angle is fixed in magnitude, and, on this account, it is made the subject with which all other angles in Geometry are compared.

Two straight lines which actually intersect one another, or which when produced would intersect, are said to be inclined to one another, and the inclination of the two lines is determined by the angle which they make with one another.

Def. x. It may be here observed that in the Elements, Euclid always assumes that when one line is perpendicular to another line, the latter is also perpendicular to the former; and always calls *a right angle*, ὀρθὴ γωνία; but *a straight line*, εὐθεῖα γραμμή.

Def. xix. This has been restored from Proclus, as it seems to have a meaning in the constructions of Prop. 14, Book II; the first case of Prop. 33, Book III, and Prop. 13, Book VI. The definition of the segment of a circle is not once alluded to in Book I, and is not required before the discussion of the properties of the circle in Book III. Proclus remarks on this definition: "Hence you may collect that the centre has three places. For it is either within the figure, as in the circle; or in its perimeter, as in the semi-circle; or without the figure, as in certain conic lines."

Def. xxiv-xxix. Triangles are divided into three classes by reference to the relations of their sides, and into three other classes by reference to their angles. A further classification may be made by considering both the relation of the sides and angles in each triangle.

In Simson's definition of the isosceles triangle, the word *only* must be omitted, as the equilateral triangle is considered isosceles in Prop. 15, Book IV. Objection has been made to the definition of an acute-angled triangle. It is said that it cannot be admitted as a definition, that all the three angles of a triangle are acute, which is supposed in Def. 29. It may be replied, that the definitions of the three kinds of angles point out and seem to supply a foundation for a similar distinction of triangles.

Def. xxx-xxxiv. The definitions of quadrilateral figures are liable to objection. All of them, except the trapezium, fall under the general idea of a parallelogram; but as Euclid has defined parallel straight lines after he had defined four-sided figures, no other arrangement could be adopted than the one he has followed; and for which there appeared to him, without doubt, some probable reasons. Sir Henry Savile, in his Seventh Lecture, remarks on some of the definitions of Euclid, "Nec dissimulandum aliquot harum in manibus exiguum esse usum in Geometriâ." A few verbal emendations have been made in some of them.

A square is a four-sided plane figure having all its sides equal, and one angle a right angle: because it is proved in Prop. 46, Book I, that if a parallelogram have one angle a right angle, all its angles are right angles.

An oblong in the same manner may be defined as a plane figure of four sides having only its opposite sides equal, and one of its angles a right angle.

A rhomboid is a four-sided plane figure having only its opposite sides equal to one another and its angles not right angles.

Sometimes an irregular four-sided figure which has two sides parallel, is called a trapezoid.

Def. xxxv. It is possible for two right lines never to meet when produced, and not be parallel.

Def. A. The term parallelogram literally implies a figure formed by parallel

straight lines, and may consist of four, six, eight, or any even number of sides, where every two of the opposite sides are parallel to one another.

In the Elements, however, the term is restricted to four-sided figures, and includes the four species of figures named in the Definitions XXX—XXXIII.

The synthetic method is followed by Euclid not only in the demonstrations of the propositions, but also in laying down the definitions. He commences with the simplest abstractions, defining a point, a line, an angle, a superficies, and their different varieties. This mode of proceeding involves the difficulty, almost insurmountable of defining satisfactorily the elementary abstractions of Geometry. Simson observes that it is necessary to consider a solid, that is a magnitude which has length, breadth, and thickness, in order to understand aright the definitions of a point, a line, and a superficies. A solid or volume considered apart from its physical properties, suggests the idea of the surfaces by which it is bounded: a surface, the idea of the line or lines which form its boundaries: and a finite line, the points which form its extremities. A solid is therefore bounded by surfaces; a surface is bounded by lines; and a line is terminated by two points. A point marks position only: a line has one dimension, length only, and defines distance: a superficies has two dimensions, length and breadth, and defines extension: and a solid has three dimensions, length, breadth, and thickness, and defines some definite portion of space.

It may also be remarked that two points are sufficient to determine the position of a straight line, and three points not in the same straight line, are necessary to fix the position of a plane.

ON THE POSTULATES.

The definitions assume the possible existence of straight lines and circles, and the postulates predicate the possibility of drawing and of producing straight lines, and of describing circles. The postulates form the principles of construction assumed in the Elements; and are, in fact, problems, the possibility of which is admitted to be self-evident, and to require no proof.

It must, however, be carefully remarked, that the third postulate only admits that when any line is given in position and magnitude, a circle may be described from either extremity of the line as a centre, and with a radius equal to the length of the line, as in Prop. 1, Book I. It does not admit the description of a circle with any other point as a centre than one of the extremities of the given line.

Prop. 2, Book I. shews how, from any given point, to draw a straight line equal to another straight line which is given in magnitude and position.

ON THE AXIOMS.

Axioms are usually defined to be self-evident truths, which cannot be rendered more evident by demonstration; in other words, the axioms of Geometry are theorems, the truth of which is admitted without proof. It is by experience we first become acquainted with the different forms of geometrical magnitudes, and the axioms, or the fundamental ideas of their equality or inequality appear to rest on the same basis. The conception of the truth of the axioms does not appear to be more removed from experience than the conception of the definitions.

These axioms, or first principles of demonstration, are such theorems as cannot be resolved into simpler theorems, and no theorem ought to be admitted as a first principle of reasoning which is capable of being demonstrated.

An axiom and its converse should both be of such a nature as that neither of them should require a formal demonstration.

The first and most simple idea, derived from experience, is, that every magnitude fills a certain space, and that several magnitudes may fill the same space.

All the knowledge we have of magnitude is purely relative, and the most simple relations are those of equality and inequality. In the comparison of magnitudes, some are considered as given or known, and the unknown are compared with the known, and conclusions are synthetically deduced with respect to the equality or inequality of the magnitudes under consideration. In this manner we form our idea of equality, which is thus formally stated in the eighth axiom: "Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another."

Every specific definition is referred to this universal principle. With regard to a few more general definitions which do not furnish an equality, it will be found that some hypothesis is always made reducing them to that principle, before any theory is built upon them. As for example, the definition of a straight line is to be referred to the tenth axiom; the definition of a right angle to the eleventh axiom; and the definition of parallel straight lines to the twelfth axiom.

It is called the principle of superposition, or, the mental process by which one Geometrical magnitude may be conceived to be placed on another, so as exactly to coincide with it, in the parts which are made the subject of comparison. Thus, if one straight line be conceived to be placed upon another, so that their extremities are coincident, the two straight lines are equal. If the directions of two lines which include one angle, coincide with the directions of the two lines which contain another angle, where the points, from which the angles diverge, coincide, then the two angles are equal: the lengths of the lines not affecting in any way the magnitudes of the angles. When one plane figure is conceived to be placed upon another, so that the boundaries of one exactly coincide with the boundaries of the other, then the two plane figures are equal. It may also be remarked, that the converse of this proposition is also true, namely, that when two magnitudes are equal, they coincide with one another.

This explanation of Geometrical equality appears to be out of its proper place. The definitions of the forms of magnitudes naturally come first, and the criterion of their equality appears as naturally to follow. If the first seven axioms are to be restricted to Geometrical magnitudes, the eighth ought to have preceded them. Perhaps Euclid intended that the first seven axioms should be applicable to numbers as well as to Geometrical magnitudes, and this is in accordance with the words of Proclus, who calls the axioms, *common notions*, not peculiar to the subject of Geometry.

The eighth axiom is properly the definition of Geometrical equality.

Axiom v. It may be observed that when equal magnitudes are taken from unequal magnitudes, the greater remainder exceeds the less remainder by as much as the greater of the unequal magnitudes exceeds the less.

Axiom IX. The whole is greater than its part, and conversely, the part is less than the whole. This axiom appears to assert the contrary of the eighth axiom, namely, that two magnitudes, of which one is greater than the other, cannot be made to coincide with one another.

Axiom X. The property of straight lines expressed by the tenth axiom, namely, "that two straight lines cannot enclose a space", is obviously implied in the definition of straight lines; for if they enclosed a space, they could not coincide between their extreme points, when the two lines are equal.

Axiom XI. This axiom has been asserted to be a demonstrable theorem. If an angle be admitted to be a species of magnitude, this axiom is only a particular application of the eighth axiom to right angles.

Axiom XII. See the notes on Prop. XXIX, Book I.

ON THE PROPOSITIONS.

THE deductive truths of Geometry are called *propositions*, which are divided into two classes, *problems* and *theorems*. A proposition, as the term imports, is something proposed; it is a *problem*, when some Geometrical *construction* is required to be *effected*: and it is a *theorem* when some Geometrical *property* is to be *demonstrated*. Every Proposition is naturally divided into two parts; a problem consists of the *data*, or *things given*; and the *quæsita*, or *things required*: a theorem, consists of the *premisses*, *hypothesis*, or *the properties admitted*; and the *conclusion*, or *predicate*, or *properties to be demonstrated*.

Hence the distinction between a problem and a theorem is this, that a problem consists of data and quæsita, and requires solution: and a theorem consists of the hypothesis and the predicate, and requires demonstration.

The connected course of reasoning by which any Geometrical truth is established is called a demonstration. It is called a *direct* demonstration when the predicate of the proposition is inferred directly from the premisses, as the conclusion of a series of successive deductions. The demonstration is called *indirect*, when the conclusion shews that the introduction of any other supposition contrary to the hypothesis stated in the proposition, necessarily leads to an absurdity.

The course pursued in the demonstrations of the propositions in Euclid's Elements of Geometry, is always to refer directly to some expressed principle, to leave nothing to be inferred from vague expressions, and to make every step of the demonstrations the object of the understanding.

It has been maintained by some philosophers that a genuine definition contains some property or properties which can form a basis for demonstration, and that the science of Geometry is deduced from the definitions, and that on them alone the demonstrations depend. Others have maintained that a definition explains only the meaning of a term, and does not embrace the nature and properties of the thing defined.

If the propositions usually called postulates and axioms are either tacitly assumed or expressly stated in the definitions; in this view, demonstrations may be said to be legitimately founded on definitions. If, on the other hand, a definition is simply an explanation of the meaning of a term, whether abstract or concrete, by such marks as may prevent a misconception of the thing defined; it will be at once obvious that some constructive and theoretic principles must be assumed besides the definitions to form the grounds of legitimate demonstration. These principles we conceive to be the postulates and axioms. The postulates describe constructions which may be admitted as possible by direct appeal to our experience; and the axioms assert general theoretic truths so simple and self-evident as to require no proof, but to be admitted as the assumed first principles of demonstration. Under this view all Geometrical reasonings proceed upon the admission of the hypotheses assumed in the definitions, and the unquestioned possibility of the postulates, and the truth of the axioms.

The general theorems of Geometry are demonstrated by means of syllogisms founded on the axioms and definitions. The form of syllogism employed in Geometrical reasonings is of the simplest character. Every syllogism consists of three propositions, of which, two are called the premisses, and the third, the conclusion. These propositions contain three terms, the subject and predicate of the conclusion, and the middle term which connects the predicate and the conclusion together. The subject of the conclusion is called *the minor*, and the predicate of the conclusion is called *the major* term, of the syllogism. The major term appears in one premiss, and the minor term in

the other, with the middle term which is in both premisses. That premiss which contains the middle term and the major term, is called the *major premiss;* and that which contains the middle term and the minor term, is called the *minor premiss* of the syllogism. As an example, we may take the first syllogism in the demonstration of Prop. 1, Book I, wherein it will be seen that the middle term is the subject of the major premiss and the predicate of the minor.

Major premiss. Because the straight line AB is equal to the straight line AC;
Minor premiss. and, because the straight line BC is equal to the straight line AB;
Conclusion. therefore the straight line BC is equal to the straight line AC.

Here, BC is the subject, and AC the predicate of the conclusion.
BC is the subject, and AB the predicate of the minor premiss.
AB is the subject, and AC the predicate of the major premiss.

Also, AC is the major term, BC the minor term, and AB the middle term of the syllogism.

In this syllogism, it may be remarked that the definition of a straight line is assumed, and the definition of the Geometrical equality of two straight lines; also that a general theoretic truth, or axiom, forms the ground of the conclusion. And further, though it be impossible to make any point, mark or sign, (σημεῖον) which has not both length and breadth, and any line which has not both length and breadth; the demonstrations in Geometry do not on this account become invalid. For they are pursued on the hypothesis that the point has no parts but position only: and the line has length only, but no breadth or thickness; also that the surface has length and breadth only, but no thickness: and all the conclusions at which we arrive are independent of every other consideration.

Every proposition, when complete, may be divided into six parts, as Proclus has pointed out in his commentary.

1. *The proposition* or *general enunciation* which states in general terms the conditions of the problem or theorem.

2. *The exposition* or *particular enunciation* which exhibits the *subject* of the proposition in particular terms as a fact, and refers it to some diagram described.

3. *The determination* contains the *predicate* in particular terms, as it is pointed out in the diagram.

4. *The construction* applies the postulates to prepare the diagram for the demonstration.

5. *The demonstration* is the connexion of syllogisms, which prove the truth or falsehood of the theorem, the possibility or impossibility of the problem, in that particular case exhibited in the diagram.

6. *The conclusion* is merely the repetition of the general enunciation, wherein the predicate is asserted as a demonstrated truth.

Prop. I. In Books I and II, the circle is employed as a mechanical instrument, in the same manner as the straight line, and the use made of it rests entirely on the third postulate. No properties of the circle are discussed or even alluded to in these books beyond the definition and the third postulate. One circle may fall within or without another entirely, or the circumferences may intersect each other, as when the centre of one circle is in the circumference of the other; and it is obvious from the two circles cutting each other, in two points, one on each side of the given line, that two equilateral triangles may be formed on the given line.

Prop. II. When the given point is neither in the line, nor in the line produced, this problem admits of eight different lines being drawn from the given point in different directions, every one of which is a solution of the problem. For 1. The given line has two extremities, to each of which a line may be drawn from the given point.

2. The equilateral triangle may be described on either side of this line. 3. And the side *BD* of the equilateral triangle *ABD* may be produced either way.

But when the given point lies either in the line or in the line produced, the distinction which arises from joining the two ends of the line with the given point no longer exists, and there are only four cases of the problem.

Prop. III. This problem admits of two solutions, and it is left undetermined from which end of the greater line the part is to be cut off.

Prop. IV. This forms the first case of equal triangles, two other cases are proved in Props. VIII and XXVI. A distinction ought to be made between equal triangles and equivalent triangles, the former including those whose sides and angles mutually coincide, the latter those whose areas only are equivalent.

The term *base* is obviously taken from the idea of a building, and the same may be said of the term *altitude*. In Geometry, however, these terms are not restricted to one particular position of a figure, as in the case of a building, but may be in any position whatever.

Prop. V. Proclus has given in his commentary a proof for the equality of the angles at the base without producing the equal sides. The construction follows the same order, taking in *AB* a point *D* and cutting off from *AC* a part *AE* equal to *AB*, and then joining *CD* and *BE*.

A corollary is a theorem which results from the demonstration of a proposition, and generally is so obvious as to require no formal proof.

Prop. VI is the converse of one part of Prop. V. One proposition is defined to be the *converse* of another when the hypothesis of the former becomes the predicate of the latter; and vice versâ.

There is besides this another kind of conversion, when a theorem has several hypotheses and one predicate; by assuming the predicate and one or more than one of the hypotheses, some one of the hypotheses may be inferred as the predicate of the converse. In this manner, Prop. VIII is the converse of Prop. IV. It may here be observed, that converse theorems are not universally true: as for instance, the following direct proposition is universally true; "If two triangles have their three sides respectively equal, the three angles of each shall be respectively equal." But the converse is not universally true; namely, "If two triangles have the three angles in each respectively equal, the three sides are respectively equal." Converse theorems require, in some instances, the consideration of other conditions than those which enter into the proof of the direct theorem. *Converse* and *contrary* propositions are by no means to be confounded, the *contrary* proposition denies what is assumed in the *direct* proposition, but the subject and predicate in each are the same.

Prop. VI is the first instance of indirect demonstrations, and they are more suited for the proof of converse propositions. All those propositions which are demonstrated ex absurdo, are properly analytical demonstrations, according to the Greek notion of analysis, which first supposed the thing required to be done, or to be true, and then shewed the consistency or inconsistency of this construction or hypothesis with truths admitted or already demonstrated.

Prop. VII. The enunciation in the text was altered into that form by Simson. Euclid's is, *'Επὶ τῆς αὐτῆς εὐθείας, δυσὶ ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ οὐ συσταθήσονται, πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη, τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις.*

Prop. VIII. When the three sides of one triangle are shewn to coincide with the three sides of any other, the equality of the triangles is at once obvious. This, however, is not stated at the conclusion of Prop. VIII or of Prop. XXVI. For the equality of the areas of two coincident triangles, reference is always made by Euclid to Prop. IV.

Prop. IX. By means of this problem, any angle may be divided into four, eight, sixteen, &c. equal angles.

Prop. X. Any finite straight line may, by this problem, be divided into four, eight, sixteen, &c. equal parts.

Prop. XI. When the point is at the extremity of the line. By the second postulate the line may be produced, and then the construction applies.

Prop. XIII. It is manifest that the lines which bisect the angles ABC and ABD are at right angles to each other.

Prop. XIV is the converse of Prop. XIII. "Upon the opposite sides of it." If these words were omitted; it is possible for two lines to make with a third, two angles, which together are equal to two right angles, in such a manner that the two lines shall not be in the same straight line.

Prop. XV is the developement of the definition of an angle. If the lines at the angular point be produced, the produced lines have the same inclination to one another as the original lines, but in a different position.

Prop. XVI. From this Prop. it follows that only one perpendicular can be drawn from a given point to a given line; and this perpendicular may be shewn to be less than any other line which can be drawn from the given point to the given line.

Prop. XVII appears to be only a corollary to the preceding proposition, and it seems to be introduced to explain Axiom XII, of which it is the converse. The exact truth respecting the angles of a triangle is proved in Prop. XXXII.

Prop. XIX is the converse of Prop. XVIII. It may be remarked, that Prop. XIX bears the same relation to Prop. XVIII, as Prop. VI does to Prop. V.

Prop. XX—XXI. "Proclus, in his commentary, relates, that the Epicureans derided this proposition, as being manifest even to asses, and needing no demonstration; and his answer is, that though the truth of it be manifest to our senses, yet it is science which must give the reason why two sides of a triangle are greater than the third. But the right answer to this objection against this and Prop. XXI, and some other plain propositions, is, that the number of axioms ought not to be increased without necessity, as it must be if these propositions be not demonstrated. Mons. Clairault, in the preface to his Elements of Geometry, published in French at Paris, 1741, says, 'that Euclid has been at the pains to prove, that the two sides of a triangle which is included within another, are together less than the two sides of the triangle which includes it.' But he has forgot to add this condition, viz. that the triangles must be upon the same base: because, unless this be added, the sides of the included triangle may be greater than the sides of the triangle which includes it, in any ratio which is less than that of two to one, as Pappus Alexandrinus has demonstrated in Prop. 3, Book III of his Mathematical Collections." Simson.

Prop. XXII. When the sum of two of the lines is equal to, and when it is less than, the third line; let the diagrams be described, and they will exhibit the impossibility implied by the restriction laid down in the proposition.

Prop. XXIII. CD might be taken equal to CE and the construction effected by means of an isosceles triangle. It would, however, be less general than Euclid's.

Prop. XXIV. Simson makes the angle EDG at D in the line ED, the side which is not the greater of the two ED, DF; otherwise, three different cases would arise, as may be seen by forming the different figures. The point G might fall below or upon the base EF produced as well as above it. Prop. XXIV and Prop. XXV bear to each other the same relation as Prop. IV and Prop. VIII.

Prop. XXVI. This forms the third case of the equality of two triangles. Every triangle has three sides and three angles, and when any three of one triangle are given equal to any three of another, the triangles may be proved to be equal to one another,

whenever the three magnitudes given in the hypothesis are independent of one another. Prop. IV contains the first case, when the hypothesis consists of two sides and the included angle of each triangle. Prop. VIII contains the second, when the hypothesis consists of the three sides of each triangle. Prop. XXVI contains the third, when the hypothesis consists of two angles, and one side either adjacent to the equal angles, or opposite to one of the equal angles in each triangle. There is another case, not proved by Euclid, when the hypothesis consists of two sides and one angle in each triangle, but these not the angles included by the two given sides in each triangle. This case however is only true under a certain restriction.

Prop. XXVII. Alternate angles are defined to be the two angles which two straight lines make with another at its extremities, but upon opposite sides of it.

Prop. XXVIII. One angle is called "the exterior angle," and another "the interior and opposite angle," when they are formed on the same side of a straight line which falls upon or intersects two other straight lines. It is also obvious that on each side of the line, there will be two exterior and two interior and opposite angles. The exterior angle EGB has the angle GHD for its corresponding interior and opposite angle: also the exterior angle FHD has the angle HGB for its interior and opposite angle.

Prop. XXIX is the converse of Prop. XXVII and Prop. XXVIII.

As the definition of parallel straight lines simply describes them by a statement of the negative property, that they never meet; it is necessary that some positive property of parallel lines should be assumed as an axiom, on which reasonings on such lines may be founded.

Euclid has assumed the statement in the twelfth axiom, which has been objected to, as not being self-evident. A stronger objection appears to be, that the converse of it forms Prop. 17, Book I; for both the assumed axiom and its converse, should be so obvious as not to require formal demonstration.

Simson has attempted to overcome the objection, not by any improved definition and axiom respecting parallel lines; but, by considering Euclid's twelfth axiom to be a theorem, and for its proof, assuming two definitions and one axiom, and then demonstrating five subsidiary Propositions.

Instead of Euclid's twelfth axiom, the following has been proposed as a more simple property for the foundation of reasonings on parallel lines; namely, "If a straight line fall on two parallel straight lines, the alternate angles are equal to one another." In whatever this may exceed Euclid's definition in simplicity, it is liable to a similar objection, being the converse of Prop. 27, Book I.

Professor Playfair has adopted in his Elements of Geometry, that "Two straight lines which intersect one another cannot be both parallel to the same straight line." This apparently more simple axiom follows as a direct inference from Prop. 30, Book I.

But one of the least objectionable of all the definitions which have been proposed on this subject, appears to be that which simply expresses the conception of equidistance. It may be formally stated thus: "Parallel lines are such as lie in the same plane, and which neither recede from, nor approach to, each other." This includes the conception stated by Euclid, that parallel lines never meet. Dr Wallis observes on this subject, "Parallelismus et æquidistantia vel idem sunt, vel certe se mutuo comitantur."

As an additional reason for this definition being preferred, it may be remarked that the meaning of the terms γραμμαὶ παράλληλοι, suggests the exact idea of such lines.

Axiom XI and XII, in some manuscripts of Euclid, are found placed respectively as the fourth and the fifth postulate.

An account of thirty methods which have been proposed at different times for avoiding the difficulty in the twelfth axiom, will be found in the appendix to Mr Thompson's "Geometry without Axioms."

Prop. XXXII. The three angles of a triangle may be shewn to be equal to two right angles without producing a side of the triangle, by drawing through any angle of the triangle a line parallel to the opposite side, as Proclus has remarked in his Commentary on this proposition. It is manifest from this proposition, that the third angle of a triangle is not independent of the sum of the other two; but is known if the sum of any two is known. Cor. 1 may be also proved by drawing lines from any one of the angles of the figure to the other angles. If any of the sides of the figure bend inwards and form what are called re-entering angles, the enunciation of these two corollaries will require some modification.

From this proposition, it is obvious that each of the angles of an equilateral triangle, is equal to two thirds of a right angle, as it is shewn in Prop. 15, Book IV. Also, if one angle of an isosceles triangle be a right angle, then each of the equal angles is half a right angle, as in Prop. 9, Book II.

Prop. XXXIV. If the other diameter be drawn, it may be shewn that the diameters of a parallelogram bisect each other, as well as bisect the area of the parallelogram. The converse of this Prop. namely, "If the opposite sides or opposite angles of a quadrilateral figure be equal, the opposite sides shall also be parallel; that is, the figure shall be a parallelogram," is not proved by Euclid.

Prop. XXXV. The latter part of the demonstration is not expressed very intelligibly. Simson, who altered the demonstration, seems in fact to consider two trapeziums of the same form and magnitude, and from one of them, to take the triangle *ABE*; and from the other, the triangle *DCF*; and then the remainders are equal by the third axiom: that is, the parallelogram *ABCD* is equal to the parallelogram *EBCF*. Otherwise, the triangle, whose base is *DE*, (fig. 2.) is taken twice from the trapezium, which would appear to be impossible, if the sense in which Euclid applies the third axiom, is to be retained here.

It may be observed, that the two parallelograms exhibited in fig. 2 partially lie on one another, and that the triangle whose base is *BC* is a common part of them, but that the triangle whose base is *DE* is entirely without both the parallelograms. After having proved the triangle *ABE* equal to the triangle *DCF*, if we take from these equals the triangle whose base is *DE*, and to each of the remainders add the triangle whose base is *BC*; perhaps the proof may appear somewhat more satisfactory.

Prop. XXXVIII. In this proposition, it is to be understood that the bases of the two triangles are in the same straight line.

Prop. XXXIX. If the vertices of all the equal triangles which can be described upon the same base, or upon the equal bases as in Prop. 40, be joined, the line thus formed will be a straight line, and is called the locus of the vertices of equal triangles upon the same base, or upon equal bases.

A locus in plane Geometry is a straight line or a plane curve, every point of which and none else satisfies a certain condition.

With the exception of the straight line and the circle, the two most simple loci; all other loci, perhaps including also the Conic Sections, may be more readily and effectually investigated algebraically by means of their rectangular or polar equations.

Prop. XLI. The converse of this proposition is not proved by Euclid; viz. If a parallelogram is double of a triangle, and they have the same base, or equal bases upon the same straight line, and towards the same parts, they shall be between the same parallels. Also, it may easily be shewn that if two equal triangles are between the same parallels; they are either upon the same base, or upon equal bases.

Prop. XLIV. A parallelogram described on a straight line is said to be *applied* to that line.

Prop. XLVII. In a right-angled triangle, the side opposite to the right angle is

called the hypothenuse, and the other two sides, the base and perpendicular, according to their position.

It is not indifferent on which sides of the lines which form the sides of the triangle the squares are described. If they were described upon the inner, instead of the outer sides of the lines, the construction would be found to fail.

By this proposition may be found a square equal to the sum of any given squares, or equal to any multiple of a given square: or equal to the difference of two given squares.

The truth of this proposition may be exhibited to the eye in some particular instances. As in the case of that right-angled triangle whose three sides are 3, 4, and 5 units respectively. If through the points of division of two contiguous sides of each of the squares upon the sides, lines be drawn parallel to the sides (see the notes on Book II. p. 68), it will be obvious, that the squares will be divided into 9, 16 and 25 small squares, each of the same magnitude; and that the number of the small squares into which the squares on the perpendicular and base are divided is equal to the number into which the square on the hypothenuse is divided.

Prop. XLVIII is the converse of Prop. XLVII. In this Prop. is assumed the Corollary that "the squares described upon two equal lines are equal," and the converse, which properly ought to have been appended to Prop. XLVI.

The first book of Euclid's Elements, it has been seen, is conversant with the construction and properties of rectilineal figures. It first lays down the definitions which limit the subjects of discussion in the first book, next the three postulates, which restrict the instruments by which the constructions in plane geometry are effected; and thirdly, the twelve axioms, which express the principles by which a comparison is made between the ideas of the things defined.

This Book may be divided into three parts. The first part treats of the origin and properties of triangles, both with respect to their sides and angles; and the comparison of these mutually, both with regard to equality and inequality. The second part treats of the generation and properties of parallelograms. The third part exhibits the connexion of the properties of triangles and parallelograms, and the equality of the squares on the base and perpendicular of a right-angled triangle to the square on the hypothenuse.

When the propositions of the first book have been read, the student is recommended to use different letters in the diagrams, and where it is possible, diagrams of a form somewhat different from those exhibited in the text, for the purpose of testing the accuracy of his knowledge of the demonstrations. And further, when he is become sufficiently familiar with the method of geometrical reasoning, he may dispense with the aid of letters altogether, and acquire the power of expressing in general terms the process of reasoning in the demonstration of any proposition. Also, should he be not satisfied with the bare knowledge of the principles of the first book which have been exhibited *synthetically*, but be also desirous of knowing how these principles may be applied to the solution of Problems *analytically* and the demonstration of Theorems; he may refer to the Geometrical Exercises on the first book, which will be found at the end of the Elements, together with some brief account of the Ancient Geometrical Analysis.

BOOK II.

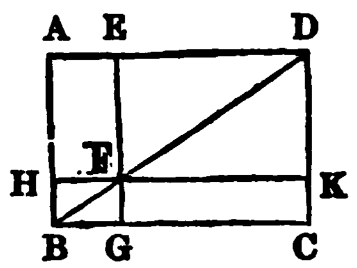
DEFINITIONS.

I.

Every right-angled parallelogram is called a *rectangle*, and is said to be contained by any two of the straight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a gnomon.



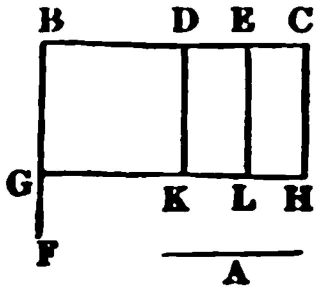
"Thus the parallelogram *HG* together with the complements *AF*, *FC*, is the gnomon, which is more briefly expressed by the letters *AGK*, or *EHC*, which are at the opposite angles of the parallelograms which make the gnomon."

PROPOSITION I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines;
and let BC be divided into any parts in the points D, E.

Then the rectangle contained by the straight lines A and BC, shall be equal to the rectangle contained by A and BD, together with that contained by A and DE, and that contained by A and EC.



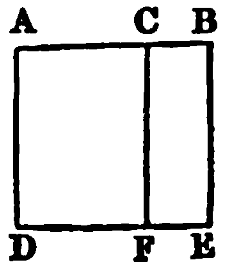
From the point B, draw BF at right angles to BC, (I. 11.)
and make BG equal to A; (I. 3.)
through G draw GH parallel to BC, (I. 31.)
and through D, E, C, draw DK, EL, CH parallel to BG.
Then the rectangle BH is equal to the rectangles BK, DL, EH.
But BH is contained by A and BC,
for it is contained by GB, BC, and GB is equal to A:
and the rectangle BK is contained by A, BD,
for it is contained by GB, BD, of which GB is equal to A:
also DL is contained by A, DE,
because DK, that is, BG, (I. 34.) is equal to A;
and in like manner the rectangle EH is contained by A, EC:
therefore the rectangle contained by A, BC, is equal to the several rectangles contained by A, BD, and by A, DE, and by A, EC.
Wherefore, if there be two straight lines, &c. Q.E.D.

PROPOSITION II. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.

Let the straight line AB be divided into any two parts in the point C.

Then the rectangle contained by AB, BC, together with that contained by AB, AC, shall be equal to the square of AB.



Upon AB describe the square $ADEB$, (I. 46.)
and through C draw CF parallel to AD or BE. (I. 31.)

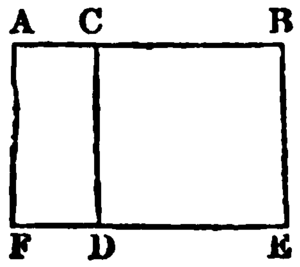
Then AE is equal to the rectangles AF, CE.
And AE is the square of AB;
and AF is the rectangle contained by BA, AC;
for it is contained by DA, AC, of which DA is equal to AB:
and CE is contained by AB, BC,
for BE is equal to AB:
therefore the rectangle contained by AB, AC, together with the rectangle AB, BC is equal to the square of AB.
If therefore a straight line, &c. Q. E. D.

PROPOSITION III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let the straight line AB be divided into any two parts in the point C.

Then the rectangle AB, BC, shall be equal to the rectangle AC, CB, together with the square of BC.



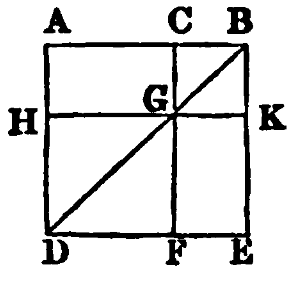
Upon BC describe the square $CDEB$, (I. 46.) and produce ED to F,
through A draw AF parallel to CD or BE. (I. 31.)
Then the rectangle AE is equal to the rectangles AD, CE;
but AE is the rectangle contained by AB, BC,
for it is contained by AB, BE, of which BE is equal to BC:
and AD is contained by AC, CB,
for CD is equal to CB:
and DB is the square of BC:
therefore the rectangle AB, BC, is equal to the rectangle AC, CB, together with the square of BC.
If therefore a straight line be divided, &c. Q. E. D.

PROPOSITION IV. THEOREM.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C.

Then the square of AB shall be equal to the squares of AC, and CB, together with twice the rectangle contained by AC, CB.



Upon AB describe the square $ADEB$, (I. 46.)
join BD, and through C draw CGF parallel to AD or BE, (I. 31.)
and through G draw HGK parallel to AB or DE.
Then, because CF is parallel to AD, and BD falls upon them,
therefore the exterior angle BGC is equal to the interior and opposite angle ADB; (I. 29.)
but the angle ADB is equal to the angle ABD, (I. 5.)
because BA is equal to AD, being sides of a square;
wherefore the angle CGB is equal to the angle CBG;
and therefore the side BC is equal to the side CG; (I. 6.)
but CB is equal also to GK, and CG to BK; (I. 34.)
wherefore the figure $CGKB$ is equilateral.
It is likewise rectangular,
for, since CG is parallel to BK, and CB meets them,
therefore the angles KBC, GCB are equal to two right angles; (I. 29.)
but the angle KBC is a right angle; (def. 30. constr.)
wherefore GCB is a right angle:
and therefore also the angles CGK, GKB, opposite to these, are right angles; (I. 34.)
wherefore $CGKB$ is rectangular:
it is also equilateral, as was demonstrated;
wherefore it is a square, and it is upon the side CB.
For the same reason HF is a square, and it is upon the side HG, which is equal to AC. (I. 34.)
Therefore the figures HF, CK, are the squares of AC, CB.
And because the complement AG is equal to the complement GE, (I. 43.)
and that AG is the rectangle contained by AC, CB,
for GC is equal to CB;
therefore GE is also equal to the rectangle AC, CB;
wherefore AG, GE are equal to twice the rectangle AC, CB;
and HF, CK are the squares of AC, CB;
wherefore the four figures HF, CK, AG, GE, are equal to the squares of AC, CB, and to twice the rectangle AC, CB:
but HF, CK, AG, GE make up the whole figure $ADEB$, which is the square of AB;
therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC, CB.
Wherefore, if a straight line be divided, &c. Q.E.D.

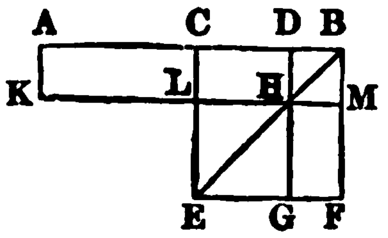
COR. From the demonstration, it is manifest, that the parallelograms about the diameter of a square are likewise squares.

PROPOSITION V. THEOREM.

If a straight line be divided into two equal parts, and also into two unequal parts; the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts in the point D.

Then the rectangle AD, DB, together with the square of CD, shall be equal to the square of CB.



Upon CB describe the square $CEFB$, (I. 46.)
join BE, and through D draw DHG parallel to CE or BF; (I. 31.)
and through H draw KLM parallel to CB or EF;
also through A draw AK parallel to CL or BM.
Then, because the complement CH is equal to the complement HF, (I. 43.)
to each of these equals add DM;
therefore the whole CM is equal to the whole DF;
but because AC is equal to CB,
therefore CM is equal to AL, (I. 36.)
therefore also AL is equal to DF:
to each of these equals add CH,
and therefore the whole AH is equal to DF and CH:
but AH is the rectangle contained by AD, DB, for DH is equal to DB;
and DF together with CH is the gnomon CMG;
therefore the gnomon CMG is equal to the rectangle AD, DB:
to each of these equals add LG, which is equal to the square of CD; (II. 4. Cor.)
therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the square of CD:
but the gnomon CMG and LG make up the whole figure $CEFB$, which is the square of CB;
therefore the rectangle AD, DB, together with the square of CD, is equal to the square of CB.

Wherefore, if a straight line, &c. Q. E. D.

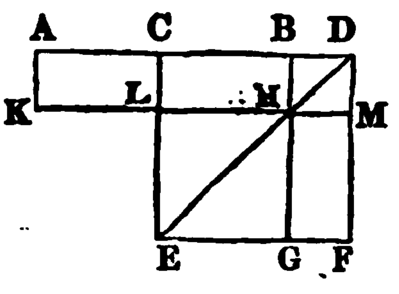
COR. From this proposition it is manifest, that the difference of the squares of two unequal lines AC, CD, is equal to the rectangle contained by their sum AD and their difference DB.

PROPOSITION VI. THEOREM.

If a straight line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D.

Then the rectangle AD, DB, together with the square of CB, shall be equal to the square of CD.



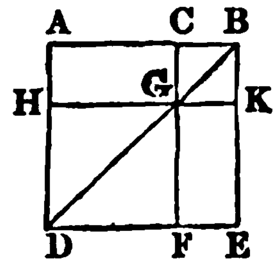
Upon CD describe the square $CEFD$, (I. 46.) and join DE,
through B draw BHG parallel to CE or DF, (I. 31.)
through H draw KLM parallel to AD or EF,
and through A draw AK parallel to CL or DM.
Then because AC is equal to CB,
therefore the rectangle AL is equal to the rectangle CH, (I. 36.)
but CH is equal to HF; (I. 43.)
therefore AL is equal to HF;
to each of these equals add CM;
therefore the whole AM is equal to the gnomon CMG:
but AM is the rectangle contained by AD, DB,
for DM is equal to DB: (II. 4. Cor.)
therefore the gnomon CMG is equal to the rectangle AD, DB:
add to each of these equals LG which is equal to the square of CB;
therefore the rectangle AD, DB, together with the square of CB, is equal to the gnomon CMG, and the figure LG;
but the gnomon CMG and LG make up the whole figure $CEFD$, which is the square of CD;
therefore the rectangle AD, DB, together with the square of CB, is equal to the square of CD.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION VII. THEOREM.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts in the point C.

Then the squares of AB, BC shall be equal to twice the rectangle AB, BC, together with the square of AC.



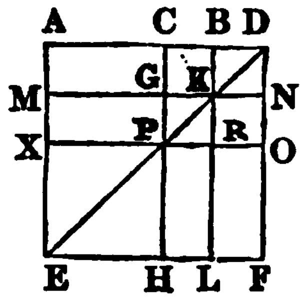
Upon AB describe the square $ADEB$, (I. 46.) and join BD;
through C draw CF parallel to AD or BE cutting BD in G, (I. 31.)
through G draw HGK parallel to AB or DE.
Then because AG is equal to GE, (I. 43.)
add to each of them CK;
therefore the whole AK is equal to the whole CE;
and therefore AK, CE, are double of AK;
but AK, CE are the gnomon AKF and the square CK;
therefore the gnomon AKF and the square CK is double of AK:
but twice the rectangle AB, BC, is double of AK,
for BK is equal to BC; (II. 4. Cor.)
therefore the gnomon AKF and the square CK, is equal to twice the rectangle AB, BC;

to each of these equals add *HF*, which is equal to the square of *AC*,
therefore the gnomon *AKF*, and the squares *CK*, *HF*, are equal to twice the rectangle *AB*, *BC*, and the square of *AC*;
but the gnomon *AKF*, together with the squares *CK*, *HF*, make up the whole figure *ADEB* and *CK*, which are the squares of *AB* and *BC*;
therefore the squares of *AB* and *BC* are equal to twice the rectangle *AB*, *BC*, together with the square of *AC*.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line, which is made up of the whole and that part.

Let the straight line *AB* be divided into any two parts in the point *C*.
Then four times the rectangle *AB*, *BC*, together with the square of *AC*, shall be equal to the square of the straight line made up of *AB* and *BC* together.



Produce *AB* to *D*, so that *BD* be equal to *CB*, (I. 3.)
upon *AD* describe the square *AEFD*, (I. 46.) and join *BE*,
through *B*, *C*, draw *BL*, *CH* parallel to *AE* or *DF*, and cutting *DE* in the points *K*, *P* respectively;
through *K*, *P*, draw *MGKN*, *XPRO* parallel to *AD* or *EF*.
Then because *CB* is equal to *BD*, *CB* to *GK*, and *BD* to *KN*;
therefore *GK* is equal to *KN*;
for the same reason, *PR* is equal to *RO*;
and because *CB* is equal to *BD*, and *GK* to *KN*,
therefore the rectangle *CK* is equal to *BN*, and *GR* to *RN*; (I. 36.)
but *CK* is equal to *RN*, (I. 43.)
because they are the complements of the parallelogram *CO*;
therefore also *BN* is equal to *GR*;
and therefore the four rectangles *BN*, *CK*, *GR*, *RN*, are equal to one another, and so are quadruple of one of them *CK*.
Again, because *CB* is equal to *BD*, and *BD* to *BK*, that is, to *CG*;
and because *CB* is equal to *GK*, that is, to *GP*;
therefore *CG* is equal to *GP*.
And because *CG* is equal to *GP*, and *PR* to *RO*,
therefore the rectangle *AG* is equal to *MP*, and *PL* to *RF*;
but the rectangle *MP* is equal to *PL*, (I. 43.)
because they are the complements of the parallelogram *ML*;
wherefore also *AG* is equal to *RF*;

therefore the four rectangles AG, MP, PL, RF, are equal to one another, and so are quadruple of one of them AG.

And it was demonstrated that the four CK, BN, GR, and RN are quadruple of CK;

therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK.

And because AK is the rectangle contained by AB, BC, for BK is equal to BC;

therefore four times the rectangle AB, BC is quadruple of AK;

but the gnomon AOH was demonstrated to be quadruple of AK;

therefore four times the rectangle AB, BC is equal to the gnomon AOH;

to each of these equals add XH, which is equal to the square of AC;

therefore four times the rectangle AB, BC, together with the square of AC, is equal to the gnomon AOH and the square XH;

but the gnomon AOH and XH make up the figure $AEFD$ which is the square of AD;

therefore four times the rectangle AB, BC together with the square of AC, is equal to the square of AD, that is, of AB and BC added together is one straight line.

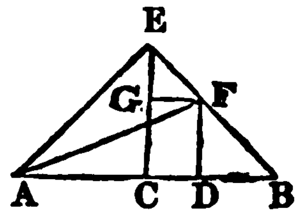
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION IX. THEOREM.

If a straight line be divided into two equal, and also into two unequal parts; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts in the point D.

Then the squares of AD, DB together, shall be double of the squares of AC, CD.



From the point C draw CE at right angles to AB, (I. 11.)

make CE equal to AC or CB, (I. 3.) and join EA, EB;

through D draw DF parallel to CE, meeting EB in F, (I. 31.)

through F draw FG parallel to BA, and join AF.

Then, because AC is equal to CE,

therefore the angle EAC is equal to the angle AEC; (I. 5.)

and because ACE is a right angle,

therefore the two other angles AEC, EAC of the triangle are together equal to a right angle; (I. 32.)

and since they are equal to one another;

therefore each of them is half of a right angle.

For the same reason, each of the angles CEB, EBC is half a right angle;

and therefore the whole AEB is a right angle.

And because the angle GEF is half a right angle,

and EGF a right angle, for it is equal to the interior and opposite angle ECB, (I. 29.)

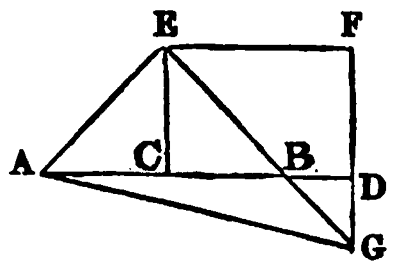
therefore the remaining angle EFG is half a right angle;
wherefore the angle GEF is equal to the angle EFG,
and the side EG equal to the side GF. (I. 6.)
Again, because the angle at B is half a right angle,
and FDB a right angle, for it is equal to the interior and opposite angle ECB, (I. 29.)
therefore the remaining angle BFD is half a right angle;
wherefore the angle at B is equal to the angle BFD,
and the side DF equal to the side DB. (I. 6.)
And because AC is equal to CE,
the square of AC is equal to the square of CE;
therefore the squares of AC, CE are double of the square of AC;
but the square of AE is equal to the squares of AC, CE, (I. 47.)
because ACE is a right angle;
therefore the square of AE is double of the square of AC.
Again, because EG is equal to GF,
the square of EG is equal to the square of GF;
therefore the squares of EG, GF are double of the square of GF;
but the square of EF is equal to the squares of EG, GF; (I. 47.)
therefore the square of EF is double of the square of GF;
and GF is equal to CD; (I. 34.)
therefore the square of EF is double of the square of CD;
but the square of AE is double of the square of AC;
therefore the squares of AE, EF are double of the squares of AC, CD;
but the square of AF is equal to the squares of AE, EF,
because AEF is a right angle: (I. 47.)
therefore the square of AF is double of the squares of AC, CD:
but the squares of AD, DF are equal to the square of AF;
because the angle ADF is a right angle; (I. 47.)
therefore the squares of AD, DF are double of the squares of AC, CD;
and DF is equal to DB;
therefore the squares of AD, DB are double of the squares of AC, CD.
If therefore a straight line be divided, &c. Q. E. D.

PROPOSITION X. THEOREM.

If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D.

Then the squares of AD, DB, shall be double of the squares of AC, CD.



From the point C draw CE at right angles to AB, (I. 11.)
make CE equal to AC or CB, (I. 3.) and join AE, EB;

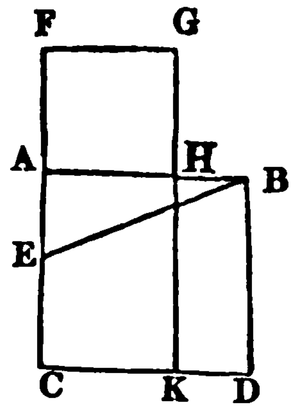
through E draw EF parallel to AB, (I. 31.)
and through D draw DF parallel to CE.
Then because the straight line EF meets the parallels CE, FD, therefore the angles CEF, EFD are equal to two right angles; (I. 29.)
and therefore the angles BEF, EFD are less than two right angles.
But straight lines, which with another straight line make the interior angles upon the same side less than two right angles, will meet if produced far enough; (ax. 12.)
therefore EB, FD will meet, if produced towards B, D;
let them meet in G, and join AG.
Then, because AC is equal to CE,
therefore the angle CEA is equal to the angle EAC; (I. 5.)
and the angle ACE is a right angle;
therefore each of the angles CEA, EAC is half a right angle. (I. 32.)
For the same reason, each of the angles CEB, EBC is half a right angle;
therefore the whole AEB is a right angle.
And because EBC is half a right angle,
therefore DBG is also half a right angle, (I. 15.)
for they are vertically opposite;
but BDG is a right angle,
because it is equal to the alternate angle DCE; (I. 29.)
therefore the remaining angle DGB is half a right angle;
and is therefore equal to the angle DBG;
wherefore also the side BD is equal to the side DG. (I. 6.)
Again, because EGF is half a right angle, and the angle at F is a right angle, being equal to the opposite angle ECD, (I. 34.)
therefore the remaining angle FEG is half a right angle,
and therefore equal to the angle EGF;
wherefore also the side GF is equal to the side FE. (I. 6.)
And because EC is equal to CA;
the square of EC is equal to the square of CA;
therefore the squares of EC, CA are double of the square of CA;
but the square of EA is equal to the squares of EC, CA; (I. 47.)
therefore the square of EA is double of the square of AC.
Again, because GF is equal to EF,
the square of GF is equal to the square of EF;
therefore the squares of GF, FE are double of the square of EF;
but the square of EG is equal to the squares of GF, EF; (I. 47.)
therefore the square of EG is double of the square of EF;
and EF is equal to CD; (I. 34.)
wherefore the square of EG is double of the square of CD;
but it was demonstrated,
that the square of EA is double of the square of AC;
therefore the squares of EA, EG are double of the squares of AC, CD;
but the square of AG is equal to the squares of EA, EG; (I. 47.)
therefore the square of AG is double of the squares of AC, CD:
but the squares of AD, DG are equal to the square of AG;
therefore the squares of AD, DG are double of the squares of AC, CD;
but DG is equal to DB;
therefore the squares of AD, DB are double of the squares of AC, CD.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XI. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square of the other part.

Let AB be the given straight line.

It is required to divide AB into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square of the other part.



Upon AB describe the square $ACDB$; (I. 46.)
bisect AC in E, (I. 10.) and join BE,
produce CA to F, and make EF equal to EB, (I. 3.)
upon AF describe the square $FGHA$. (I. 46.)

Then AB shall be divided in H, so that the rectangle AB, BH is equal to the square of AH.

Produce GH to meet CD in K.

Then because the straight line AC is bisected in E, and produced to F,
therefore the rectangle CF, FA, together with the square of AE, is equal to the square of EF; (II. 6.)
but EF is equal to EB;
therefore the rectangle CF, FA together with the square of AE, is equal to the square of EB;
but the squares of BA, AE are equal to the square of EB, (I. 47.)
because the angle EAB is a right angle;
therefore the rectangle CF, FA, together with the square of AE, is equal to the squares of BA, AE;
take away the square of AE, which is common to both;
therefore the rectangle contained by CF, FA is equal to the square of BA.
But the figure FK is the rectangle contained by CF, FA,
for FA is equal to FG;
and AD is the square of AB;
therefore the figure FK is equal to AD;
take away the common part AK,
therefore the remainder FH is equal to the remainder HD;
but HD is the rectangle contained by AB, BH,
for AB is equal to BD;
and FH is the square of AH;
therefore the rectangle AB, BH, is equal to the square of AH.

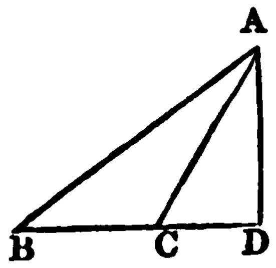
Wherefore the straight line AB is divided in H, so that the rectangle AB, BH is equal to the square of AH. Q.E.F.

PROPOSITION XII. THEOREM.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle, is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Let *ABC* be an obtuse-angled triangle, having the obtuse angle *ACB*, and from the point *A* let *AD* be drawn perpendicular to *BC* produced.

Then the square of *AB* shall be greater than the squares of *AC, CB*, by twice the rectangle *BC, CD*.



Because the straight line *BD* is divided into two parts in the point *C*,
therefore the square of *BD* is equal to the squares of *BC, CD*, and twice the rectangle *BC, CD*; (II. 4.)
to each of these equals add the square of *DA*;
therefore the squares of *BD, DA* are equal to the squares of *BC, CD, DA*, and twice the rectangle *BC, CD*;
but the square of *BA* is equal to the squares of *BD, DA*, (I. 47.)
because the angle at *D* is a right angle;
and the square of *CA* is equal to the squares of *CD, DA*;
therefore the square of *BA* is equal to the squares of *BC, CA*, and twice the rectangle *BC, CD*;
that is, the square of *BA* is greater than the squares of *BC, CA*, by twice the rectangle *BC, CD*.
Therefore in obtuse-angled triangles, &c. Q.E.D.

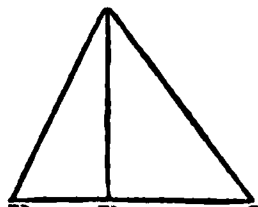
PROPOSITION XIII. THEOREM.

In every triangle, the square of the side subtending either of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular let fall upon it from the opposite angle.

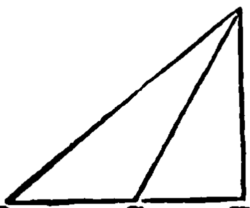
Let *ABC* be any triangle, and the angle at *B* one of its acute angles, and upon *BC*, one of the sides containing it, let fall the perpendicular *AD* from the opposite angle. (I. 12.)

Then the square of *AC* opposite to the angle *B*, shall be less than the squares of *CB, BA*, by twice the rectangle *CB, DB*.

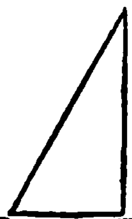
First, let *AD* fall within the triangle *ABC*.



Then, because the straight line CB is divided into two parts in D,
the squares of CB, BD are equal to twice the rectangle contained by CB, BD, and the square of DC; (II. 7.)
to each of these equals add the square of AD;
therefore the squares of CB, BD, DA, are equal to twice the rectangle CB, BD, and the squares of AD, DC;
but the square of AB is equal to the squares of BD, DA, (I. 47.)
because the angle BDA is a right angle;
and the square of AC is equal to the squares of AD, DC;
therefore the squares of CB, BA are equal to the square of AC, and twice the rectangle CB, BD;
that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB, BD.
Secondly, let AD fall without the triangle ABC.



Then, because the angle at D is a right angle,
the angle ACB is greater than a right angle; (I. 16.)
and therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle BC, CD; (II. 12.)
to each of these equals add the square of BC;
therefore the squares of AB, BC are equal to the square of AC, twice the square of BC, and twice the rectangle BC, CD;
but because BD is divided into two parts in C,
therefore the rectangle DB, BC is equal to the rectangle BC, CD, and the square of BC; (II. 3.)
and the doubles of these are equal;
that is, twice the rectangle DB, BC is equal to twice the rectangle BC, CD and twice the square of BC:
wherefore the squares of AB, BC are equal to the square of AC, and twice the rectangle DB, BC:
therefore the square of AC alone is less than the squares of AB, BC, by twice the rectangle DB, BC.
Lastly, let the side AC be perpendicular to BC.

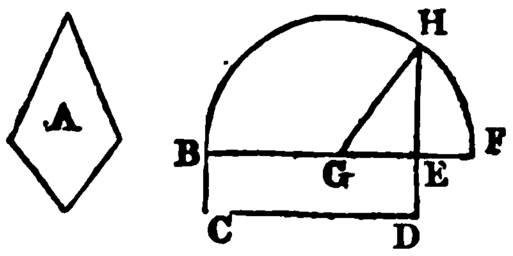


Then BC is the straight line between the perpendicular and the acute angle at B;
and it is manifest, that the squares of AB, BC, are equal to the square of AC, and twice the square of BC. (I. 47.
Therefore in any triangle, &c. Q. E. D.

PROPOSITION XIV. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure.
It is required to describe a square that shall be equal to A.



Describe the rectangular parallelogram $BCDE$ equal to the rectilineal figure A. (I. 45.)

Then, if the sides of it, BE, ED, are equal to one another, it is a square, and what was required is now done.

But if they are not equal,
produce one of them BE to F, and make EF equal to ED,
bisect BF in G; (I. 10.)
from the centre G, at the distance GB, or GF, describe the semicircle BHF,
and produce DE to meet the circumference in H.

The square described upon EH shall be equal to the given rectilineal figure A.

Join GH.

Then because the straight line BF is divided into two equal parts in the point G, and into two unequal parts in the point E;
therefore the rectangle BE, EF, together with the square of EG, are equal to the square of GF; (II. 5.)
but GF is equal to GH; (def. 15.)
therefore the rectangle BE, EF, together with the square of EG, is equal to the square of GH;
but the squares of HE, EG are equal to the square of GH; (I. 47.)
therefore the rectangle BE, EF, together with the square of EG, are equal to the squares of HE, EG;
take away the square of EG, which is common to both;
therefore the rectangle BE, EF is equal to the square of HE.

But the rectangle contained by BE, EF is the parallelogram BD, because EF is equal to ED;
therefore BD is equal to the square of EH;
but BD is equal to the rectilineal figure A; (constr.)
therefore the square of EH is equal to the rectilineal figure A.

Wherefore a square has been made equal to the given rectilineal figure A, namely, the square described upon EH. Q.E.F.

NOTES TO BOOK II.

In Book I, Geometrical magnitudes of the same kind, lines, angles and surfaces, more particularly triangles and parallelograms, are compared, either as being absolutely equal or unequal to, one another.

In Book II, the properties of right-angled parallelograms, but without reference to their magnitudes are demonstrated, besides an important extension is made of Prop. 47, Book I, to acute-angled and obtuse-angled triangles. Euclid has given no definition of *a rectangular parallelogram*, or *rectangle:* probably, because the Greek expression παραλληλόγραμμον ὀρθογώνιον, or ὀρθογώνιον simply, is a definition of the figure. In English, the term *rectangle* has no signification, and therefore ought to be defined before its properties are demonstrated. A rectangle is defined to be a parallelogram having one angle a right angle; and a square is a rectangle having all its sides equal.

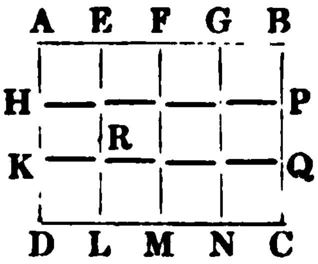
In Prop. 35, Book I, it may be seen that there may be an indefinite number of parallelograms on the same base and between the same parallels whose areas are always equal to one another; but that one of them has all its angles right angles, and the length of its boundary less than the boundary of any other parallelogram upon the same base and between the same parallels. The area of this rectangular parallelogram is therefore determined by the two lines which contain one of its right angles. Hence it is stated in Def. 1, that every right-angled parallelogram *is said to be contained* by any two of the straight lines which contain one of the right angles. No distinction is made in Book II. between *equality* and *identity*, as the rectangle may be said to be contained by two lines which are equal respectively to the two which contain one right angle of the figure. It may be remarked that the rectangle itself *is bounded* by four straight lines.

It is of primary importance to discriminate the Geometrical conception of a rectangle from the Arithmetical or Algebraical representation of it. The subject of Geometry is *magnitude* not *number*, and therefore it would be a departure from strict reasoning on space, to substitute in Geometrical demonstrations, the Arithmetical or Algebraical representation of a rectangle for the rectangle itself. It is, however, absolutely necessary that the connection of *number* and *magnitude* be clearly understood, as far as regards the representation of lines and areas.

All lines are measured by lines, and all surfaces by surfaces. Some one line of definite length is arbitrarily assumed as the linear unit, and the length of every other line is represented by the number of linear units contained in it. The square is the figure assumed for the measure of surfaces. The square unit or the unit of area is assumed to be that square, the side of which is one linear unit in length, and the magnitude of every surface is represented by the number of square units contained in it. But here it may be remarked, that Euclid has not informed us that the two lines by which every rectangle is contained, are capable of being represented by any multiples of the same linear unit. If, however, in the present case, the sides of rectangles are supposed to be divisible into an exact number of linear units, a numerical representation for the area of the rectangle may be deduced.

On two lines at right angles to each other, take AB equal to 4, and AD equal to 3 linear units.

Complete the rectangle $ABCD$, and through the points of division of AB, AD, draw EL, FM, GN parallel to AD; and HP, KQ parallel to AB respectively.



Then the whole rectangle AC is divided into squares, all equal to each other.

And AC is equal to the sum of the rectangles AL, EM, FN, GC; (II. 1.)

also these rectangles are equal to one another, (I. 36.)

therefore the whole AC is equal to four times one of them AL.

Again, the rectangle AL is equal to the rectangles EH, HR, RD, and these rectangles, by construction, are squares described upon the equal lines AH, HK, KD, and are equal to one another.

therefore the rectangle AL is equal to 3 times the square AH,

but the whole rectangle AC is equal to 4 times the rectangle AL,

therefore the rectangle AC is 4×3 times the square AH, or 12 square units:

that is, the product of the two numbers which express the number of linear units in the two sides, will give the number of square units in the rectangle, and therefore will be an arithmetical representation of its area.

And generally, if AB, AD, instead of 4 and 3, consisted of a and b linear units respectively, it may be shewn in a similar manner, that the area of the rectangle AC would contain ab square units; and therefore the product ab is a proper representation for the area of the rectangle AC.

Hence, it follows, that the term *rectangle* in Geometry corresponds to the term *product* in Arithmetic and Algebra, and that a similar comparison may be made between the products of the two numbers which represent the sides of rectangles, as between the areas of the rectangles themselves. This forms the basis of what are called Arithmetical or Algebraical proofs of Geometrical properties.

If the two sides of the rectangle be equal, or if b be equal to a, the figure is a square, and the area is represented by aa or a^2.

Also, since a triangle is equal to the half of a parallelogram of the same base and altitude.

Therefore the area of a triangle will be represented by half the rectangle which has the same base and altitude as the triangle: in other words, if the length of the base be a units, and the altitude be b units.

Then the area of the triangle is algebraically represented by $\frac{1}{2}ab$, or $\dfrac{ab}{2}$.

The leading idea which runs through the demonstrations of the first eight propositions, is the obvious axiom, that, "the whole area of every figure in each case, is equal to all the parts of it taken together."

Prop. I. For the sake of brevity of expression, "the rectangle contained by the straight lines AB, BC," is called "the rectangle AB, BC;" and sometimes "the rectangle ABC."

The method of reasoning on the properties of rectangles by means of the products which indicate the number of square units contained in their areas is foreign to Euclid's ideas of rectangles, as discussed in his second book, which have no reference to any particular unit of length or measure of surface.

Prop. I. Algebraically. (fig. Prop. I.)

Let the line BC contain a linear units, and A, b linear units of the same length.

Also suppose the parts BD, DE, EC to contain m, n, p linear units respectively.

Then $a = m + n + p$,

multiply these equals by b,

$$\text{therefore} \quad ab = bm + bn + bp.$$

That is, the product of two numbers, one of which is divided into any number of parts, is equal to the sum of the products of the undivided number, and the several parts of the other;

or, if the Geometrical interpretation of the products be restored.

The number of square units expressed by the product ab, is equal to the number of square units expressed by the sum of the products bm, bn, bp.

Prop. II. Algebraically. (fig. Prop. II.)

Let AB contain a linear units, and AC, CB, m and n linear units respectively.

$$\text{Then} \quad m + n = a,$$

multiplying these equals by a,

$$\text{therefore} \quad am + an = a^2.$$

That is, if a number be divided into any two parts, the sum of the products of the whole and each of the parts is equal to the square of the whole number.

Prop. III. Algebraically. (fig. Prop. III.)

Let AB contain a linear units, and let BC contain m, and AC, n linear units.

$$\text{Then} \quad a = m + n,$$

and multiplying these equals by m,

$$\text{therefore} \quad ma = m^2 + mn.$$

That is, if a number be divided into any two parts, the product of the whole number and one of the parts, is equal to the square of that part, and the product of the two parts.

Prop. IV. might have been deduced from the two preceding propositions; but Euclid has rather preferred the method of exhibiting, in all the demonstrations of the second book, the equality of the spaces compared.

In the corollary to Prop. XLVI. Book I, it is stated that a parallelogram which has one right angle, has all its angles right angles. By applying this corollary, the demonstration of Prop. IV. may be considerably shortened.

If the two parts be equal, then the square of the whole line is equal to four times the square of half the line.

Also, if a line be divided into any three parts, the square of the whole line is equal to the squares of the three parts, and twice the rectangle contained by every two parts.

Prop. IV. Algebraically. (fig. Prop. IV.)

Let the line AB contain a linear units, and the parts of it AC and BC, m and n linear units respectively.

$$\text{Then} \quad a = m + n,$$

$$\text{squaring these equals,} \quad \therefore a^2 = (m+n)^2,$$

$$\text{or} \quad a^2 = m^2 + 2mn + n^2.$$

That is, if a number be divided into any two parts, the square of the number is equal to the squares of the two parts together with twice the product of the two parts.

Prop. V. It must be kept in mind, that the sum of two straight lines in Geometry, means the straight line formed by joining the two lines together, so that both may be in the same straight line.

The following simple properties respecting the equal and unequal division of a line are worthy of being remembered.

I. Since $AB = 2BC = 2(BD + DC) = 2BD + 2DC$. (fig. Prop. V.)

$$\text{and} \quad AB = AD + DB;$$

$$\therefore 2CD + 2DB = AD + DB,$$

and by subtracting $2DB$ from these equals,

$$\therefore\ 2CD = AD - DB,$$

$$\text{and}\quad CD = \tfrac{1}{2}(AD - DB).$$

That is, if a line AB is divided into two equal parts in C, and into two unequal parts in D, the part CD of the line between the points of section is equal to half the difference of the unequal parts AD and DB.

II. Here $AD = AC + CD$, the sum of the unequal parts, (fig. Prop. v.)

and $DB = AC - CD$ their difference.

Hence by adding these equals together,

$$\therefore\ AD + DB = 2AC,$$

or the sum and difference of two lines AC, CD, are together equal to twice the greater line.

And the halves of these equals are equal,

$$\therefore\ \tfrac{1}{2}.AD + \tfrac{1}{2}.DB = AC,$$

or, half the sum of two unequal lines AC, CD added to half their difference is equal to the greater line AC.

III. Again, since $AD = AC + CD$, and $DB = AC - CD$,

by subtracting these equals,

$$\therefore\ AD - DB = 2CD,$$

or, the difference between the sum and difference of two unequal lines is equal to twice the less line.

And the halves of these equals are equal,

$$\therefore\ \tfrac{1}{2}.AD - \tfrac{1}{2}.DB = CD,$$

or, half the difference of two lines subtracted from half their sum is equal to the less of the two lines.

IV. Since $AC - CD = DB$ the difference,

$$\therefore\ AC = CD + DB,$$

and adding CD the less to each of these equals,

$$\therefore\ AC + CD = 2CD + DB,$$

or, the sum of two unequal lines is equal to twice the greater line together with the difference between the lines.

Prop. v. Algebraically.

Let AB contain $2a$ linear units,

its half BC will contain a linear units.

And let CD the line between the points of section contain m linear units.

Then AD the greater of the two unequal parts, contains $a + m$ linear units;

and DB the less contains $a - m$ units.

Also m is half the difference of $a + m$ and $a - m$;

$$\therefore\ (a + m)(a - m) = a^2 - m^2,$$

to each of these equals add m^2;

$$\therefore\ (a + m)(a - m) + m^2 = a^2.$$

That is, if a number be divided into two equal parts, and also into two unequal parts, the product of the unequal parts and the square of half their difference, is equal to the square of half the number.

Bearing in mind that AC, CD are respectively half the sum and half the difference of the two lines AD, DB; the corollary to this proposition may be expressed in the following form: "The rectangle contained by two straight lines is equal to the difference of the squares of half their sum and half their difference."

The rectangle contained by AD and DB, and the square of BC are each bounded by the same extent of line, but the spaces enclosed differ by the square of CD.

Prop. VI. Algebraically.

Let AB contain $2a$ linear units, then its half BC contains a units; and let BD contain m units,

$$\text{Then } AD \text{ contains } 2a + m \text{ units,}$$
$$\text{and } \therefore\ (2a + m)m = 2am + m^2;$$
$$\text{to each of these equals add } a^2,$$
$$\therefore\ (2a + m)\,m + a^2 = a^2 + 2am + m^2.$$
$$\text{But } a^2 + 2am + n^2 = (a + m)^2,$$
$$\therefore\ (2a + m)\,m + a^2 = (a + m)^2.$$

That is, If a number be divided into two equal numbers, and another number be added to the whole and to one of the parts; the product of the whole number thus increased and the other number together with the square of half the given number, is equal to the square of the number which is made up of half the given number increased.

The Algebraical results of Prop. V and Prop. VI are identical, as it is obvious that the difference of $a + m$ and $a - m$ in Prop. V is equal to the difference of $2a + m$ and m in Prop. VI, and one algebraical result expresses the truth of both propositions.

This arises from the two ways in which the difference between two unequal lines may be represented geometrically, when they are in the same direction.

In the diagram (fig. to Prop. V.), the difference DB of the two unequal lines AC and CD is exhibited by producing the less line CD, and making CB equal to AC the greater.

Then the part produced DB is the difference between AC and CD,
for AC is equal to CB, and taking CD from each,
the difference of AC and CD is equal to the difference of CB and CD.

In the diagram (fig. to Prop. VI), the difference DB of the two unequal lines CD and CA is exhibited by cutting off from CD the greater, a part CB equal to CA the less.

Prop. VII. Algebraically.

Let AB contain a linear units, and let the parts AC and CB contain m and n linear units respectively.

$$\text{Then } a = m + n;$$
$$\therefore \text{ squaring these equals,}$$
$$\therefore\ a^2 = m^2 + 2mn + n^2,$$
$$\text{add } n^2 \text{ to each of these equals,}$$
$$\therefore\ a^2 + n^2 = m^2 + 2mn + 2n^2.$$
$$\text{But } 2mn + 2n^2 = 2(m + n)\,n = 2an,$$
$$\therefore\ a^2 + n^2 = 2an + m^2.$$

That is, If a number be divided into any two parts, the square of the whole number and of one of the parts is equal to twice the product of the whole number and that part, together with the square of the other part.

Prop. VIII. Algebraically.

Let the whole line AB contain a linear units of which the parts AC, CB contain m, n units respectively.

$$\text{Then } m + n = a,$$
$$\text{and subtracting or taking } n \text{ from each,}$$
$$\therefore\ m = a - n,$$
$$\text{squaring these equals,}$$
$$\therefore\ m^2 = a^2 - 2an + n^2,$$
$$\text{and adding } 4an \text{ to each of these equals,}$$
$$\therefore\ 4an + m^2 = a^2 + 2an + n^2.$$

$$\text{But } a^2 + 2an + n^2 = (a+n)^2,$$
$$\therefore\ 4an + m^2 = (a+n)^2.$$

That is, If a number be divided into any two parts, four times the product of the whole number and one of the parts, together with the square of the other part, is equal to the square of the number made up of the whole and that part.

Prop. IX. Algebraically.

Let AB contain $2a$ linear units, its half AC or BC will contain a units; and let CD the line between the points of section contain m units.

Also AD the greater of the two unequal parts contains $a+m$ units,
and DB the less contains $a-m$ units.

$$\text{Then } (a+m)^2 = a^2 + 2am + m^2,$$
$$\text{and } (a-m)^2 = a^2 + 2am + m^2,$$
Hence by adding these equals,
$$\therefore\ (a+m)^2 + (a-m)^2 = 2a^2 + 2m^2.$$

That is, If a number be divided into two equal parts, and also into two unequal parts, the sum of the squares of the two unequal parts is equal to twice the square of half the number and twice the square of half the difference of the unequal parts.

Prop. X. Algebraically.

Let the line AB contain $2a$ linear units, of which its half AC or CB will contain a units;

and let BD contain m units.

Then the whole line and the part produced will contain $2a+m$ units,
and half the line and the part produced will contain $a+m$ units,

$$\therefore\ (2a+m)^2 = 4a^2 + 4am + m^2,$$
add m^2 to each of these equals,
$$\therefore\ (2a+m)^2 + m^2 = 4a^2 + 4am + 2m^2.$$
$$\text{Again } (a+m)^2 = a^2 + 2am + m^2,$$
add a^2 to each of these equals,
$$\therefore\ (a+m)^2 + a^2 = 2a^2 + 2am + m^2,$$
and doubling these equals,
$$\therefore\ 2(a+m)^2 + 2a^2 = 4a^2 + 4am + m^2.$$
$$\text{But } (2a+m)^2 + m^2 = 4a^2 + 4am + m^2.$$
$$\text{Hence } \therefore\ (2a+m)^2 + m^2 = 2a^2 + 2(a+m)^2.$$

That is, If a number be divided into two equal parts, and the whole number and one of the parts be increased by the addition of another number, the squares of the whole number thus increased, and of the number by which it is increased, are equal to double the squares of half the number, and of half the number increased.

The algebraical results of Prop. IX, and Prop. X, are identical, the enunciations of the two Props. arising, as in Prop. V and Prop. VI, from the two ways of exhibiting the difference between two lines.

To solve Prop. XI algebraically, or to find the point H in AB such that the rectangle contained by the whole line AB and the part HB shall be equal to the square of the other part AH.

Let AB contain a linear units, and AH one of the unknown parts contain x units,

then the other part HB contains $a-x$ units.

And $\therefore\ a(a-x) = x^2$, by the problem,

or $x^2 + ax = a^2$, a quadratic equation.

$$\text{Hence } x = \frac{\pm a\sqrt{5} - a}{2}.$$

The former of these values of x determines the point H.

$$\text{So that} \quad x = \frac{\sqrt{5}-1}{2} \,.\, AB = AH, \text{ one part,}$$

$$\text{and} \quad a - x = a - AH = \frac{3-\sqrt{5}}{2} \,.\, AB = HB, \text{ the other part.}$$

It may be observed, that the parts AH and HB cannot be numerically expressed by any rational number. Approximation to their true values in terms of AB, may be made to any required degree of accuracy, by extending the extraction of the square root of 5 to any number of decimals.

To ascertain the meaning of the other result $x = -\frac{\sqrt{5}+1}{2} \,.\, a$.

$$\text{In the equation} \quad a(a - x) = x^2,$$

$$\text{for } x \text{ write } -x, \text{ then } a(a + x) = x^2,$$

which when translated into words gives the following problem.

To find the length a given line must be produced so that the rectangle contained by the given line and the line made up of the given line and the part produced, may be equal to the square of the part produced.

Or, the problem may also be expressed as follows:

To find two lines having a given difference, such that the rectangle contained by the difference and one of them may be equal to the square of the other.

Prop. XII. Algebraically.

Assuming the truth of Prop. 47, Book I, Algebraically.

Let BC, CA, AB contain a, b, c linear units respectively,

and let CD, DA, contain m, n units,

then BD contains $a + m$ units.

And therefore, $c^2 = (a + m)^2 + n^2$, from the right-angled triangle ABD,

also $b^2 = m^2 + n^2$ from ACD;

$$\begin{aligned} \therefore \; c^2 - b^2 &= (a + m)^2 - m^2 \\ &= a^2 + 2am + m^2 - m^2 \\ &= a^2 + 2am, \end{aligned}$$

$$\therefore \; c^2 = b^2 + a^2 + 2am,$$

that is, c^2 is greater than $b^2 + a^2$ by $2am$.

Prop. XIII. Case II may be proved more simply as follows, in the same manner as Case I.

Since BD is divided into two parts in the point D,

therefore the squares of CB, BD are equal to twice the rectangle contained by CB, BD and the square of CD; (II. 7.)

add the square of AD to each of these equals;

therefore the squares of CB, BD, DA are equal to twice the rectangle CB, BD, and the squares of CD and DA,

but the squares of BD, DA are equal to the square of AB, (I. 47.)

and the squares of CD, DA are equal to the squares of AC,

therefore the squares of CB, BA are equal to the square of AC, and twice the rectangle CB, BD.

That is, &c.

Prop. XIII. Algebraically.

Let BC, CA, AB contain respectively a, b, c linear units, and let BD and AD also contain m and n units.

Case I. Then DC contains $a - m$ units.

Therefore $c^2 = n^2 + m^2$ from the right-angled triangle ABD,
and $b^2 = n^2 + (a - m)^2$ from ADC;

$$\therefore\ c^2 - b^2 = m^2 - (a - m)^2$$
$$= m^2 - a^2 + 2am - m^2$$
$$= -a^2 + 2am,$$
$$\therefore\ a^2 + c^2 = b^2 + 2am,$$
or $b^2 + 2am = a^2 + c^2$,

that is, b^2 is less than $a^2 + c^2$ by $2am$.

Case II. $DC = m - a$ units,

$\therefore\ c^2 = m^2 + n^2$ from the right-angled triangle ABD,
and $b^2 = (m - a)^2 + n^2$ from ACD,

$$\therefore\ c^2 - b^2 = m^2 - (m - a)^2$$
$$= m^2 - m^2 + 2am - a^2$$
$$= 2am - a^2,$$
$$\therefore\ a^2 + c^2 = b^2 + 2am,$$
or $b^2 + 2am = a^2 + c^2$,

that is, b^2 is less than $a^2 + c^2$ by $2am$.

Case III. Here m is equal to a.

And $b^2 + a^2 = c^2$, from the right-angled triangle ABC.

Add to each of these equals a^2,

$$\therefore\ b^2 + 2a^2 = c^2 + a^2,$$

that is, b^2 is less than $c^2 + a^2$ by $2a^2$, or $2aa$.

BOOK III.

DEFINITIONS.

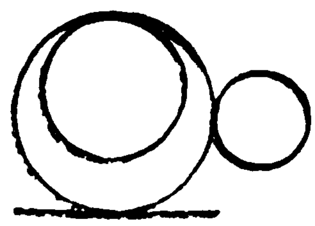
I.

Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.

II.

A straight line is said to touch a circle, when it meets the circle, and being produced does not cut it.

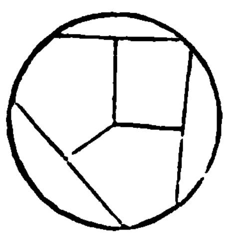


III.

Circles are said to touch one another, which meet, but do not cut one another.

IV.

Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



V.

And the straight line on which the greater perpendicular falls, is said to be farther from the centre.

VI.

A segment of a circle is the figure contained by a straight line, and the circumference which it cuts off.

VII.

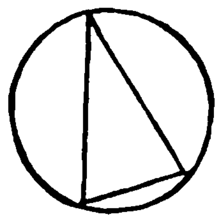
The angle of a segment is that which is contained by the straight line and the circumference.

VIII.

An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the straight line which is the base of the segment.

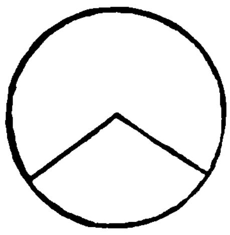
IX.

An angle is said to insist or stand upon the circumference intercepted between the straight lines that contain the angle.



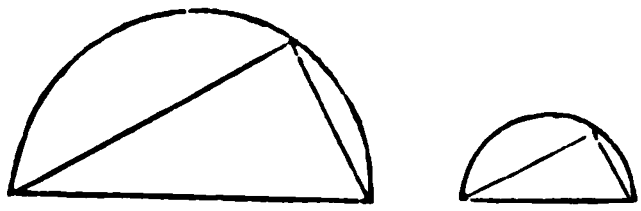
X.

A sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



XI.

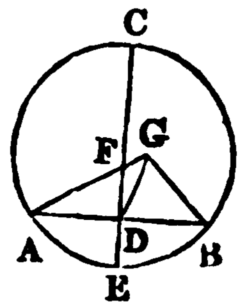
Similar segments of circles are those in which the angles are equal, or which contain equal angles.



PROPOSITION I. PROBLEM.

To find the centre of a given circle.

Let ABC be the given circle.
It is required to find its centre.



Draw within it any straight line AB, and bisect AB in D; (I. 10.)
from the point D draw DC at right angles to AB, (I. 11.)
produce CD to E, and bisect CE in F.
Then the point F shall be the centre of the circle ABC.
For, if it be not, let, if possible, G be the centre, and join GA, GD, GB.

Then, because DA is equal to DB, (constr.) and DG common to the two triangles ADG, BDG, the two sides AD, DG, are equal to the two BD, DG, each to each;
and the base GA is equal to the base GB, (I. def. 15.)
because they are drawn from the centre G:
therefore the angle ADG is equal to the angle GDB: (I. 8.)
but when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of the angles is a right angle; (I. def. 10.)
therefore the angle GDB is a right angle:
but FDB is likewise a right angle; (constr.)
wherefore the angle FDB is equal to the angle GDB, (ax. 1.)
the greater equal to the less, which is impossible;
therefore G is not the centre of the circle ABC.

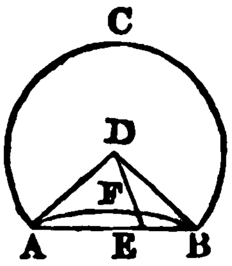
In the same manner it can be shewn that no other point out of the line CE is the centre;
and since CE is bisected in F,
any other point in CE divides CE into unequal parts, and cannot be the centre.
Therefore no point but F is the centre of the circle ABC.
Which was to be found.

Cor. From this it is manifest, that if in a circle a straight line bisects another at right angles, the centre of the circle is in the line which bisects the other.

PROPOSITION II. THEOREM.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A, B any two points in the circumference.
The straight line drawn from A to B shall fall within the circle.



For if AB do not fall within the circle,
let it fall, if possible, without, as AEB;
find D the centre of the circle ABC, (III. 1.) and join DA, DB;
in the circumference AB take any point F, join DF, and produce it to meet AB in E.
Then, because DA is equal to DB, (I. def. 15.)
therefore the angle DAB is equal to the angle DBA; (I. 5.)
and because AE, a side of the triangle DAE, is produced to B,
the exterior angle DEB is greater than the interior and opposite angle DAE; (I. 16.)
but DAE was proved equal to the angle DBE;
therefore the angle DEB is greater than the angle DBE;
but to the greater angle the greater side is opposite, (I. 19.)
therefore DB is greater than DE:
but DB is equal to DF; (I. def. 15.)
wherefore DF is greater than DE, the less than the greater, which is impossible;

therefore the straight line drawn from A to B does not fall without the circle.

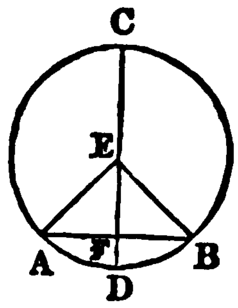
In the same manner, it may be demonstrated that it does not fall upon the circumference;
therefore it falls within it.
Wherefore, if any two points, &c. Q.E.D.

PROPOSITION III. THEOREM.

If a straight line drawn through the centre of a circle bisect a straight line in which it does not pass through the centre, it shall cut it at right angles: and conversely, if it cuts it at right angles, it shall bisect it.

Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, in the point F.

Then CD shall cut AB at right angles.



Take E the centre of the circle, (III. 1.) and join EA, EB.

Then, because AF is equal to FB, (hyp.) and FE common to the two triangles AFE, BFE,
there are two sides in the one equal to two sides in the other, each to each;
and the base EA is equal to the base EB; (I. def. 15.)

therefore the angle AFE is equal to the angle BFE: (I. 8.)

but when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of them is a right angle; (I. def. 10.)

therefore each of the angles AFE, BFE, is a right angle:

wherefore the straight line CD, drawn through the centre, bisecting another AB that does not pass through the centre, cuts the same at right angles.

But let CD cut AB at right angles.

Then CD shall also bisect AB, that is, AF shall be equal to FB.

The same construction being made.

Because EA, EB, from the centre are equal to one another, (I. def. 15.)

the angle EAF is equal to the angle EBF; (I. 5.)

and the right angle AFE is equal to the right angle BFE: (I. def. 10.)

therefore, in the two triangles, EAF, EBF,

there are two angles in the one equal to two angles in the other, each to each;

and the side EF, which is opposite to one of the equal angles in each, is common to both;

therefore the other sides are equal; (I. 26.)

therefore AF is equal to FB.

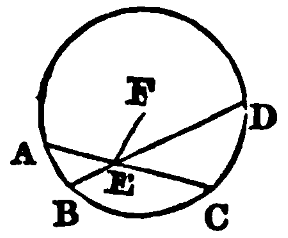
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION IV. THEOREM.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Let $ABCD$ be a circle, and AC, BD two straight lines in it which cut one another in the point E, and do not both pass through the centre.

Then AC, BD, shall not bisect one another.



For, if it is possible, let AE be equal to EC, and BE to ED.

If one of the lines pass through the centre,

it is plain that it cannot be bisected by the other which does not pass through the centre:

but if neither of them pass through the centre,

take F the centre of the circle, (III. 1.) and join EF.

Then because FE, a straight line drawn through the centre, bisects another AC which does not pass through the centre, (hyp.)

therefore FE cuts AC at right angles: (III. 3.)

wherefore FEA is a right angle.

Again, because the straight line FE bisects the straight line BD, which does not pass through the centre, (hyp.)

therefore FE cuts BD at right angles: (III. 3.)

wherefore FEB is a right angle:

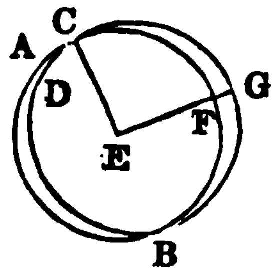
but FEA was shewn to be a right angle;

therefore the angle FEA is equal to the angle FEB, (ax. 1.)
the less equal to the greater, which is impossible:
therefore AC, BD do not bisect one another..
Wherefore, if in a circle, &c. Q.E.D.

PROPOSITION V. THEOREM.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG, cut one another in the points B, C.
They shall not have the same centre.

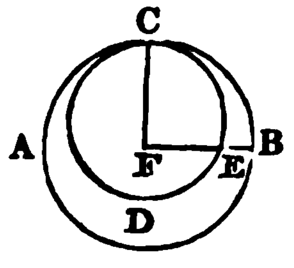


For, if it be possible, let E be their centre:
join EC, and draw any straight line EFG meeting them in F and G.
And because E is the centre of the circle ABC,
therefore EC is equal to EF: (I. def. 15.)
again, because E is the centre of the circle CDG,
therefore EC is equal to EG: (I. def. 15.)
but EC was shewn to be equal to EF;
therefore EF is equal to EG, (ax. 1.)
the equal less to the greater, which is impossible.
Therefore E is not the centre of the circles ABC, CDG.
Wherefore, if two circles, &c. Q.E.D.

PROPOSITION VI. THEOREM.

If one circle touch another internally, they shall not have the same centre.

Let the circle CDE touch the circle ABC internally in the point C.
They shall not have the same centre.



For, if they have, let it be F:
join FC, and draw any straight line FEB, meeting them in E and B.
And because F is the centre of the circle ABC,
FC is equal to FB; (I. def. 15.)
also, because F is the centre of the circle CDE,
FC is equal to FE: (I. def. 15.)
but FC was shewn to be equal to FB;

therefore *FE* is equal to *FB*, (ax. 1.)
the less equal to the greater, which is impossible:
therefore *F* is not the centre of the circles *ABC*, *CDE*.
Therefore, if two circles, &c. Q. E. D.

PROPOSITION VII. THEOREM.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote: and from the same point there can be drawn only two equal straight lines to the circumference, one upon each side of the diameter.

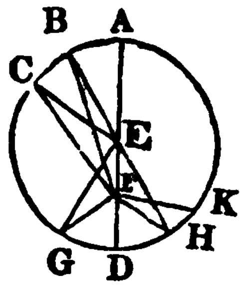
Let *ABCD* be a circle, and *AD* its diameter, in which let any point *F* be taken which is not the centre:

let the centre be *E*.

Then, of all the straight lines *FB*, *FC*, *FG*, &c. that can be drawn from *F* to the circumference,

FA, that in which the centre is, shall be the greatest, and *FD*, the other part of the diameter *AD*, shall be the least:

and of the others, *FB*, the nearer to *FA*, shall be greater than *FC*, the more remote, and *FC* greater than *FG*.



Join *BE*, *CE*, *GE*.
And because two sides of a triangle are greater than the third, (I. 20.)
therefore *BE*, *EF* are greater than *BF*:
but *AE* is equal to *BE*; (I. def. 15.)
therefore *AE*, *EF*, that is, *AF* is greater than *BF*.
Again, because *BE* is equal to *CE*,
and *FE* common to the triangles *BEF*, *CEF*,
the two sides *BE*, *EF* are equal to the two *CE*, *EF*, each to each;
but the angle *BEF* is greater than the angle *CEF*; (ax. 9.)
therefore the base *BF* is greater than the base *CF*. (I. 24.)
For the same reason *CF* is greater than *GF*.
Again, because *GF*, *FE* are greater than *EG*, (I. 20.)
and *EG* is equal to *ED*;
therefore *GF*, *FE* are greater than *ED*:
take away the common part *FE*,
and the remainder *GF* is greater than the remainder *FD*. (ax. 5.)

Therefore, *FA* is the greatest, and *FD* the least of all the straight lines from *F* to the circumference;

and *BF* is greater than *CF*, and *CF* than *GF*.

Also, there can be drawn only two equal straight lines from the point *F* to the circumference, one upon each side of the diameter.

At the point E, in the straight line EF, make the angle FEH equal to the angle FEG, (I. 23.) and join FH.

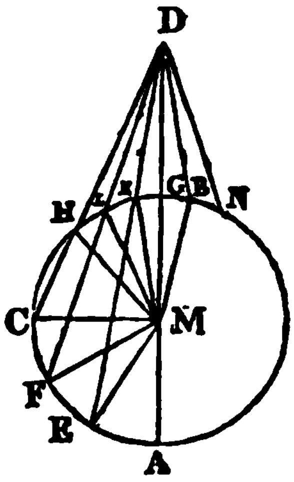
Then, because GE is equal to EH, (I. def. 15.)
and EF common to the two triangles GEF, HEF;
the two sides GE, EF are equal to the two HE, EF, each to each;
and the angle GEF is equal to the angle HEF; (constr.)
therefore, the base FG is equal to the base FH: (I. 4.)
but, besides FH, no other straight line can be drawn from F to the circumference equal to FG:
for, if there can, let it be FK:
and, because FK is equal to FG, and FG to FH,
therefore FK is equal to FH; (ax. 1.)
that is, a line nearer to that which passes through the centre, is equal to one which is more remote;
which has been proved to be impossible.
Therefore, if any point be taken, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote: but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote: and only two equal straight lines can be drawn from the same point to the circumference, one upon each side of the line which passes through the centre.

Let ABC be a circle, and D any point without it, from which let the straight lines DA, DE, DF, DC be drawn to the circumference, whereof DA passes through the centre.

Of those which fall upon the concave part of the circumference $AEFC$,
the greatest shall be DA, which passes through the centre;
and any line nearer to it shall be greater than one more remote,
viz. DE, shall be greater than DF, and DF greater than DC:
but of those which fall upon the convex circumference $HLKG$,
the least shall be DG between the point D and the diameter AG;
and any line nearer to it shall be less than one more remote,
viz. DK less than DL, and DL less than DH.



Take M the centre of the circle ABC, (III. 1.)
and join ME, MF, MC, MK, ML, MH.
And because AM is equal to ME,
add MD to each of these equals,
therefore AD is equal to EM, MD: (ax. 2.)
but EM, MD are greater than ED; (I. 20.)
therefore also AD is greater than ED.

Again, because ME is equal to MF, and MD common to the triangles EMD, FMD;
EM, MD, are equal to FM, MD, each to each:
but the angle EMD is greater than the angle FMD; (ax. 9.)
therefore the base ED is greater than the base FD. (I. 24.)
In like manner it may be shewn that FD is greater than CD.
Therefore, DA is the greatest;
and DE greater than DF, and DF greater than DC.
And, because MK, KD are greater than MD, (I. 20.)
and MK is equal to MG, (I. def. 15.)
the remainder KD is greater than the remainder GD, (ax. 5.)
that is, GD is less than KD:

and because MLD is a triangle, and from the points M, D, the extremities of its side MD, the straight lines MK, DK are drawn to the point K within the triangle,
therefore MK, KD are less than ML, LD: (I. 21.)
but MK is equal to ML; (I. def. 15.)
therefore, the remainder DK is less than the remainder DL. (ax. 5.)
In like manner it may be shewn, that DL is less than DH.
Therefore, DG is the least, and DK less than DL, and DL less than DH.

Also, there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the line through the centre.

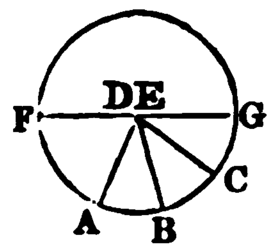
At the point M, in the straight line MD,
make the angle DMB equal to the angle DMK, (I. 23.) and join DB.

And because MK is equal to MB, and MD common to the triangles KMD, BMD,
the two sides KM, MD are equal to the two BM, MD, each to each;
and the angle KMD is equal to the angle BMD; (constr.)
therefore the base DK is equal to the base DB: (I. 4.)
but, besides DB, there can be no straight line drawn from D to the circumference equal to DK:
for, if there can, let it be DN:
and because DK is equal to DN, and also to DB,
therefore DB is equal to DN;
that is, a line nearer to the least is equal to one more remote, which has been proved to be impossible.
If, therefore, any point, &c. Q.E.D.

PROPOSITION IX. THEOREM.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC,
from which to the circumference there fall more than two equal straight lines, viz. DA, DB, DC.
Then the point D shall be the centre of the circle.

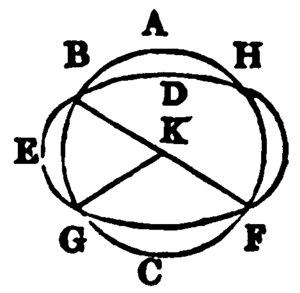


For, if not, let E be the centre:
join DE, and produce it to the circumference in F, G;
then FG is a diameter of the circle ABC: (I. def. 17.)
and because in FG, the diameter of the circle ABC, there is taken the point D, which is not the centre,
therefore DG is the greatest line from it to the circumference,
and DC is greater than DB, and DB greater than DA: (III. 7.)
but they are likewise equal, (hyp.) which is impossible:
therefore E is not the centre of the circle ABC.
In like manner it may be demonstrated,
that no other point but D is the centre;
D therefore is the centre.
Wherefore, if a point be taken, &c. Q.E.D.

PROPOSITION X. THEOREM.

One circumference of a circle cannot cut another in more than two points.

If it be possible, let the circumference FAB cut the circumference DEF in more than two points, viz. in B, G, F.



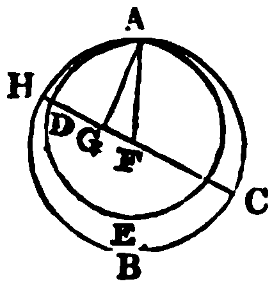
Take the centre K of the circle ABC, (III. 3.) and join KB, KG, KF.
Then because K is the centre of the circle ABC,
therefore KB, KG, KF are all equal to each other: (I. def. 15.)
and because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF,
therefore the point K is the centre of the circle DEF: (III. 9.)
but K is also the centre of the circle ABC; (constr.)
therefore the same point is the centre of two circles that cut one another, which is impossible. (III. 5.)
Therefore, one circumference of a circle cannot cut another in more than two points. Q.E.D.

PROPOSITION XI. THEOREM.

If one circle touch another internally in any point, the straight line which joins their centres being produced shall pass through that point of contact.

Let the circle ADE touch the circle ABC internally in the point A;

and let F be the centre of the circle ABC, and G the centre of the circle ADE;
then the straight line which joins the centres F, G, being produced, shall pass through the point A.

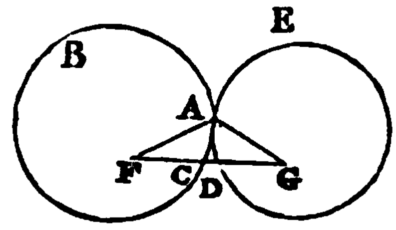


For, if FG produced do not pass through the point A,
let it fall otherwise, if possible, as $FGDH$, and join AF, AG.
Then, because two sides of a triangle are together greater than the third side, (I. 20.)
therefore FG, GA are greater than FA:
but FA is equal to FH; (I. def. 15.)
therefore FG, GA are greater than FH:
take away from these unequals the common part FG;
therefore the remainder AG is greater than the remainder GH; (ax. 5.)
but AG is equal to GD; (I. def. 15.)
therefore GD is greater than GH,
the less than the greater, which is impossible.
Therefore the straight line which joins the points F, G, being produced, cannot fall otherwise than upon the point A,
that is, it must pass through it.
Therefore, if one circle, &c. Q.E.D.

PROPOSITION XII. THEOREM.

If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point of contact.

Let the two circles ABC, ADE, touch each other externally in the point A;
and let F be the centre of the circle ABC, and G the centre of ADE.
Then the straight line which joins the points F, G, shall pass through the point of contact A.



For, if not, let it pass otherwise, if possible, as $FCDG$, and join FA, AG.
And because F is the centre of the circle ABC,
FA is equal to FC:
also, because G is the centre of the circle ADE,
GA is equal to GD:
therefore FA, AG are equal to FC, DG; (ax. 2.)
wherefore the whole FG is greater than FA, AG:
but FG is less than FA, AG; (I. 20.) which is impossible:

therefore the straight line which joins the points F, G, cannot pass otherwise than through the point of contact A,

that is, FG must pass through the point A.

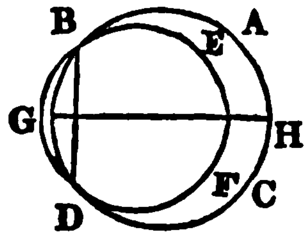
Therefore, if two circles, &c. Q.E.D.

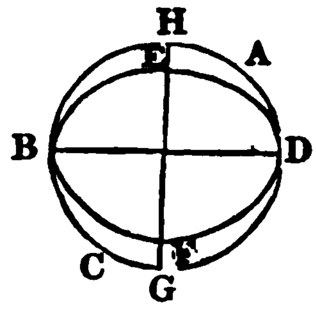
PROPOSITION XIII. THEOREM.

One circle cannot touch another in more points than one, whether it touches it on the inside or outside.

For, if it be possible, let the circle EBF touch the circle ABC in more points than one,

and first on the inside, in the points B, D.





Join BD, and draw GH bisecting BD at right angles. (I. 11.)

Because the points B, D are in the circumferences of each of the circles, therefore the straight line BD falls within each of them; (III. 2.)

therefore their centres are in the straight line GH which bisects BD at right angles; (III. Cor. 1.)

therefore GH passes through the point of contact: (III. 11.)

but it does not pass through it,

because the points B, D are without the straight line GH;

which is absurd:

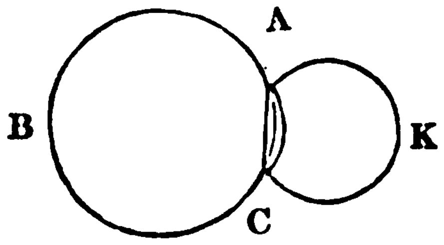
therefore one circle cannot touch another on the inside in more points than one.

Nor can two circles touch one another on the outside in more than one point.

For, if it be possible,

let the circle ACK touch the circle ABC in the points A, C;

join AC.



Because the two points A, C are in the circumference of the circle ACK,

therefore the straight line AC which joins them, falls within the circle ACK: (III. 2.)

but the circle ACK is without the circle ABC; (hyp.)

therefore the straight line AC is without this last circle:

but, because the points A, C are in the circumference of the circle ABC,

the straight line AC must be within the same circle, (III. 2.)

which is absurd;

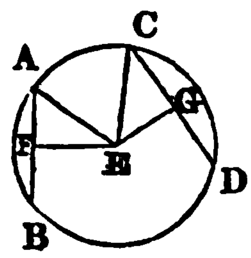
therefore one circle cannot touch another on the outside in more than one point:
and it has been shewn, that they cannot touch on the inside in more points than one.
Therefore, one circle, &c. Q.E.D.

PROPOSITION XIV. THEOREM.

Equal straight lines in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre, are equal to one another.

Let the straight lines *AB, CD,* in the circle *ABDC,* be equal to one another.

Then *AB* and *CD* shall be equally distant from the centre.



Take *E* the centre of the circle *ABDC,* (III. 1.)
from *E* draw *EF, EG* perpendiculars to *AB, CD,* (I. 12.) and join *EA, EC.*
Then, because the straight line *EF,* passing through the centre, cuts the straight line *AB,* which does not pass through the centre, at right angles,
it also bisects it: (III. 3.)
therefore *AF* is equal to *FB,* and *AB* double of *AF.*
For the same reason *CD* is double of *CG*:
but *AB* is equal to *CD*: (hyp.)
therefore *AF* is equal to *CG.* (ax. 7.)
And because *AE* is equal to *EC,* (I. def. 15.)
the square of *AE* is equal to the square of *EC*:
but the squares of *AF, FE* are equal to the square of *AE,* (I. 47.)
because the angle *AFE* is a right angle;
and, for the like reason, the squares of *EG, GC* are equal to the square of *EC*;
therefore the squares of *AF, FE* are equal to the squares of *CG, GE*: (ax. 1.)
but the square of *AF* is equal to the square of *CG,*
because *AF* is equal to *CG*;
therefore the remaining square of *EF* is equal to the remaining square of *EG,* (ax. 3.)
and the straight line *EF* is therefore equal to *EG*:
but straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal: (III. def. 4.)
therefore *AB, CD* are equally distant from the centre.
Next, let the straight lines *AB, CD* be equally distant from the centre, (III. def. 4.)
that is, let *FE* be equal to *EG*;

then AB shall be equal to CD.

For, the same construction being made, it may, as before, be demonstrated,

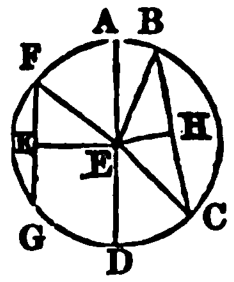
that AB is double of AF, and CD double of CG,
and that the squares of EF, FA are equal to the squares of EG, GC:
but the square of FE is equal to the square of EG,
because FE is equal to EG; (hyp.)
therefore the remaining square of AF is equal to the remaining square of CG: (ax. 3.)
and the straight line AF is therefore equal to CG:
but AB was shewn to be double of AF, and CD double of CG;
wherefore AB is equal to CD. (ax. 6.)
Therefore equal straight lines, &c. Q.E.D.

PROPOSITION XV. THEOREM.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote: and the greater is nearer to the centre than the less.

Let $ABCD$ be a circle, of which the diameter is AD, and the centre E;
and let BC be nearer to the centre than FG.

Then AD shall be greater than any straight line BC, which is not a diameter, and BC shall be greater than FG.



From the centre E draw EH, EK perpendiculars to BC, FG, (I. 12.)
and join EB, EC, EF.
And because AE is equal to EB, and ED to EC, (I. def. 15.)
therefore AD is equal to EB, EC: (ax. 2.)
but EB, EC are greater than BC; (I. 20.)
wherefore also AD is greater than BC.
And, because BC is nearer to the centre than FG, (hyp.)
therefore EH is less than EK: (III. def. 5.)
but, as was demonstrated in the preceding proposition,
BC is double of BH, and FG double of FK,
and the squares of EH, HB are equal to the squares of EK, KF:
but the square of EH is less than the square of EK,
because EH is less than EK;
therefore the square of BH is greater than the square of FK,
and the straight line BH greater than FK,
and therefore BC is greater than FG.

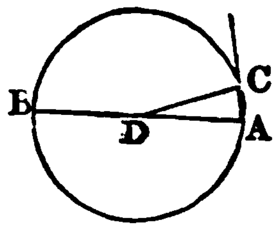
Next, let BC be greater than FG;
then BC shall be nearer to the centre than FG, that is, the same construction being made, EH shall be less than EK. (III. def. 5.)
Because BC is greater than FG,
BH likewise is greater than KF:
and the squares of BH, HE are equal to the squares of FK, KE;

of which the square of BH is greater than the square of FK,
because BH is greater than FK:
therefore the square of EH is less than the square of EK,
and the straight line EH less than EK:
and therefore BC is nearer to the centre than FG. (III. def. 5.)
Wherefore the diameter, &c. Q.E.D.

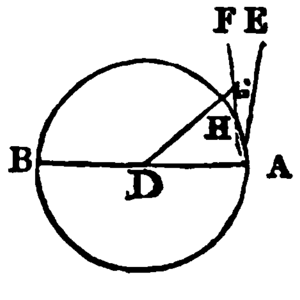
PROPOSITION XVI. THEOREM.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity between that straight line and the circumference, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.

Let ABC be a circle, the centre of which is D, and the diameter AB.
Then the straight line drawn at right angles to AB from its extremity A, shall fall without the circle.



For, if it does not, let it fall, if possible, within the circle, as AC;
and draw DC to the point C, where it meets the circumference.
And because DA is equal to DC, (I. def. 15.)
the angle DAC is equal to the angle ACD: (I. 5.)
but DAC is a right angle; (hyp.)
therefore ACD is a right angle;
and therefore the angles DAC, ACD are equal to two right angles;
which is impossible: (I. 17.)
therefore the straight line drawn from A at right angles to BA, does not fall within the circle.
In the same manner it may be demonstrated,
that it does not fall upon the circumference;
therefore it must fall without the circle, as AE.
Also, between the straight line AE and the circumference, no straight line can be drawn from the point A which does not cut the circle.
For, if possible, let AF be between them.



From the point D draw DG perpendicular to AF, (I. 12.)
and let it meet the circumference in H.
And because AGD is a right angle, and DAG less than a right angle, (I. 17.)

therefore DA is greater than DG: (I. 19.)
but DA is equal to DH; (I. def. 15.)
therefore DH is greater than DG,
the less than the greater, which is impossible:
therefore no straight line can be drawn from the point A, between AE and the circumference, which does not cut the circle:
or, which amounts to the same thing, however great an acute angle a straight line makes with the diameter at the point A, or however small an angle it makes with AE, the circumference must pass between that straight line and the perpendicular AE.

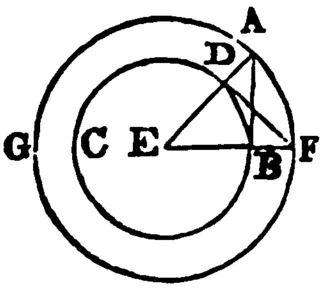
"And this is all that is to be understood, when, in the Greek text, and translations from it, the angle of the semicircle is said to be greater than any acute rectilineal angle, and the remaining angle less than any rectilineal angle." Q.E.D.

COR. From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; (III. def. 2.) and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it. (III. 2.) "Also, it is evident, that there can be but one straight line which touches the circle in the same point."

PROPOSITION XVII. PROBLEM.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, let A be a given point without the given circle BCD;
it is required to draw a straight line from A which shall touch the circle.



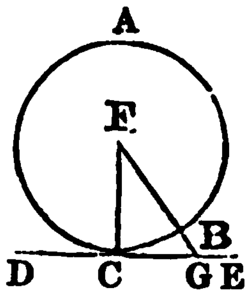
Find the centre E of the circle, (III. 1.) and join AE;
and from the centre E, at the distance EA, describe the circle AFG;
from the point D draw DF at right angles to EA, (I. 11.) and join EBF, AB.
Then AB shall touch the circle BCD in the point B.
Because E is the centre of the circles BCD, AFG,
therefore EA is equal to EF, (I. def. 15.) and ED to EB;
therefore the two sides AE, EB, are equal to the two FE, ED, each to each;
and they contain the angle at E common to the two triangles AEB, FED;
therefore the base DF is equal to the base AB, (I. 4.)
and the triangle FED to the triangle AEB,
and the other angles to the other angles:
therefore the angle EBA is equal to the angle EDF:
but EDF is a right angle, (constr.)
wherefore EBA is a right angle: (ax. 1.)
and EB is drawn from the centre:

but a straight line drawn from the extremity of a diameter, at right angles to it, touches the circle: (III. 16. Cor.)
therefore AB touches the circle;
and it is drawn from the given point A.
But if the given point be in the circumference of the circle, as the point D,
draw DE to the centre E, and DF at right angles to DE:
DF touches the circle. (III. 16. Cor.) Q.E.F.

PROPOSITION XVIII. THEOREM.

If a straight line touches a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC in the point C;
take the centre F, and draw the straight line FC. (III. 1.)
Then FC shall be perpendicular to DE.

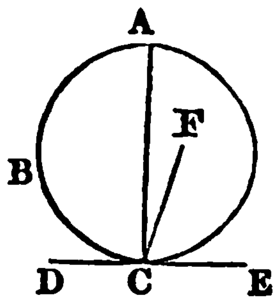


For, if it be not, from the point F draw FBG perpendicular to DE.
And because FGC is a right angle,
therefore GCF is an acute angle; (I. 17.)
and to the greater angle the greater side is opposite: (I. 19.)
therefore FC is greater than FG:
but FC is equal to FB; (I. def. 15.)
therefore FB is greater than FG,
the less than the greater, which is impossible:
therefore FG is not perpendicular to DE.
In the same manner it may be shewn,
that no other is perpendicular to DE besides FC,
that is, FC is perpendicular to DE.
Therefore, if a straight line, &c. Q.E.D.

PROPOSITION XIX. THEOREM.

If a straight line touches a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC in C,
and from C let CA be drawn at right angles to DE.
Then the centre of the circle shall be in CA.



For, if not, let F be the centre, if possible, and join CF.

Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact,

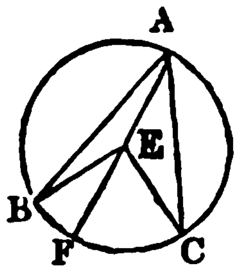
therefore FC is perpendicular to DE; (III. 18.)
therefore FCE is a right angle:
but ACE is also a right angle; (hyp.)
therefore the angle FCE is equal to the angle ACE, (ax. 1.)
the less to the greater, which is impossible:
therefore F is not the centre of the circle ABC.
In the same manner it may be shewn,
that no other point which is not in CA, is the centre;
that is, the centre of the circle is in CA.
Therefore, if a straight line, &c. Q.E.D.

PROPOSITION XX. THEOREM.

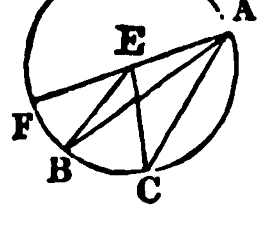
The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same circumference BC for their base.

Then the angle BEC shall be double of the angle BAC.



Join AE, and produce it to F.
First, let the centre of the circle be within the angle BAC.
Because EA is equal to EB,
therefore the angle EAB is equal to the angle EBA; (I. 5.)
therefore the angles EAB, EBA are double of the angle EAB:
but the angle BEF is equal to the angles EAB, EBA; (I. 32.)
therefore also the angle BEF is double of the angle EAB:
for the same reason, the angle FEC is double of the angle EAC:
therefore the whole angle BEC is double of the whole angle BAC.
Again, let the centre of the circle be without the angle BAC.

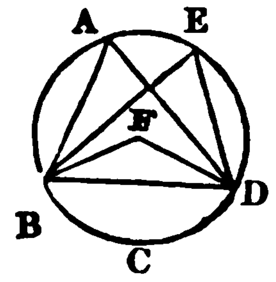


It may be demonstrated, as in the first case,
that the angle FEC is double of the angle FAC,
and that FEB, a part of the first, is double of FAB, a part of the other;
therefore the remaining angle BEC is double of the remaining angle BAC.
Therefore the angle at the centre, &c. Q.E.D.

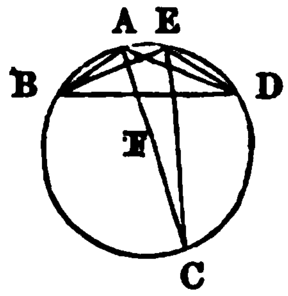
PROPOSITION XXI. THEOREM.

The angles in the same segment of a circle are equal to one another.

Let *ABCD* be a circle,
and *BAD, BED* angles in the same segment *BAED*.
Then the angles *BAD, BED* shall be equal to one another.
First, let the segment *BAED* be greater than a semicircle.



Take *F*, the centre of the circle *ABCD*, (III. 1.) and join *BF, FD*.
And because the angle *BFD* is at the centre, and the angle *BAD* at the circumference, and that they have the same part of the circumference, viz. *BCD* for their base;
therefore the angle *BFD* is double of the angle *BAD*: (III. 20.)
for the same reason the angle *BFD* is double of the angle *BED*:
therefore the angle *BAD* is equal to the angle *BED*. (ax. 7.)
Next, let the segment *BAED* be not greater than a semicircle.

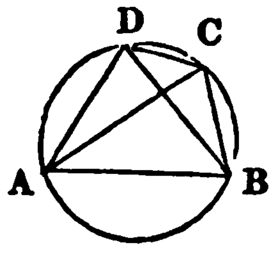


Draw *AF* to the centre, and produce it to *C*, and join *CE*.
Therefore the segment *BADC* is greater than a semicircle;
and the angles in it *BAC, BEC* are equal, by the first case:
for the same reason, because *CBED* is greater than a semicircle,
the angles *CAD, CED*, are equal:
therefore the whole angle *BAD* is equal to the whole angle *BED*. (ax. 2.)
Wherefore the angles in the same segment, &c. Q. E. D.

PROPOSITION XXII. THEOREM.

The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.

Let *ABCD* be a quadrilateral figure in the circle *ABCD*.
Then any two of its opposite angles shall together be equal to two right angles.



Join *AC*, *BD*.

And because the three angles of every triangle are equal to two right angles, (I. 32.)
the three angles of the triangle *CAB*, viz. the angles *CAB*, *ABC*, *BCA*, are equal to two right angles:
but the angle *CAB* is equal to the angle *CDB*, (III. 21.)
because they are in the same segment *CDAB*;
and the angle *ACB* is equal to the angle *ADB*,
because they are in the same segment *ADCB*:
therefore the two angles *CAB*, *ACB* are together equal to the whole angle *ADC*: (ax. 2.)
to each of these equals add the angle *ABC*;
therefore the three angles *ABC*, *CAB*, *BCA* are equal to the two angles *ABC*, *ADC*: (ax. 2.)
but *ABC*, *CAB*, *BCA*, are equal to two right angles;
therefore also the angles *ABC*, *ADC* are equal to two right angles. (ax. 1.)

In the same manner, the angles *BAD*, *DCB*, may be shewn to be equal to two right angles.

Therefore, the opposite angles, &c. Q.E.D.

PROPOSITION XXIII. THEOREM.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, upon the same straight line *AB*, and upon the same side of it, let there be two similar segments of circles, *ACB*, *ADB*, not coinciding with one another.

Then, because the circle *ACB* cuts the circle *ADB* in the two points *A*, *B*,
they cannot cut one another in any other point: (III. 10.)
therefore one of the segments must fall within the other:
let *ACB* fall within *ADB*:
draw the straight line *BCD*, and join *CA*, *DA*.
And because the segment *ACB* is similar to the segment *ADB*, (hyp.)
and that similar segments of circles contain equal angles; (III. def. 11.)
therefore the angle *ACB* is equal to the angle *ADB*,
the exterior to the interior, which is impossible. (I. 16.)

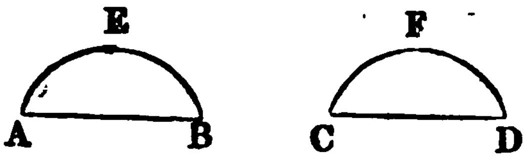
Therefore, there cannot be two similar segments of circles upon the same side of the same line, which do not coincide. Q.E.D.

PROPOSITION XXIV. THEOREM.

Similar segments of circles upon equal straight lines are equal to one another.

Let *AEB*, *CFD* be similar segments of circles upon the equal straight lines *AB*, *CD*.

Then the segment *AEB* shall be equal to the segment *CFD*.

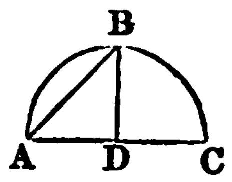


For if the segment AEB be applied to the segment CFD,
so that the point A may be on C, and the straight line AB upon CD,
then the point B shall coincide with the point D,
because AB is equal to CD:
therefore, the straight line AB coinciding with CD,
the segment AEB must coincide with the segment CFD, (III. 23.)
and therefore is equal to it. (ax. 8.)
Wherefore similar segments, &c. Q.E.D.

PROPOSITION XXV. PROBLEM.

A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle.
It is required to describe the circle of which it is the segment.
Bisect AC in D, (I. 10.) and from the point D draw DB at right angles to AC, (I. 11.) and join AB.
First, let the angles ABD, BAD be equal to one another:



then the straight line BD is equal to DA, (I. 6.) and therefore, to DC;
and because the three straight lines DA, DB, DC are all equal,
therefore D is the centre of the circle. (III. 9.)
From the centre D, at the distance of any of the three, DA, DB, DC, describe a circle;
this shall pass through the other points;
and the circle of which ABC is a segment is described:
and because the centre D is in AC, the segment ABC is a semicircle.
But if the angles ABD, BAD are not equal to one another.

At the point A, in the straight line AB, make the angle BAE equal to the angle ABD, (I. 23.)
and produce BD, if necessary, to meet AE in E, and join EC.
And because the angle ABE is equal to the angle BAE,
therefore the straight line BE is equal to EA: (I. 6.)
and because AD is equal to DC, and DE common to the triangles ADE, CDE,
the two sides AD, DE, are equal to the two CD, DE, each to each;
and the angle ADE is equal to the angle CDE,

for each of them is a right angle; (constr.)
therefore the base AE is equal to the base EC: (I. 4.)
but AE was shewn to be equal to EB;
wherefore also BE is equal to EC; (ax. 1.)
and therefore the three straight lines AE, EB, EC are equal to one another:
wherefore E is the centre of the circle. (III. 9.)

From the centre E, at the distance of any of the three AE, EB, EC, describe a circle;
this shall pass through the other points;
and the circle, of which ABC is a segment, is described.

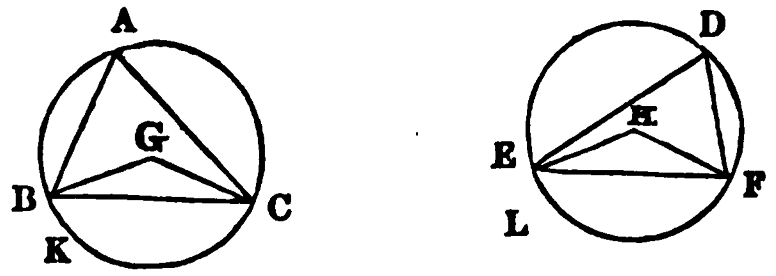
And it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which therefore is less than a semicircle:
but if the angle ABD be less than BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle.

Wherefore a segment of a circle being given, the circle is described of which it is a segment. Q.E.F.

PROPOSITION XXVI. THEOREM.

In equal circles, equal angles *stand upon equal circumferences, whether they be at the centres or circumferences.*

Let ABC, DEF be equal circles,
and let the angles BGC, EHF at their centres,
and BAC, EDF at their circumferences be equal to each other.
The circumference BKC shall be equal to the circumference ELF.

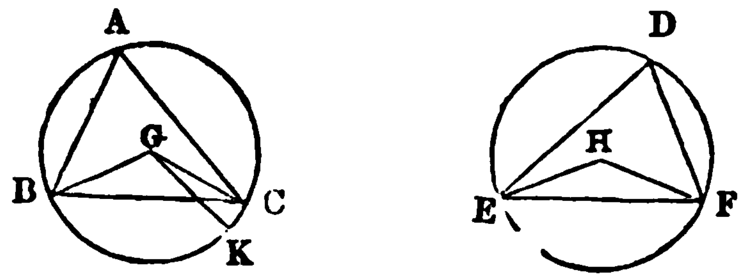


Join BC, EF.
And because the circles ABC, DEF are equal,
the straight lines drawn from their centres are equal: (III. def. 1.)
therefore the two sides BG, GC, are equal to the two EH, HF, each to each:
and the angle at G is equal to the angle at H; (hyp.)
therefore the base BC is equal to the base EF. (I. 4.)
And because the angle at A is equal to the angle at D, (hyp.)
the segment BAC is similar to the segment EDF; (III. def. 11.)
and they are upon equal straight lines BC, EF:
but similar segments of circles upon equal straight lines, are equal to one another, (III. 24.)
therefore the segment BAC is equal to the segment EDF:
but the whole circle ABC is equal to the whole DEF; (hyp.)
therefore the remaining segment BKC is equal to the remaining segment ELF, (ax. 3.)
and the circumference BKC to the circumference ELF.
Wherefore, in equal circles, &c. Q.E.D.

PROPOSITION XXVII. THEOREM.

In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let *ABC, DEF* be equal circles,
and let the angles *BGC, EHF* at their centres,
and *BAC, EDF* at their circumferences, stand upon the equal circumferences *BC, EF*.
The angle *BGC* shall be equal to the angle *EHF*, and the angle *BAC* to the angle *EDF*.

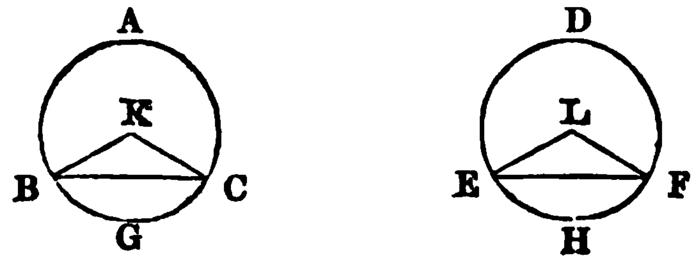


If the angle *BGC* be equal to the angle *EHF*,
it is manifest that the angle *BAC* is also equal to *EDF*. (III. 20. and I. ax. 7.)
But, if not, one of them must be greater than the other:
let *BGC* be the greater,
and at the point *G*, in the straight line *BG*, make the angle *BGK* equal to the angle *EHF*. (I. 23.)
Then because the angle *BGK* is equal to the angle *EHF*,
and that equal angles stand upon equal circumferences, when they are at the centres; (III. 26.)
therefore the circumference *BK* is equal to the circumference *EF*:
but *EF* is equal to *BC*; (hyp.)
therefore also *BK* is equal to *BC*, the less to the greater,
which is impossible: (ax. 1.)
therefore the angle *BGC* is not unequal to the angle *EHF*;
that is, it is equal to it:
but the angle at *A* is half of the angle *BGC*, (III. 20.)
and the angle at *D* half of the angle *EHF*;
therefore the angle at *A* is equal to the angle at *D*. (ax. 7.)
Wherefore, in equal circles, &c. Q.E.D.

PROPOSITION XXVIII. THEOREM.

In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

Let *ABC, DEF* be equal circles,
and *BC, EF* equal straight lines in them, which cut off the two greater circumferences *BAC, EDF*, and the two less *BGC, EHF*.
Then the greater circumference *BAC* shall be equal to the greater *EDF*, and the less *BGC* to the less *EHF*.

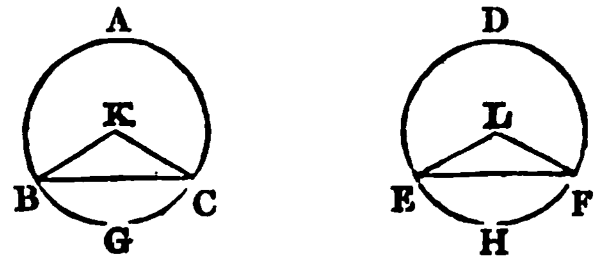


Take K, L, the centres of the circles, (III. 1.) and join BK, KC, EL, L
Because the circles ABC, DEF are equal, the straight lines fro
their centres are equal: (III. def. 1.)
therefore BK, KC are equal to EL, LF, each to each:
and the base BC is equal to the base EF; (hyp.)
therefore the angle BKC is equal to the angle ELF: (I. 8.)
but equal angles stand upon equal circumferences, when they a
at the centres; (III. 26.)
therefore the circumference BGC is equal to the circumference EHF
but the whole circle ABC is equal to the whole EDF; (hyp.)
therefore the remaining part of the circumference,
viz. BAC, is equal to the remaining part EDF. (ax. 3.)
Therefore, in equal circles, &c. Q.E.D.

PROPOSITION XXIX. THEOREM.

In equal circles, equal circumferences are subtended by equal straight line

Let ABC, DEF be equal circles,
and let the circumferences BGC, EHF also be equal;
and join BC, EF.
Then the straight line BC shall be equal to the straight line EF.

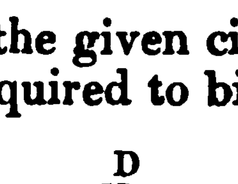


Take K, L, (III. 1.) the centres of the circles, and join BK, KC, EL, LF
and because the circumference BGC is equal to the circumference EHF
therefore the angle BKC is equal to the angle ELF: (III. 27.)
and because the circles ABC, DEF, are equal,
the straight lines from their centres are equal; (III. def. 1.)
therefore BK, KC, are equal to EL, LF, each to each:
and they contain equal angles;
therefore the base BC is equal to the base EF. (I. 4.)
Therefore, in equal circles, &c. Q.E.D.

PROPOSITION XXX. PROBLEM.

To bisect a given circumference, that is, to divide it into two equal part

Let ADB be the given circumference.
It is required to bisect it.



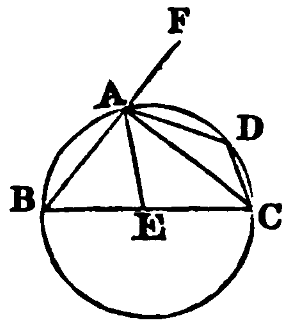
Join AB, and bisect it in C; (I. 10.)
from the point C draw CD at right angles to AB. (I. 11.)
Then the circumference ADB shall be bisected in the point D.
Join AD, DB.

And because AC is equal to CB, and CD common to the triangles ACD, BCD,
the two sides AC, CD are equal to the two BC, CD, each to each;
and the angle ACD is equal to the angle BCD,
because each of them is a right angle:
therefore the base AD is equal to the base BD. (I. 4.)
But equal straight lines cut off equal circumferences, (III. 28.)
the greater equal to the greater, and the less to the less;
and AD, DB are each of them less than a semicircle;
because DC passes through the centre: (III. 1. Cor.)
therefore the circumference AD is equal to the circumference DB.
Therefore the given circumference is bisected in D. Q.E.F.

PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let $ABCD$ be a circle, of which the diameter is BC, and centre E;
and draw CA, dividing the circle into the segments ABC, ADC,
and join BA, AD, DC.
Then the angle in the semicircle BAC shall be a right angle;
and the angle in the segment ABC, which is greater than a semicircle, shall be less than a right angle;
and the angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.



Join AE, and produce BA to F.
And because BE is equal to EA, (I. def. 15.)
the angle EAB is equal to EBA; (I. 5.)
also, because AE is equal to EC,
the angle EAC is equal to ECA;
wherefore the whole angle BAC is equal to the two angles ABC, ACB: (ax. 2.)
but FAC, the exterior angle of the triangle ABC, is equal to the two angles ABC, ACB; (I. 32.)
therefore the angle BAC is equal to the angle FAC; (ax. 1.)
and therefore each of them is a right angle: (I. def. 10.)
wherefore the angle BAC in a semicircle is a right angle.
And because the two angles ABC, BAC of the triangle ABC are together less than two right angles, (I. 17.)
and that BAC has been proved to be a right angle;
therefore ABC must be less than a right angle:
and therefore the angle in a segment ABC greater than a semicircle, is less than a right angle.
And because $ABCD$ is a quadrilateral figure in a circle,

any two of its opposite angles are equal to two right angles: (III. 22.
therefore the angles ABC, ADC, are equal to two right angles:
and ABC has been proved to be less than a right angle;
wherefore the other ADC is greater than a right angle.

Besides, it is manifest, that the circumference of the greater segment ABC falls without the right angle CAB; but the circumference of the less segment ADC falls within the right angle CAF. "And this is all that is meant, when in the Greek text, and the translations from it, the angle of the greater segment is said to be greater, and the angle of the less segment is said to be less, than a right angle."

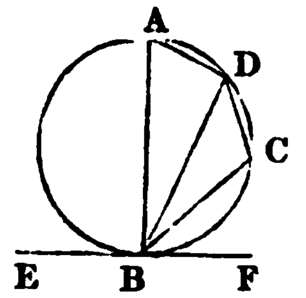
Cor. From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle: because the angle adjacent to it is equal to the same two; (I. 32.) and when the adjacent angles are equal, they are right angles. (I. def. 10.)

PROPOSITION XXXII. THEOREM.

If a straight line touches a circle, and from the point of contact a straight line be drawn cutting the circle; the angles which this line makes with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

Let the straight line EF touch the circle $ABCD$ in B,
and from the point B let the straight line BD be drawn, cutting the circle.

Then the angles which BD makes with the touching line EF, shall be equal to the angles in the alternate segments of the circle;
that is, the angle DBF shall be equal to the angle which is in the segment DAB,
and the angle DBE shall be equal to the angle in the segment DCB.



From the point B draw BA at right angles to EF, (I. 11.)
and take any point C in the circumference DB, and join AD, DC, CB.

And because the straight line EF touches the circle $ABCD$ in the point B,
and BA is drawn at right angles to the touching line from the point of contact B,
the centre of the circle is in BA: (III. 19.)
therefore the angle ADB in a semicircle is a right angle: (III. 31.)
and consequently the other two angles BAD, ABD, are equal to a right angle; (I. 32.)
but ABF is likewise a right angle; (constr.)
therefore the angle ABF is equal to the angles BAD, ABD: (ax. 1.)
take from these equals the common angle ABD:
therefore the remaining angle DBF is equal to the angle BAD, (ax. 3.)
which is in the alternate segment of the circle.

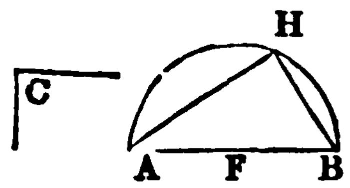
And because $ABCD$ is a quadrilateral figure in a circle,
the opposite angles BAD, BCD are equal to two right angles: (III. 22.

but the angles DBF, DBE are likewise equal to two right angles; (I. 13.)
therefore the angles DBF, DBE are equal to the angles BAD, BCD, (ax. 1.)
and DBF has been proved equal to BAD;
therefore the remaining angle DBE is equal to the angle BCD in the alternate segment of the circle. (ax. 2.)
Wherefore, if a straight line, &c. Q.E.D.

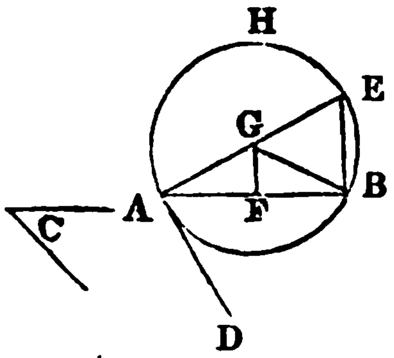
PROPOSITION XXXIII. PROBLEM.

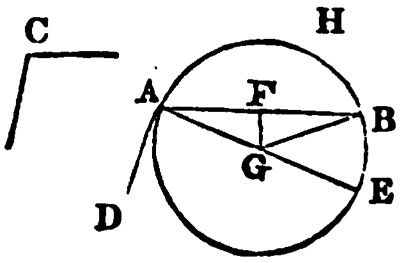
Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

Let AB be the given straight line,
and the angle at C the given rectilineal angle.
It is required to describe upon the given straight line AB, a segment of a circle, which shall contain an angle equal to the angle C.
First, let the angle at C be a right angle.



Bisect AB in F, (I. 10.)
and from the centre F, at the distance FB, describe the semicircle AHB.
Therefore the angle AHB in a semicircle is equal to the right angle at C. (III. 31.)
But, if the angle C be not a right angle.





At the point A, in a straight line AB, make the angle BAD equal to the angle C, (I. 23.)
and from the point A draw AE at right angles to AD; (I. 11.)
bisect AB in F, (I. 10.)
and from F draw FG at right angles to AB, (I. 11.) and join GB.
And because AF is equal to FB, and FG common to the triangles AFG, BFG,
the two sides AF, FG are equal to the two BF, FG, each to each:
and the angle AFG is equal to the angle BFG; (I. def. 10.)
therefore the base AG is equal to the base GB; (I. 4.)
and therefore the circle described from the centre G, at the distance GA, shall pass through the point B:
let this be the circle AHB.
The segment AHB shall contain an angle equal to the given rectilineal angle C.
Because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE,
therefore AD touches the circle: (III. 16. Cor.)

and because AB, drawn from the point of contact A, cuts the circle, the angle DAB is equal to the angle in the alternate segment AHB: (III. 32.)

but the angle DAB is equal to the angle C; (constr.)

therefore the angle C is equal to the angle in the segment AHB. (ax. 1.)

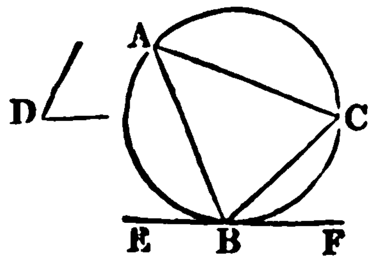
Wherefore, upon the given straight line AB, the segment AHB of a circle is described, which contains an angle equal to the given angle at C. Q.E.F.

PROPOSITION XXXIV. PROBLEM.

From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle.

It is required to cut off from the circle ABC a segment that shall contain an angle equal to the given angle D.



Draw the straight line EF touching the circle ABC in the point B, (III. 17.)

and at the point B, in the straight line BF,

make the angle FBC equal to the angle D. (I. 23.)

Then the segment BAC shall contain an angle equal to the given angle D.

Because the straight line EF touches the circle ABC,

and BC is drawn from the point of contact B,

therefore the angle FBC is equal to the angle in the alternate segment BAC of the circle: (III. 32.)

but the angle FBC is equal to the angle D; (constr.)

therefore the angle in the segment BAC is equal to the angle D. (ax. 1.)

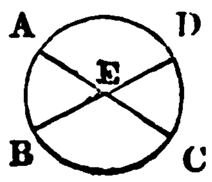
Wherefore from the given circle ABC, the segment BAC is cut off, containing an angle equal to the given angle D. Q.E.F.

PROPOSITION XXXV. THEOREM.

If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them, is equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD, cut one another in the point E, within the circle $ABCD$.

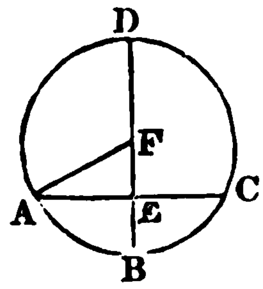
Then the rectangle contained by AE, EC shall be equal to the rectangle contained by BE, ED.



If *AC, BD* pass each of them through the centre, so that *E* is the centre.

It is evident that since *AE, EC, BE, ED,* being all equal, (I. def. 15.) therefore the rectangle *AE, EC,* is equal to the rectangle *BE, ED.*

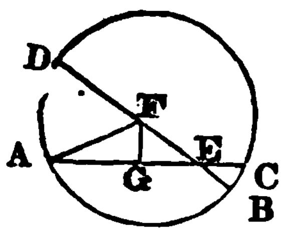
But let one of them *BD* pass through the centre, and cut the other *AC,* which does not pass through the centre, at right angles, in the point *E.*



Then, if *BD* be bisected in *F*,
F is the centre of the circle *ABCD*:
join *AF.*

And because *BD* which passes through the centre, cuts the straight line *AC,* which does not pass through the centre, at right angles in *E,*
therefore *AE* is equal to *EC*: (III. 3.)
and because the straight line *BD* is cut into two equal parts in the point *F,* and into two unequal parts in the point *E,*
therefore the rectangle *BE, ED,* together with the square of *EF,* is equal to the square of *FB*; (II. 5.)
that is, to the square of *FA*:
but the squares of *AE, EF,* are equal to the square of *FA*; (I. 47.)
therefore the rectangle *BE, ED,* together with the square of *EF,* is equal to the squares of *AE, EF*: (ax. 1.)
take away the common square of *EF,*
and the remaining rectangle *BE, ED* is equal to the remaining square of *AE*; (ax. 3.)
that is, to the rectangle *AE, EC.*

Next, let *BD,* which passes through the centre, cut the other *AC,* which does not pass through the centre, in *E,* but not at right angles.



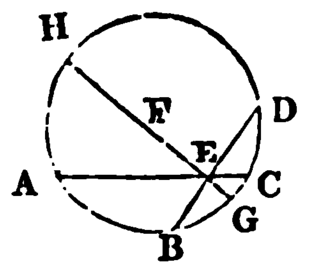
Then, as before, if *BD* be bisected in *F,*
F is the centre of the circle.
Join *AF,* and from *F* draw *FG* perpendicular to *AC*; (I. 12.)
therefore *AG* is equal to *GC*; (III. 3.)
wherefore the rectangle *AE, EC,* together with the square of *EG,* is equal to the square of *AG*: (II. 5.)
to each of these equals add the square of *GF*;
therefore the rectangle *AE, EC,* together with the squares of *EG, GF,* is equal to the squares of *AG, GF*; (ax. 2.)
but the squares of *EG, GF,* are equal to the square of *EF*; (I. 47.)
and the squares of *AG, GF* are equal to the square of *AF*:
therefore the rectangle *AE, EC,* together with the square of *EF,* is equal to the square of *AF*;
that is, to the square of *FB*:
but the square of *FB* is equal to the rectangle *BE, ED,* together with the square of *EF*; (II. 5.)

therefore the rectangle AE, EC, together with the square EF, is equal to the rectangle BE, ED, together with the square EF; (ax. 1.)

take away the common square of EF,

and the remaining rectangle AE, EC, is therefore equal to remaining rectangle BE, ED. (ax. 3.)

Lastly, let neither of the straight lines AC, BD pass through the cent



Take the centre F, (III. 1.)

and through E the intersection of the straight lines AC, DB, draw the diameter $GEFH$.

And because the rectangle AE, EC, is equal as has been shown the rectangle GE, EH;

and for the same reason, the rectangle BE, ED is equal to t same rectangle GE, EH;

therefore the rectangle AE, EC is equal to the rectangle BE, ED. (ax.

Wherefore if two straight lines, &c. Q.E.D.

PROPOSITION XXXVI. THEOREM.

If from any point without a circle two straight lines be drawn, one which cuts the circle, and the other touches it; the rectangle contained the whole line which cuts the circle, and the part of it without the circ shall be equal to the square of the line which touches it.

Let D be any point without the circle ABC,

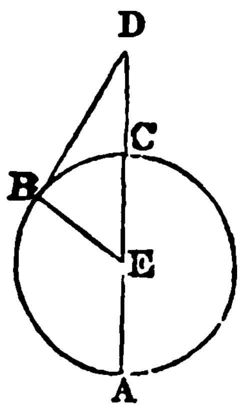
and let DCA, DB be two straight lines drawn from it,

of which DCA cuts the circle, and DB touches the same.

Then the rectangle AD, DC shall be equal to the square of DB.

Either DCA passes through the centre, or it does not:

first, let it pass through the centre E.



Join EB,

therefore the angle EBD is a right angle. (III. 18.)

And because the straight line AC is bisected in E, and produced the point D,

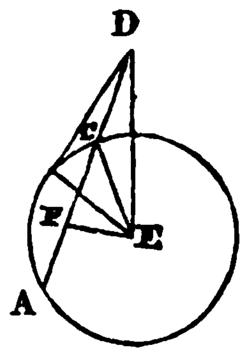
therefore the rectangle AD, DC, together with the square of E is equal to the square of ED: (II. 6.)

but CE is equal to EB;

therefore the rectangle AD, DC, together with the square of E is equal to the square of ED:

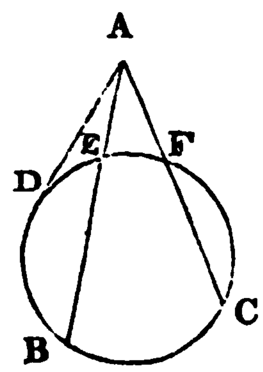
but the square of ED is equal to the squares of EB, BD, (I. 47.)

because EBD is a right angle:
therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD: (ax. 1.)
take away the common square of EB;
therefore the remaining rectangle AD, DC is equal to the square of the tangent, DB. (ax. 3.)
But if DCA does not pass through the centre of the circle ABC.
Take E the centre of the circle, (III. 1.)



draw EF perpendicular to AC, (I. 12.) and join EB, EC, ED.
And because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles.
Therefore EF also bisects AC; (III. 3.)
therefore AF is equal to FC:
and because the straight line AC is bisected in F, and produced to D, the rectangle AD, DC, together with the square of FC is equal to the square of FD: (II. 6.)
to each of these equals add the square of FE;
therefore the rectangle AD, DC, together with the squares of CF, FE is equal to the squares of DF, FE: (ax. 2.)
but the square of ED is equal to the squares of DF, FE, (I. 47.)
because EFD is a right angle;
and for the same reason,
the square of EC is equal to the squares of CF, FE;
therefore the rectangle AD, DC, together with the square of EC, is equal to the square of ED: (ax. 1.)
but CE is equal to EB;
therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED:
but the squares of EB, BD are equal to the square of ED, (I. 47.)
because EBD is a right angle:
therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD;
take away the common square of EB;
therefore the remaining rectangle AD, DC is equal to the square of DB. (ax. 3.)
Wherefore, if from any point, &c. Q.E.D.

COR. If from any point without a circle, there be drawn two



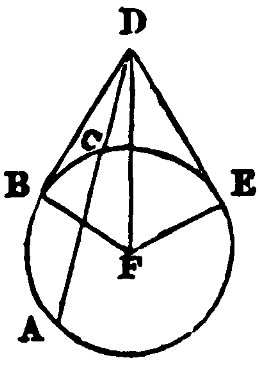
straight lines cutting it, as *AB, AC,* the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle *BA, AE,* to the rectangle *CA, AF*: for each of them is equal to the square of the straight line *AD,* which touches the circle.

PROPOSITION XXXVII. THEOREM.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets shall touch the circle.

Let any point *D* be taken without the circle *ABC,*
and from it let two straight lines *DCA* and *DB* be drawn, of which *DCA* cuts the circle, and *DB* meets it.

If the rectangle *AD,DC* be equal to the square of *DB*; then *DB* shall touch the circle.



Draw the straight line *DE,* touching the circle *ABC,* in the point *B*; (III. 17.)
find *F,* the centre of the circle, (III. 1.)
and join *FE, FB, FD.*
Then *FED* is a right angle: (III. 18.)
and because *DE* touches the circle *ABC,* and *DCA* cuts it,
therefore the rectangle *AD, DC* is equal to the square of *DE*: (III. 36.)
but the rectangle *AD, DC* is, by hypothesis, equal to the square of *DB*;
therefore the square of *DE* is equal to the square of *DB*; (ax. 1.)
and the straight line *DE* equal to the straight line *DB*:
and *FE* is equal to *FB*; (I. def. 15.)
wherefore *DE, EF* are equal to *DB, BF,* each to each;
and the base *FD* is common to the two triangles *DEF, DBF*;
therefore the angle *DEF* is equal to the angle *DBF*: (I. 8.)
but *DEF* was shewn to be a right angle;
therefore also *DBF* is a right angle: (ax. 1.)
and *BF,* if produced, is a diameter;
and the straight line which is drawn at right angles to a diameter,
from the extremity of it, touches the circle; (III. 16. Cor.)
therefore *DB* touches the circle *ABC.*
Wherefore, if from a point, &c. Q.E.D.

NOTES TO BOOK III.

In the third Book of the Elements are demonstrated the properties of the circle, assuming all the properties of figures demonstrated in the first and second books.

A new conception is introduced in the third book, namely, that of *similarity*, and applied to the proof of properties connected with similar segments of circles.

It may be worthy of remark, that the word *circle* will be found sometimes taken to mean *the surface* included within the circumference, and sometimes *the circumference itself*. A circle is said to be given in position when the position of its centre is known, and in magnitude when its radius is known.

Def. I states the criterion of equal circles. Simson calls it a theorem; and Euclid seems to have considered it as one of those theorems, or definitions involving an axiom, which might be admitted as a basis for reasoning on the equality of circles.

Def. VI, X. An arc of a circle is any portion of the circumference; and a chord is the straight line joining the extremities of an arc. Every chord except a diameter divides a circle into two unequal segments, one greater than, and the other less than a semicircle. And in the same manner, two radii drawn from the centre to the circumference, divide the circle into two unequal sectors, which become equal when the two radii are in the same straight line. A quadrant is a sector whose radii are perpendicular to one another, and contains a fourth part of the circle.

Prop. I. If the point G be in the diameter CE, but not coinciding with the point F, the demonstration given in the text does not hold good. At the same time, it is obvious that G cannot be the centre of the circle, because GC is not equal to GE.

Prop. II. In this proposition, the circumference of a circle is proved to be essentially different from a straight line, by shewing that every straight line joining any two points in the arc falls entirely within the circle, and can neither coincide with any part of the circumference, nor meet it except in the two assumed points. From which follows the corollary, that, "a straight line cannot cut the circumference of a circle in more points than two."

Commandine's direct demonstration of Prop. II depends on the following axiom, "If a point be taken nearer to the centre of a circle than the circumference, that point falls within the circle."

Take any point E in AB, and join DA, DE, DB.
Then because DA is equal to DB in the triangle DAB;
therefore the angle DAB is equal to the angle DEB; (v. 1.)
but since the side AE of the triangle DAE is produced to D,
therefore the exterior angle DEB is greater than the interior and opposite angle DAE;
but the angle DAE is equal to the angle DBE,
therefore the angle DEB is greater than the angle DBE.
And in every triangle the greater side is subtended by the greater angle;
therefore the side DB is greater than the side DE;
but DB from the centre meets the circumference of the circle,
therefore DE does not meet it.
Wherefore the point E falls within the circle.
And E is any point in the straight line AB,
therefore the straight line AB falls within the circle.

Prop. VII and Prop. VIII exhibit the same property; in the former, the point is taken in the diameter, and in the latter, in the diameter produced.

Prop. XI and Prop. XII. In the enunciations it is not asserted that the contact of two circles is confined to a single point. The meaning appears to be, that supposing two circles touch each other in any point, the straight line which joins their centres being produced shall pass through that point in which the circles touch each other.

In Prop. XIII, it is proved that a circle cannot touch another in more points than one, by assuming two points of contact, and proving that this is impossible.

Prop. XV and XVI. The converse of these propositions are not proved by Euclid.

Prop. XVI may be demonstrated *directly* by assuming the following axiom; "If a point be taken further from the centre of a circle than the circumference, that point falls without the circle."

Prop. XVII. When the given point is without the circumference of the given circle, it is obvious that two equal tangents may be drawn from the given point to touch the circle, as may be seen from the diagram to Prop. VIII.

Circles are called *concentric circles* when they have the same centre.

Prop. XVIII appears to be nothing more than a corollary to Prop. XVI. Because a tangent to any point of the circumference of a circle is a straight line at right angles at the extremity of the diameter which meets the circumference in that point.

In Prop. XVI, AE is proved to be perpendicular to AB, and in Prop. XVIII, AB is proved to be perpendicular to AE; which is the same thing.

Prop. XX. This proposition is proved by Euclid only in the case in which the angle at the circumference is less than a right angle, and the demonstration is free from objection. If, however, the angle at the circumference be a right angle, the angle at the centre disappears by the two straight lines from the centre to the extremities of the arc becoming one straight line. And, if the angle at the circumference be an obtuse angle, the angle formed by the two lines from the centre, does not stand on the same arc, but upon the arc which the assumed arc wants of the whole circumference.

If Euclid's definition of an angle be strictly observed, Prop. XX is geometrically true only when the angle at the centre is less than a right angle. If, however, the defect of an angle from four right angles may be regarded as an angle, the proposition is universally true, as may be proved by drawing a line from the angle in the circumference through the centre, and thus forming two angles at the centre in Euclid's strict sense of the term.

In the first case, it is assumed that, if there be four magnitudes, such that the first is double of the second, and the third double of the fourth, then the first and third together shall be double of the second and fourth together: also in the second case, that if one magnitude be double of another, and a part taken from the first be double of a part taken from the second, the remainder of the first shall be double the remainder of the second, which is, in fact, a particular case of Prop. 5, Book V.

Prop. XXI. Hence, the locus of the vertices of all triangles upon the same base, and which have the same vertical angle, is a circular arc.

Prop. XXXI suggests a method of drawing a line at right angles to another when the given point is at the extremity of the given line.

Prop. XXXV. It is possible to prove the most general case of this proposition, and from it to deduce the other cases. The converse of Prop. XXXV is not proved by Euclid.

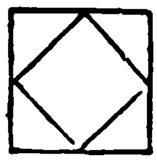
The properties of the circle demonstrated in the third book may be divided under three heads. 1. Those which relate to the centre. 2. To similar segments. 3. To the equal rectangles contained by the segments of the lines which intersect each other within and without the circle.

BOOK IV.

DEFINITIONS.

I.

A RECTILINEAL figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

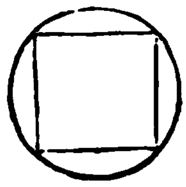


II.

In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

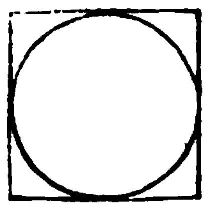
III.

A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.



IV.

A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

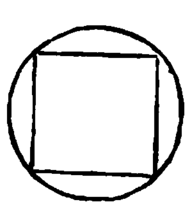


V.

In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI.

A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.



VII.

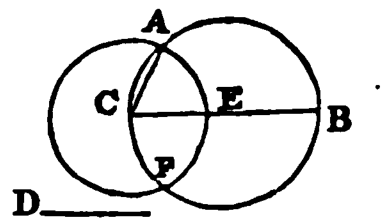
A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROPOSITION I. PROBLEM.

In a given circle to place a straight line, equal to a given strai line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line, : greater than the diameter of the circle.

It is required to place in the circle ABC a straight line equal to



Draw BC the diameter of the circle ABC.
Then, if BC is equal to D, the thing required is done;
for in the circle ABC a straight line BC is placed equal to D.
But, if it is not, BC is greater than D; (hyp.)
make CE equal to D, (I. 3.)
and from the centre C, at the distance CE, describe the circle AE
and join CA.
Then CA shall be equal to D.
Because C is the centre of the circle AEF,
therefore CA is equal to CE: (I. def. 15.)
but CE is equal to D; (constr.)
therefore D is equal to CA. (ax. 1.)

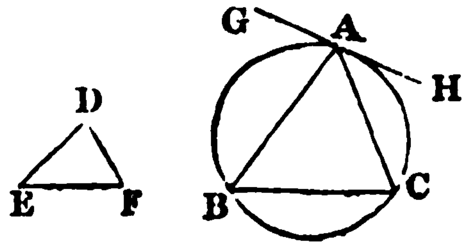
Wherefore in the circle ABC, a straight line CA is placed equal the given straight line D, which is not greater than the diameter of circle. Q.E.F.

PROPOSITION II. PROBLEM.

In a given circle to inscribe a triangle equiangular to a given trian

Let ABC be the given circle, and DEF the given triangle.

It is required to inscribe in the circle ABC a triangle equiang to the triangle DEF.



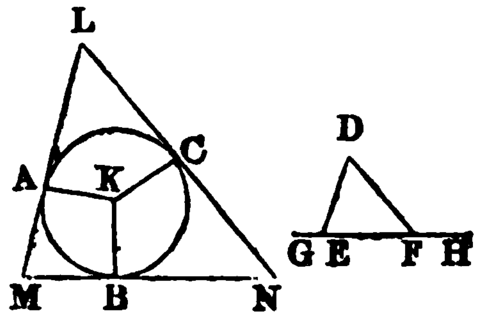
Draw the straight line GAH touching the circle in the point A, (III.
and at the point A, in the straight line AH,
make the angle HAC equal to the angle DEF; (I. 23.)
and at the point A, in the straight line AG,
make the angle GAB equal to the angle DFE;
and join BC: then ABC shall be the triangle required.
Because HAG touches the circle ABC,
and AC is drawn from the point of contact,

therefore the angle HAC is equal to the angle ABC in the alternate segment of the circle: (III. 32.)
but HAC is equal to the angle DEF; (constr.)
therefore also the angle ABC is equal to DEF: (ax. 1.)
for the same reason, the angle ABC is equal to the angle DFE:
therefore the remaining angle BAC is equal to the remaining angle EDF: (I. 32 and ax. 1.)
wherefore the triangle ABC is equiangular to the triangle DEF,
and it is inscribed in the circle ABC. Q.E.F.

PROPOSITION III. PROBLEM.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle.
It is required to describe a triangle about the circle ABC equiangular to the triangle DEF.



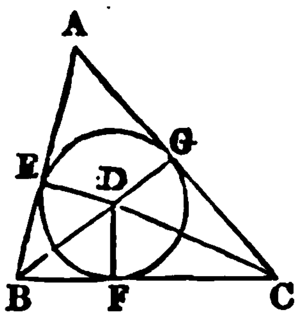
Produce EF both ways to the points G, H;
find the centre K of the circle ABC, (III. 1.)
and from it draw any straight line KB;
at the point K in the straight line KB,
make the angle BKA equal to the angle DEG, (I. 23.)
and the angle BKC equal to the angle DFH;
and through the points A, B, C, draw the straight lines LAM, MBN, NCL, touching the circle ABC. (III. 17.)
Then LMN shall be the triangle required.
Because LM, MN, NL touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC,
therefore the angles at the points A, B, C are right angles: (III. 18.)
and because the four angles of the quadrilateral figure $AMBK$ are equal to four right angles,
for it can be divided into two triangles;
and that two of them KAM, KBM are right angles,
therefore the other two AKB, AMB are equal to two right angles: (ax. 3.)
but the angles DEG, DEF are likewise equal to two right angles; (I. 13.)
therefore the angles AKB, AMB are equal to the angles DEG, DEF; (ax. 1.)
of which AKB is equal to DEG; (constr.)
wherefore the remaining angle AMB is equal to the remaining angle DEF. (ax. 3.)
In like manner, the angle LNM may be demonstrated to be equal to DFE;

and therefore the remaining angle *MLN* is equal to the remaining angle *EDF*: (I. 32 and ax. 3.)
therefore the triangle *LMN* is equiangular to the triangle *DEF*:
and it is described about the circle *ABC*. Q.E.F.

PROPOSITION IV. PROBLEM.

To inscribe a circle in a given triangle.

Let the given triangle be *ABC*.
It is required to inscribe a circle in *ABC*.

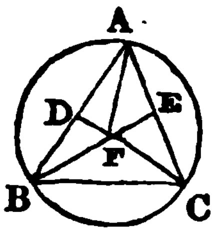


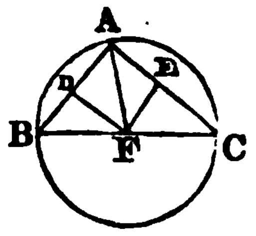
Bisect the angles *ABC, BCA* by the straight lines *BD, CD* meeting one another in the point *D*, (I. 9.)
from which draw *DE, DF, DG* perpendiculars to *AB, BC, CA*. (I. 12.)
And because the angle *EBD* is equal to the angle *FBD*,
for the angle *ABC* is bisected by *BD*,
and that the right angle *BED* is equal to the right angle *BFD*; (ax. 11.)
therefore the two triangles *EBD, FBD* have two angles of the one equal to two angles of the other, each to each;
and the side *BD*, which is opposite to one of the equal angles in each, is common to both;
therefore their other sides are equal; (I. 26.)
wherefore *DE* is equal to *DF*:
for the same reason, *DG* is equal to *DF*:
therefore *DE* is equal to *DG*: (ax. 1.)
therefore the three straight lines *DE, DF, DG* are equal to one another;
and the circle described from the center *D*, at the distance of any of them, will pass through the extremities of the other two, and touch the straight lines *AB, BC, CA*,
because the angles at the points *E, F, G* are right angles,
and the straight line which is drawn from the extremity of a diameter at right angles to it, touches the circle: (III. 16.)
therefore the straight lines *AB, BC, CA* do each of them touch the circle,
and therefore the circle *EFG* is inscribed in the triangle *ABC*. Q.E.F.

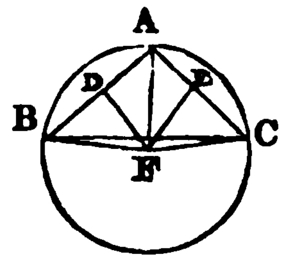
PROPOSITION V. PROBLEM.

To describe a circle about a given triangle.

Let the given triangle be *ABC*.
It is required to describe a circle about *ABC*.







Bisect AB, AC in the points D, E, (I. 10.)
and from these points draw DF, EF at right angles to AB, AC; (I. 11.)
DF, EF produced meet one another:
for, if they do not meet, they are parallel,
wherefore AB, AC, which are at right angles to them, are parallel;
which is absurd:
let them meet in F, and join FA;
also, if the point F be not in BC, join BF, CF.

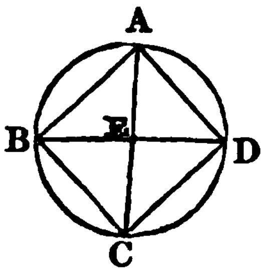
Then, because AD is equal to DB, and DF common, and at right angles to AB,
therefore the base AF is equal to the base FB. (I. 4.)
In like manner, it may be shewn that CF is equal to FA;
and therefore BF is equal to FC; (ax. 1.)
and FA, FB, FC are equal to one another:
wherefore the circle described from the centre F, at the distance of one of them, will pass through the extremities of the other two, and be described about the triangle ABC. Q.E.F.

COR.—And it is manifest, that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, (III. 31.) each of them being in a segment greater than a semicircle; but, when the centre is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, (III. 31.) is a right angle; and, if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, (III. 31.) is greater than a right angle: therefore, conversely, if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

PROPOSITION VI. PROBLEM.

To inscribe a square in a given circle.

Let $ABCD$ be the given circle.
It is required to inscribe a square in $ABCD$.



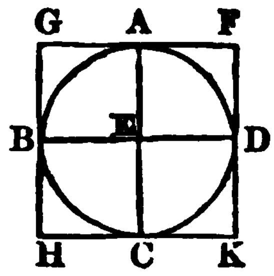
Draw the diameters, AC, BD, at right angles to one another, (III. 1 and I. 11.)
and join AB, BC, CD, DA.
The figure $ABCD$ shall be the square required.

Because BE is equal to ED, for E is the centre, and that EA is common, and at right angles to BD;
the base BA is equal to the base AD: (I. 4.)
and, for the same reason, BC, CD are each of them equal to BA, or AD;
therefore the quadrilateral figure $ABCD$ is equilateral.
It is also rectangular;
for the straight line BD being the diameter of the circle $ABCD$, BAD is a semicircle;
wherefore the angle BAD is a right angle: (III. 31.)
for the same reason, each of the angles ABC, BCD, CDA is a right angle:
therefore the quadrilateral figure $ABCD$ is rectangular:
and it has been shewn to be equilateral;
therefore it is a square; (I. def. 30.)
and it is inscribed in the circle $ABCD$. Q.E.F.

PROPOSITION VII. PROBLEM.

To describe a square about a given circle.

Let $ABCD$ be the given circle.
It is required to describe a square about it.



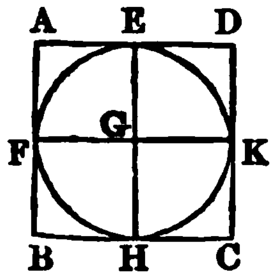
Draw two diameters AC, BD of the circle $ABCD$, at right angles to one another,
and through the points A, B, C, D, draw FG, GH, HK, KF touching the circle. (III. 17.)
The figure $GHKF$ shall be the square required.
Because FG touches the circle $ABCD$, and EA is drawn from the centre E to the point of contact A,
therefore the angles at A are right angles: (III. 18.)
for the same reason, the angles at the points B, C, D are right angles:
and because the angle AEB is a right angle, as likewise is EBG,
therefore GH is parallel to AC: (I. 28.)
for the same reason AC is parallel to FK:
and in like manner GF, HK may each of them be demonstrated to be parallel to BED:
therefore the figures GK, GC, AK, FB, BK are parallelograms;
and therefore GF is equal to HK, and GH to FK: (I. 34.)
and because AC is equal to BD, and that AC is equal to each of the two GH, FK;
and BD to each of the two GF, HK;
GH, FK are each of them equal to GF, or HK:
therefore the quadrilateral figure $FGHK$ is equilateral.
It is also rectangular;
for $GBEA$ being a parallelogram, and AEB a right angle,
therefore AGB is likewise a right angle: (I. 34.)

and in the same manner it may be shewn that the angles at *H, K, F* are right angles:
therefore the quadrilateral figure *FGHK* is rectangular:
and it was demonstrated to be equilateral;
therefore it is a square; (I. def. 30.)
and it is described about the circle *ABCD*. Q.E.F.

PROPOSITION VIII. PROBLEM.

To inscribe a circle in a given square.

Let *ABCD* be the given square.
It is required to inscribe a circle in *ABCD*.

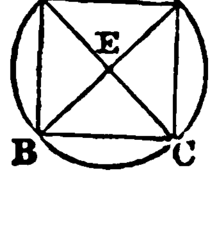


Bisect each of the sides *AB, AD* in the points *F, E,* (I. 10.)
and through *E* draw *EH* parallel to *AB* or *DC*, (I. 31.)
and through *F* draw *FK* parallel to *AD* or *BC*:
therefore each of the figures *AK, KB, AH, HD, AG, GC, BG, GD* is a right-angled parallelogram;
and their opposite sides are equal: (I. 34.)
and because *AD* is equal to *AB*, (I. def. 30.) and that *AE* is the half of *AD*, and *AF* the half of *AB*,
therefore *AE* is equal to *AF*; (ax. 7.)
wherefore the sides opposite to these are equal, viz. *FG* to *GE*:
in the same manner it may be demonstrated that *GH, GK* are each of them equal to *FG* or *GE*:
therefore the four straight lines *GE, GF, GH, GK* are equal to one another;
and the circle described from the centre *G* at the distance of one of them, will pass through the extremities of the other three, and touch the straight lines *AB, BC, CD, DA*;
because the angles at the points *E, F, H, K*, are right angles, (I. 29.)
and that the straight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle: (III. 16. Cor.)
therefore each of the straight lines *AB, BC, CD, DA* touches the circle,
which therefore is inscribed in the square *ABCD*. Q.E.F.

PROPOSITION IX. PROBLEM.

To describe a circle about a given square.

Let *ABCD* be the given square.
It is required to describe a circle about *ABCD*.

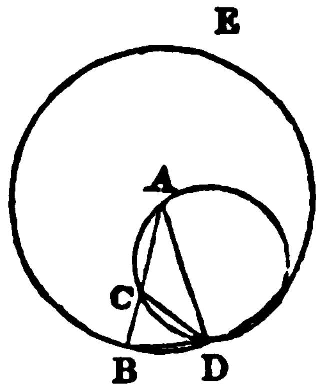


Join AC, BD, cutting one another in E:
and because DA is equal to AB, and AC common to the triangles DAC, BAC, (I. def. 30.)
the two sides DA, AC are equal to the two BA, AC, each to each;
and the base DC is equal to the base BC;
wherefore the angle DAC is equal to the angle BAC, (I. 8.)
and the angle DAB is bisected by the straight line AC:
in the same manner it may be demonstrated that the angles ABC, BCD, CDA are severally bisected by the straight lines BD, AC:
therefore, because the angle DAB is equal to the angle ABC, (I. def. 30.)
and that the angle EAB is the half of DAB, and EBA the half of ABC;
therefore the angle EAB is equal to the angle EBA; (ax. 7.)
wherefore the side EA is equal to the side EB: (I. 6.)
in the same manner it may be demonstrated, that the straight lines EC, ED are each of them equal to EA or EB:
therefore the four straight lines EA, EB, EC, ED are equal to one another;
and the circle described from the centre E, at the distance of one of them, will pass through the extremities of the other three, and be described about the square $ABCD$. Q.E.F.

PROPOSITION X. PROBLEM.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Take any straight line AB, and divide it in the point C, (II. 11.)
so that the rectangle AB, BC may be equal to the square of CA;
and from the centre A, at the distance AB, describe the circle BDE,
in which place the straight line BD equal to AC, which is not greater than the diameter of the circle BDE; (IV. 1.)
and join DA.
Then the triangle ABD shall be such as is required,
that is, each of the angles ABD, ADB shall be double of the angle BAD.



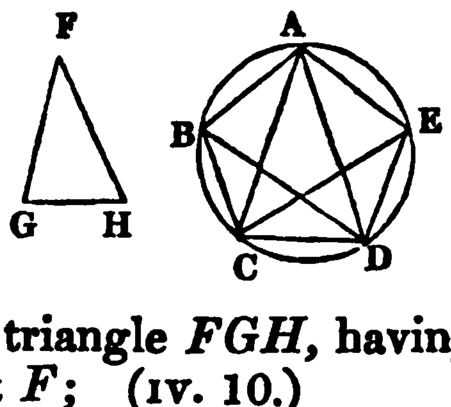
Join DC, and about the triangle ADC describe the circle ACD. (IV. 5.)
And because the rectangle AB, BC is equal to the square of AC,
and that AC is equal to BD, (constr.)
the rectangle AB, BC is equal to the square of BD: (ax. 1.)
and because from the point B, without the circle ACD, two straight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC, contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of BD which meets it;
therefore the straight line BD touches the circle ACD: (III. 37.)

and because BD touches the circle, and DC is drawn from the point of contact D,
the angle BDC is equal to the angle DAC in the alternate segment of the circle: (III. 32.)
to each of these add the angle CDA;
therefore the whole angle BDA is equal to the two angles CDA, DAC: (ax. 2.)
but the exterior angle BCD is equal to the angles CDA, DAC; (I. 32.)
therefore also BDA is equal to BCD: (ax. 1.)
but BDA is equal to the angle CBD, (I. 5.)
because the side AD is equal to the side AB;
therefore CBD, or DBA, is equal to BCD; (ax. 1.)
and consequently the three angles BDA, DBA, BCD are equal to one another:
and because the angle DBC is equal to the angle BCD,
the side BD is equal to the side DC: (I. 6.)
but BD was made equal to CA;
therefore also CA is equal to CD, (ax. 1.)
and the angle CDA equal to the angle DAC; (I. 5.)
therefore the angles CDA, DAC together, are double of the angle DAC:
but BCD is equal to the angles CDA, DAC; (I. 32.)
therefore also BCD is double of DAC:
and BCD was proved to be equal to each of the angles BDA, DBA;
therefore each of the angles BDA, DBA is double of the angle DAB.
Wherefore an isosceles triangle ABD is described, having each of the angles at the base double of the third angle. Q. E. F.

PROPOSITION XI. PROBLEM.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let $ABCDE$ be the given circle.
It is required to inscribe an equilateral and equiangular pentagon in the circle $ABCDE$.



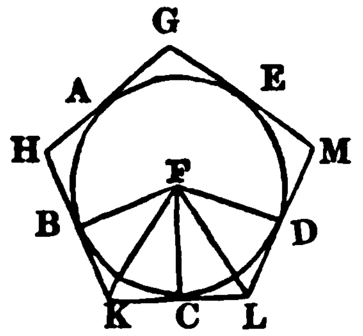
Describe an isosceles triangle FGH, having each of the angles at G, H double of the angle at F; (IV. 10.)
and in the circle $ABCDE$ inscribe the triangle ACD equiangular to the triangle FGH, (IV. 2.)
so that the angle CAD may be equal to the angle at F,
and each of the angles ACD, CDA equal to the angle at G or H;
wherefore each of the angles ACD, CDA is double of the angle CAD.
Bisect the angles ACD, CDA by the straight lines CE, DB; (I. 9.)
and join AB, BC, DE, EA.
Then $ABCDE$ shall be the pentagon required.
Because each of the angles ACD, CDA is double of CAD,
and that they are bisected by the straight lines CE, DB;

therefore the five angles DAC, ACE, ECD, CDB, BDA are equal to one another:
but equal angles stand upon equal circumferences; (III. 26.)
therefore the five circumferences AB, BC, CD, DE, EA are equal to one another:
and equal circumferences are subtended by equal straight lines; (III. 29.)
therefore the five straight lines AB, BC, CD, DE, EA are equal to one another.
Wherefore the pentagon $ABCDE$ is equilateral.
It is also equiangular:
for, because the circumference AB is equal to the circumference DE,
if to each be added BCD,
the whole $ABCD$ is equal to the whole $EDCB$: (ax. 2.)
but the angle AED stands on the circumference $ABCD$;
and the angle BAE on the circumference $EDCB$;
therefore the angle BAE is equal to the angle AED: (III. 27.)
for the same reason, each of the angles ABC, BCD, CDE is equal to the angle BAE, or AED:
therefore the pentagon $ABCDE$ is equiangular;
and it has been shewn that it is equilateral:
wherefore, in the given circle, an equilateral and equiangular pentagon has been described. Q.E.F.

PROPOSITION XII. PROBLEM.

To describe an equilateral and equiangular pentagon about a given circle.

Let $ABCDE$ be the given circle.
It is required to describe an equilateral and equiangular pentagon about the circle $ABCDE$.



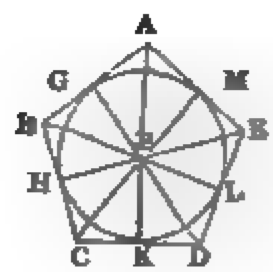
Let the angles of a pentagon, inscribed in the circle, by the last proposition, be in the points A, B, C, D, E,
so that the circumferences AB, BC, CD, DE, EA are equal; (IV. 11.)
and through the points A, B, C, D, E draw GH, HK, KL, LM, MG touching the circle; (III. 17.)
the figure $GHKLM$ shall be the pentagon required.
Take the centre F, and join FB, FK, FC, FL, FD.
And because the straight line KL touches the circle $ABCDE$ in the point C, to which FC is drawn from the centre F,
FC is perpendicular to KL, (III. 18.)
therefore each of the angles at C is a right angle:
for the same reason, the angles at the points B, D are right angles:
and because FCK is a right angle,
the square of FK is equal to the squares of FC, CK: (I. 47.)

for the same reason, the square of FK is equal to the squares of FB, BK:
therefore the squares of FC, CK are equal to the squares of FB, BK; (ax. 1.)
of which the square of FC is equal to the square of FB;
therefore the remaining square of CK is equal to the remaining square of BK, (ax. 3.)
and the straight line CK equal to BK:
and because FB is equal to FC, and FK common to the triangles BFK, CFK,
the two BF, FK are equal to the two CF, FK, each to each;
and the base BK was proved equal to the base KC;
therefore the angle BFK is equal to the angle KFC, (I. 8.)
and the angle BKF to FKC: (I. 4.)
wherefore the angle BFC is double of the angle KFC,
and BKC double of FKC:
for the same reason, the angle CFD is double of the angle CFL,
and CLD double of CLF:
and because the circumference BC is equal to the circumference CD,
the angle BFC is equal to the angle CFD; (III. 27.)
and BFC is double of the angle KFC,
and CFD double of CFL;
therefore the angle KFC is equal to the angle CFL: (ax. 7.)
and the right angle FCK is equal to the right angle FCL;
therefore, in the two triangles FKC, FLC, there are two angles of the one equal to two angles of the other, each to each;
and the side FC, which is adjacent to the equal angles in each, is common to both;
therefore the other sides are equal to the other sides, and the third angle to the third angle: (I. 26.)
therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC:
and because KC is equal to CL,
KL is double of KC.
In the same manner it may be shewn that HK is double of BK:
and because BK is equal to KC, as was demonstrated,
and that KL is double of KC, and HK double of BK,
therefore HK is equal to KL: (ax. 6.)
in like manner it may be shewn that GH, GM, ML are each of them equal to HK, or KL:
therefore the pentagon $GHKLM$ is equilateral.
It is also equiangular:
for, since the angle FKC is equal to the angle FLC,
and that the angle HKL is double of the angle FKC,
and KLM double of FLC, as was before demonstrated;
therefore the angle HKL is equal to KLM: (ax. 6.)
and in like manner it may be shewn,
that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM:
therefore the five angles GHK, HKL, KLM, LMG, MGH, being equal to one another,
the pentagon $GHKLM$ is equiangular:
and it is equilateral, as was demonstrated;
and it is described about the circle $ABCDE$. Q.E.F.

PROPOSITION XIII. PROBLEM.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let $ABCDE$ be the given equilateral and equiangular pentagon.
It is required to inscribe a circle in the pentagon $ABCDE$.



Bisect the angles BCD, CDE by the straight lines CF, DF, (I. 9.)
and from the point F, in which they meet, draw the straight lines FB, FA, FE:
therefore since BC is equal to CD, (hyp.) and CF common to the triangles BCF, DCF,
the two sides BC, CF are equal to the two DC, CF, each to each;
and the angle BCF is equal to the angle DCF; (constr.)
therefore the base BF is equal to the base FD, (I. 4.)
and the other angles to the other angles, to which the equal sides are opposite;
therefore the angle CBF is equal to the angle CDF:
and because the angle CDE is double of CDF,
and that CDE is equal to CBA, and CDF to CBF;
CBA is also double of the angle CBF;
therefore the angle ABF is equal to the angle CBF;
wherefore the angle ABC is bisected by the straight line BF:
in the same manner it may be demonstrated,
that the angles BAE, AED, are bisected by the straight lines AF, FE.

From the point F, draw FG, FH, FK, FL, FM perpendiculars to the straight lines AB, BC, CD, DE, EA: (I. 12.)
and because the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC;
therefore in the triangles FHC, FKC, there are two angles of the one equal to two angles of the other, each to each;
and the side FC, which is opposite to one of the equal angles in each, is common to both;
therefore the other sides are equal, each to each; (I. 26.)
wherefore the perpendicular FH is equal to the perpendicular FK:
in the same manner it may be demonstrated, that FL, FM, FG are each of them equal to FH, or FK:
therefore the five straight lines FG, FH, FK, FL, FM are equal to one another:

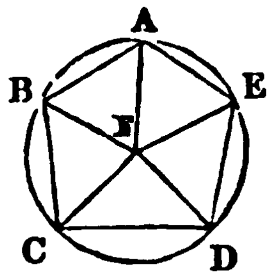
wherefore the circle described from the centre F, at the distance of one of these five, will pass through the extremities of the other four, and touch the straight lines AB, BC, CD, DE, EA,
because the angles at the points G, H, K, L, M are right angles,
and that a straight line drawn from the extremity of the diameter of a circle at right angles to it, touches the circle: (III. 16.)

therefore each of the straight lines *AB, BC, CD, DE, EA* touches the circle:
wherefore it is inscribed in the pentagon *ABCDE*. Q.E.F.

PROPOSITION XIV. PROBLEM.

To describe a circle about a given equilateral and equiangular pentagon.

Let *ABCDE* be the given equilateral and equiangular pentagon.
It is required to describe a circle about *ABCDE*.



Bisect the angles *BCD, CDE* by the straight lines *CF, FD,* (I. 9.)
and from the point *F*, in which they meet, draw the straight lines *FB, FA, FE,* to the points *B, A, E.*

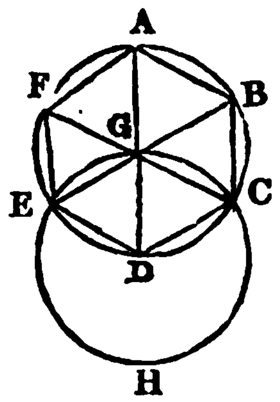
It may be demonstrated, in the same manner as in the preceding proposition,
that the angles *CBA, BAE, AED* are bisected by the straight lines *FB, FA, FE.*

And because the angle *BCD* is equal to the angle *CDE,*
and that *FCD* is the half of the angle *BCD,*
and *CDF* the half of *CDE;*
therefore the angle *FCD* is equal to *FDC;* (ax. 7.)
wherefore the side *CF* is equal to the side *FD:* (I. 6.)
in like manner it may be demonstrated,
that *FB, FA, FE,* are each of them equal to *FC* or *FD*:
therefore the five straight lines *FA, FB, FC, FD, FE* are equal to one another;
and the circle described from the centre *F*, at the distance of one of them, will pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon *ABCDE*. Q.E.F.

PROPOSITION XV. PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let *ABCDEF* be the given circle.
It is required to inscribe an equilateral and equiangular hexagon in it.



Find the centre *G* of the circle *ABCDEF*, and draw the diameter *AGD;* (III. 1.)

and from D, as a centre, at the distance DG, describe the circle $EGCH$,
join EG, CG, and produce them to the points B, F;
and join AB, BC, CD, DE, EF, FA:
the hexagon $ABCDEF$ shall be equilateral and equiangular.
Because G is the centre of the circle $ABCDEF$,
GE is equal to GD:
and because D is the centre of the circle $EGCH$,
DE is equal to DG:
wherefore GE is equal to ED, (ax. 1.)
and the triangle EGD is equilateral;
and therefore its three angles EGD, GDE, DEG, are equal to one another: (I. 5. Cor.)
but the three angles of a triangle are equal to two right angles; (I. 32.)
therefore the angle EGD is the third part of two right angles:
in the same manner it may be demonstrated,
that the angle DGC is also the third part of two right angles:
and because the straight line GC makes with EB the adjacent angles EGC, CGB equal to two right angles; (I. 13.)
the remaining angle CGB is the third part of two right angles:
therefore the angles EGD, DGC, CGB are equal to one another:
and to these are equal the vertical opposite angles BGA, AGF, FGE: (I. 15.)
therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another:
but equal angles stand upon equal circumferences; (III. 26.)
therefore the six circumferences AB, BC, CD, DE, EF, FA are equal to one another:
and equal circumferences are subtended by equal straight lines: (III. 29.)
therefore the six straight lines are equal to one another,
and the hexagon $ABCDEF$ is equilateral.
It is also equiangular:
for, since the circumference AF is equal to ED,
to each of these equals add the circumference $ABCD$;
therefore the whole circumference $FABCD$ is equal to the whole $EDCBA$:
and the angle FED stands upon the circumference $FABCD$,
and the angle AFE upon $EDCBA$;
therefore the angle AFE is equal to FED: (III. 27.)
in the same manner it may be demonstrated
that the other angles of the hexagon $ABCDEF$ are each of them equal to the angle AFE or FED:
therefore the hexagon is equiangular;
and it is equilateral, as was shewn;
and it is inscribed in the given circle $ABCDEF$. Q.E.F.

COR.—From this it is manifest, that the side of the hexagon is equal to the straight line from the centre, that is, to the semi-diameter of the circle.

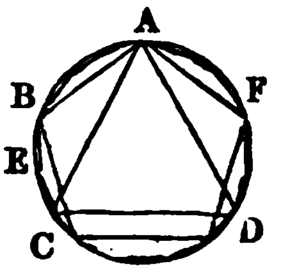
And if through the points A, B, C, D, E, F there be drawn straight lines touching the circle, an equilateral and equiangular hexagon will be described about it, which may be demonstrated from what has been said of the pentagon: and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

PROPOSITION XVI. PROBLEM.

o inscribe an equilateral and equiangular quindecagon in a given circle.

Let $ABCD$ be the given circle.

It is required to inscribe an equilateral and equiangular quindecagon i the circle $ABCD$.



et AC be the side of an equilateral triangle inscribed in the circle, (IV. 2.)
and AB the side of an equilateral and equiangular pentagon inscribed in the same: (IV. 11.)
therefore, of such equal parts as the whole circumference $ABCDF$ contains fifteen,
he circumference ABC, being the third part of the whole, contains five;
and the circumference AB, which is the fifth part of the whole, contains three;
therefore BC, their difference, contains two of the same parts:
bisect BC in E; (III. 30.)
therefore BE, EC are, each of them, the fifteenth part of the whole circumference $ABCD$:
therefore if the straight lines BE, EC be drawn, and straight lines qual to them be placed round in the whole circle, (IV. 1.) an equilateral nd equiangular quindecagon will be inscribed in it. Q.E.F.

And in the same manner as was done in the pentagon, if through the oints of division made by inscribing the quindecagon, straight lines be rawn touching the circle, an equilateral and equiangular quindecagon vill be described about it: and likewise, as in the pentagon, a circle may e inscribed in a given equilateral and equiangular quindecagon, and ircumscribed about it.

NOTES TO BOOK IV.

THE fourth book of the Elements contains some particular cases of four general problems on the inscription, and the circumscription of triangles and regular figures in and about circles. Euclid has not given any instances of the inscription or circumscription of rectilinear figures in and about other rectilinear figures.

Any rectilinear figure, of five sides and angles, is called a pentagon; of seven sides and angles, a heptagon; of eight sides and angles, an octagon; of nine sides and angles, a nonagon; of ten sides and angles, a decagon; of twelve sides and angles, a duodecagon; of fifteen sides and angles, a quindecagon, &c.

These figures are included under the general name of *polygons*; and are called *equilateral*, when their sides are equal; and *equiangular*, when their angles are equal; also when both their sides and angles are equal, they are called *regular polygons*.

Prop. III. An objection has been raised to the construction of this problem. It is said that in this and other instances of a similar kind, the lines which touch the circle at A, B, and C, should be proved to meet one another. This may be done by joining AB, and then since the angles KAM, KBM are equal to two right angles (III. 18.), therefore the angles BAM, ABM are less than two right angles, and consequently (ax. 12.), AM and BM must meet one another, when produced far enough. Similarly, it may be shewn that AL and CL, as also CN and BN meet one another.

Prop. V. The corollary to this proposition appears to have been already demonstrated in Prop. 31, Book III.

Prop. VI, VII. It is obvious that the square described about a circle is equal to double the square inscribed in the same circle. Also that the circumscribed square is equal to the square of the diameter, or four times the square of the radius of the circle.

Prop. VII. It is manifest that a square is the only right-angled parallelogram which can be circumscribed about a circle, but that both a rectangle and a square may be inscribed in a circle.

Prop. X. By means of this proposition, a right angle may be divided into five equal parts.

Prop. XVI. The arc subtending a side of the quindecagon, may be found by placing in the circle from the same point, two lines respectively equal to the sides of the regular hexagon and pentagon.

The centres of the inscribed and circumscribed circles of any regular polygon are coincident.

Besides the circumscription and inscription of triangles and regular polygons about and in circles, some very important problems are solved in the constructions respecting the division of the circumferences of circles into equal parts.

By inscribing an equilateral triangle, a square, a pentagon, a hexagon, &c. in a circle, the circumference is divided into three, four, five, six, &c. equal parts. In Prop. XXVI, Book III, it has been shewn that equal angles at the centres of equal circles, and therefore at the centre of the same circle, subtend equal arcs; by bisecting the angles at the centre, the arcs which are subtended by them are also bisected, and hence, a sixth, eighth, tenth, twelfth, &c. part of the circumference of a circle may be found.

By the aid of the first corollary to Prop. 32, Book I, may be found the magnitude of an interior angle of any regular polygon whatever.

Let θ denote the magnitude of one of the interior angles of a regular polygon of n sides,

then $n\theta$ is the sum of all the interior angles.

But all the interior angles of any rectilinear figure together with four right angles, are equal to twice as many right angles as the figure has sides,

that is, if we agree to assume π to designate two right angles,

$$\therefore\ n\theta + 2\pi = n\pi,$$

$$\text{and}\quad n\theta = n\pi - 2\pi = (n-2)\,.\,\pi,$$

$$\therefore\ \theta = \frac{(n-2)}{n}\,.\,\pi,$$

the magnitude of an interior angle of a regular polygon of n sides.

By taking $n = 3, 4, 5, 6$, &c. may be found the magnitude, in terms of two right angles, of an interior angle of any regular polygon whatever.

Pythagoras was the first, as Proclus informs us in his commentary, who discovered that a multiple of the angles of three regular figures only, namely, the trigon, the square, and the hexagon, can fill up space round a point in a plane.

It has been shewn that the interior angle of any regular polygon of n sides in terms of two right angles, is expressed by the equation

$$\theta = \frac{n-2}{n}\,.\,\pi.$$

Let θ_3 denote the magnitude of the interior angle of a regular figure of 3 sides.

$$\text{Then}\quad \theta_3 = \frac{3-2}{3} = \frac{\pi}{3} = \text{one third of two right angles},$$

$$\therefore\ 3\theta_3 = \pi,$$

$$\text{and}\quad 6\theta_3 = 2\pi,$$

that is, six angles each equal to the interior angle of an equilateral triangle are equal to four right angles, and therefore six equilateral triangles may be placed so as completely to fill up the space round the point at which they meet in a plane.

In a similar way, it may be shewn that four squares and three hexagons may be placed so as completely to fill up the space round a point.

Also it will appear from the results deduced, that no other regular figures besides these three, can be made to fill up the space round a point: for any multiple of the interior angles of any other regular polygon, will be found to be in excess above, or in defect from four right angles.

The equilateral triangle or trigon, the square or tetragon, the pentagon, and the hexagon, were the only regular polygons known to the Greeks, capable of being inscribed in circles, besides those which may be derived from them.

M. Gauss in his Disquisitiones Arithmeticæ, has extended the number by shewing that in general, a regular polygon of $2^n + 1$ sides is capable of being inscribed in a circle by means of straight lines and circles, in those cases in which $2^n + 1$ is a prime number.

BOOK V.

DEFINITIONS.

I.

A LESS magnitude is said to be *a part* of a greater magnitude when the less measures the greater; that is, 'when the less is contained a certain number of times exactly in the greater.'

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less, that is, 'when the greater contains the less a certain number of times exactly.'

III.

"Ratio is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity."

IV.

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth: or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth: or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

VI.

Magnitudes which have the same ratio are called proportionals. 'N.B. When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second, as the third to the fourth.'

VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

VIII.

"Analogy, or proportion, is the similitude of ratios."

IX.

Proportion consists in three terms at least.

X.

When three magnitudes are proportionals, the first is said to have to he third the duplicate ratio of that which it has to the second.

XI.

When four magnitudes are continual proportionals, the first is said o have to the fourth the triplicate ratio of that which it has to the econd, and so on, quadruplicate, &c. increasing the denomination still y unity, in any number of proportionals.

Definition *A*, to wit, of compound ratio.

When there are any number of magnitudes of the same kind, the irst is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has o the third, and of the ratio which the third has to the fourth, and so n unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said have to the last D the ratio compounded of the ratio of A to B, and of the ratio of to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of e ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F; and B to C the same ratio that has to H; and C to D the same that K has to L; then, by this definition, A is said have to D the ratio compounded of ratios which are the same with the ratios of to F, G to H, and K to L. And the same thing is to be understood when it is ore briefly expressed by saying, A has to D the ratio compounded of the ratios of to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which has to D; then, for shortness sake, M is said to have to N the ratio compounded of e ratios of E to F, G to H, and K to L.

XII.

In proportionals, the antecedent terms are called homologous to one nother, as also the consequents to one another.

'Geometers make use of the following technical words, to signify ertain ways of changing either the order or magnitude of proportionals, that they continue still to be proportionals.'

XIII.

Permutando, or alternando, by permutation or alternately. This ord is used when there are four proportionals, and it is inferred that he first has the same ratio to the third which the second has to the ourth; or that the first is to the third as the second to the fourth: as is hown in Prop. XVI. of this fifth book.

XIV.

Invertendo, by inversion; when there are four proportionals, and it s inferred, that the second is to the first, as the fourth to the third. Prop. B. Book. v.

XV.

Componendo, by composition; when there are four proportionals, and it is inferred that the first together with the second, is to the second, as the third together with the fourth, is to the fourth. Prop. 18, Book v.

XVI.

Dividendo, by division; when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second,

as the excess of the third above the fourth, is to the fourth. Prop. 17, Book v.

XVII.

Convertendo, by conversion; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. Prop. E. Book v.

XVIII.

Ex æquali (sc. distantiâ), or ex æquo, from equality of distance: when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: 'Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two.'

XIX.

Ex æquali, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in Prop. 22, Book v.

XX.

Ex æquali in proportione perturbatâ seu inordinatâ, from equality in perturbate or disorderly proportion*. This term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in a cross order: and the inference is as in the 18th definition. It is demonstrated in Prop. 23, Book v.

AXIOMS.

I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

II.

Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

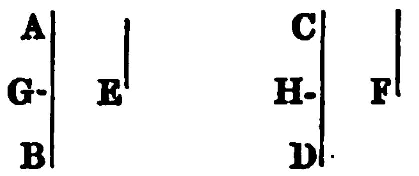
* Prop. 4. Lib. II. Archimedis de sphærâ et cylindro.

PROPOSITION I. THEOREM.

If any number of magnitudes be equimultiples of as many, each of ach; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.

Let any number of magnitudes AB, CD be equimultiples of as many thers E, F, each of each.

Then whatsoever multiple AB is of E,
the same multiple shall AB and CD together be of E and F together.



Because AB is the same multiple of E that CD is of F,
as many magnitudes as there are in AB equal to E, so many are there in CD equal to F.
Divide AB into magnitudes equal to E, viz. AG, GB;
and CD into CH, HD, equal each of them to F:
therefore the number of the magnitudes CH, HD shall be equal to the number of the others AG, GB:
and because AG is equal to E, and CH to F,
therefore AG and CH together are equal to E and F together: (I. ax. 2.)
for the same reason, because GB is equal to E, and HD to F;
GB and HD together are equal to E and F together;
wherefore as many magnitudes as there are in AB equal to E,
so many are there in AB, CD together, equal to E and F together:
therefore, whatsoever multiple AB is of E,
the same multiple is AB and CD together, of E and F together.

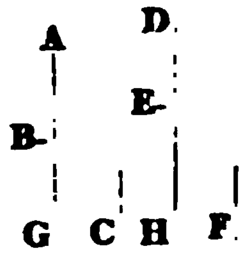
Therefore, if any magnitudes, how many soever, be equimultiples of as many, each of each; whatsoever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the others: 'For the same demonstration holds in any number of magnitudes, which was here applied to two.' Q.E.D.

PROPOSITION II. THEOREM.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

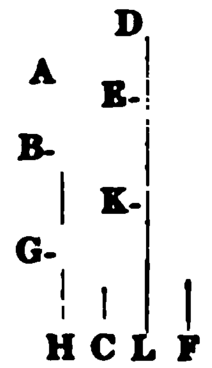
Let AB the first be the same multiple of C the second, that DE the third is of F the fourth:
and BG the fifth the same multiple of C the second, that EH the sixth is of F the fourth.

Then shall AG, the first together with the fifth, be the same multiple of C the second, that DH, the third together with the sixth, is of F the fourth.



Because AB is the same multiple of C that DE is of F;
there are as many magnitudes in AB equal to C, as there are in DE equal to F:
in like manner, as many as there are in BG equal to C, so many are there in EH equal to F:
therefore as many as there are in the whole AG equal to C,
so many are there in the whole DH equal to F:
therefore AG is the same multiple of C that DH is of F;
that is, AG, the first and fifth together, is the same multiple of the second C,
that DH, the third and sixth together, is of the fourth F.
If therefore, the first be the same multiple, &c. Q. E. D.

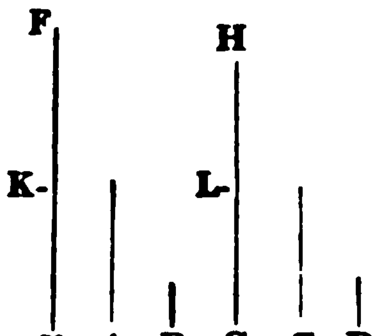
Cor. From this it is plain, that if any number of magnitudes AB, BG, GH be multiples of another C;
and as many DE, EK, KL be the same multiples of F, each of each:
then the whole of the first, viz. AH, is the same multiple of C,
that the whole of the last, viz. DL, is of F.



PROPOSITION III. THEOREM.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples; these shall be equimultiples, the one of the second, and the other of the fourth.

Let A the first be the same multiple of B the second, that C the third is of D the fourth;
and of A, C let equimultiples EF, GH be taken.
Then EF shall be the same multiple of B, that GH is of D.



Because EF is the same multiple of A, that GH is of C,
there are as many magnitudes in EF equal to A, as there are in GH equal to C:

let EF be divided into the magnitudes EK, KF, each equal to A;
and GH into GL, LH, each equal to C:
therefore the number of the magnitudes EK, KF shall be equal to the number of the others GL, LH:
and because A is the same multiple of B, that C is of D,
and that EK is equal to A, and GL equal to C;
therefore EK is the same multiple of B, that GL is of D:
for the same reason, KF is the same multiple of B, that LH is of D:
and so, if there be more parts in EF, GH, equal to A, C:
therefore, because the first EK is the same multiple of the second B, which the third GL is of the fourth D,
and that the fifth KF is the same multiple of the second B, which the sixth LH is of the fourth D;
EF the first, together with the fifth, is the same multiple of the second B, (v. 2.)
which GH the third, together with the sixth, is of the fourth D.
If, therefore, the first, &c. Q. E. D.

PROPOSITION IV. THEOREM.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. 'the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.'

Let A the first have to B the second, the same ratio which the third C has to the fourth D;
and of A and C let there be taken any equimultiples whatever E, F;
and of B and D any equimultiples whatever G, H.
Then E shall have the same ratio to G, which F has to H.

Take of E and F any equimultiples whatever K, L,
and of G, H any equimultiples whatever M, N:
then because E is the same multiple of A, that F is of C;
and of E and F have been taken equimultiples K, L;
therefore K is the same multiple of A, that L is of C: (v. 3.)
for the same reason, M is the same multiple of B, that N is of D.

And because, as A is to B, so is C to D, (hyp.)
and of A and C have been taken certain equimultiples K, L,
and of B and D have been taken certain equimultiples M, N;
therefore if K be greater than M, L is greater than N;
and if equal, equal; if less, less: (v. def. 5.)
but K, L are any equimultiples whatever of E, F, (constr.)
and M, N any whatever of G, H;
therefore as E is to G, so is F to H. (v. def. 5.)
Therefore, if the first, &c. Q.E.D.

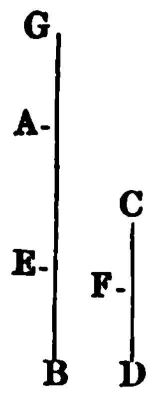
COR. Likewise, if the first has the same ratio to the second, which the third has to the fourth, then also any equimultiples whatever of the first and third shall have the same ratio to the second and fourth; and in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.

Let A the first have to B the second the same ratio which the third C has to the fourth D,
and of A and C let E and F be any equimultiples whatever.
Then E shall be to B as F to D.
Take of E, F any equimultiples whatever K, L,
and of B, D any equimultiples whatever G, H:
then it may be demonstrated, as before, that K is the same multiple of A, that L is of C:
and because A is to B, as C is to D, (hyp.)
and of A and C certain equimultiples have been taken, viz. K and L;
and of B and D certain equimultiples G, H;
therefore, if K be greater than G, L is greater than H;
and if equal, equal; if less, less: (v. def. 5.)
but K, L are any equimultiples whatever of E, F, (constr.)
and G, H any whatever of B, D;
therefore, as E is to B, so is F to D. (v. def. 5.)
And in the same way the other case is demonstrated.

PROPOSITION V. THEOREM.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder, that the whole is of the whole.

Let the magnitude AB be the same multiple of CD, that AE taken from the first, is of CF taken from the other.
The remainder EB shall be the same multiple of the remainder FD, that the whole AB is of the whole CD.



Take AG the same multiple of FD, that AE is of CF:
therefore AE is the same multiple of CF, that EG is of CD: (v. 1.)

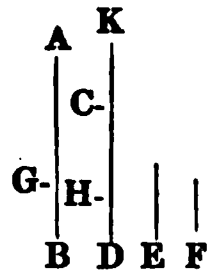
but AE, by the hypothesis, is the same multiple of CF, that AB is of CD;
therefore EG is the same multiple of CD that AB is of CD;
wherefore EG is equal to AB: (v. ax. 1.)
take from each of them the common magnitude AE;
and the remainder AG is equal to the remainder EB.
Wherefore, since AE is the same multiple of CF, that AG is of FD, (constr.)
and that AG has been proved equal to EB;
therefore AE is the same multiple of CF, that EB is of FD:
but AE is the same multiple of CF that AB is of CD: (hyp.)
therefore EB is the same multiple of FD, that AB is of CD.
Therefore, if one magnitude, &c. Q.E.D.

PROPOSITION VI. THEOREM.

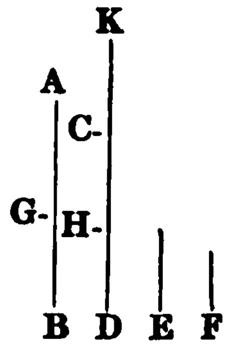
If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others, or equimultiples of them.

Let the two magnitudes AB, CD be equimultiples of the two E, F, and let AG, CH taken from the first two be equimultiples of the same E, F.

Then the remainders GB, HD shall be either equal to E, F, or equimultiples of them.



First, let GB be equal to E:
HD shall be equal to F.
Make CK equal to F:
and because AG is the same multiple of E, that CH is of F: (hyp.)
and that GB is equal to E, and CK to F;
therefore AB is the same multiple of E, that KH is of F:
but AB, by the hypothesis, is the same multiple of E, that CD is of F;
therefore KH is the same multiple of F, that CD is of F:
wherefore KH is equal to CD: (v. ax. 1.)
take away the common magnitude CH,
then the remainder KC is equal to the remainder HD:
but KC is equal to F; (constr.)
therefore HD is equal to F.
Next let GB be a multiple of E.
Then HD shall be the same multiple of F.



Make CK the same multiple of F, that GB is of E:
and because AG is the same multiple of E, that CH is of F; (hyp.)
and GB the same multiple of E, that CK is of F;
therefore AB is the same multiple of E, that KH is of F: (v. 2.)
but AB is the same multiple of E, that CD is of F; (hyp.)
therefore KH is the same multiple of F, that CD is of F;
wherefore KH is equal to CD: (v. ax. 1.)
take away CH from both;
therefore the remainder KC is equal to the remainder HD:
and because GB is the same multiple of E, that KC is of F, (constr.)
and that KC is equal to HD;
therefore HD is the same multiple of F, that GB is of E.
If, therefore, two magnitudes, &c. Q. E. D.

PROPOSITION A. THEOREM.

If the first of four magnitudes has the same ratio to the second, which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Take any equimultiples of each of them, as the doubles of each:
then, by def. 5th of this book, if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth:
but, if the first be greater than the second,
the double of the first is greater than the double of the second;
wherefore also the double of the third is greater than the double of the fourth;
therefore the third is greater than the fourth:
in like manner, if the first be equal to the second, or less than it,
the third can be proved to be equal to the fourth, or less than it.
Therefore, if the first, &c. Q. E. D.

PROPOSITION B. THEOREM.

If four magnitudes are proportionals, they are proportionals also when taken inversely.

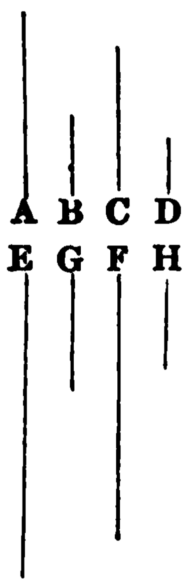
Let A be to B, as C is to D.
Then also inversely, B shall be to A, as D to C.

Take of B and D any equimultiples whatever E and F;
and of A and C any equimultiples whatever G and H.
First, let E be greater than G, then G is less than E:
and because A is to B, as C is to D, (hyp.)
nd of A and C, the first and third, G and H are equimultiples;
l of B and D, the second and fourth, E and F are equimultiples;
d that G is less than E, therefore H is less than F; (v. def. 5.)
that is, F is greater than H;
if, therefore, E be greater than G,
F is greater than H;
in like manner, if E be equal to G,
F may be shewn to be equal to H;
and if less, less;
ut E, F, are any equimultiples whatever of B and D, (constr.)
and G, H any whatever of A and C;
therefore, as B is to A, so is D to C. (v. def. 5.)
Therefore, if four magnitudes, &c. Q.E.D.

PROPOSITION C. THEOREM.

f the first be the same multiple of the second, or the same part of it, the third is of the fourth; the first is to the second, as the third is to ourth.

Let the first A be the same multiple of the second B,
that the third C is of the fourth D.
Then A shall be to B as C is to D.



Take of A and C any equimultiples whatever E and F;
and of B and D any equimultiples whatever G and H.
en, because A is the same multiple of B that C is of D; (hyp.)
nd that E is the same multiple of A, that F is of C; (constr.)
therefore E is the same multiple of B, that F is of D; (v. 3.)
that is, E and F are equimultiples of B and D:
but G and H are equimultiples of B and D; (constr.)
therefore, if E be a greater multiple of B than G is of B,
F is a greater multiple of D than H is of D;
that is, if E be greater than G,
F is greater than H:
in like manner, if E be equal to G, or less than it,
F may be shewn to be equal to H, or less than it:

but E, F are equimultiples, any whatever, of A, C; (cc
and G, H any equimultiples whatever of B, D;
therefore A is to B, as C is to D. (v. def. 5.)

Next, let the first A be the same part of the second B, tha
C is of the fourth D.

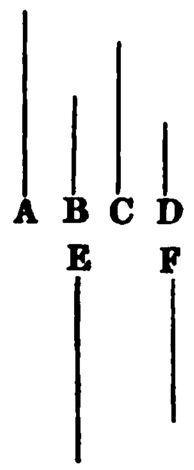
Then A shall be to B, as C is to D.

For since A is the same part of B that C is of D,
therefore B is the same multiple of A, that D is of C
wherefore, by the preceding case, B is to A, as D is to
and therefore inversely A is to B, as C is to D. (v. I
Therefore, if the first be the same multiple, &c. Q. E.

PROPOSITION D. THEOREM.

If the first be to the second as the third to the fourth, a first be a multiple, or a part of the second; the third is the sam or the same part of the fourth.

Let A be to B as C is to D:
and first let A be a multiple of B.
Then C shall be the same multiple of D.



Take E equal to A,
and whatever multiple A or E is of B, make F the same mul
then, because A is to B, as C is to D; (hyp.)
and of B the second, and D the fourth, equimultiples
taken, E and F;
therefore A is to E, as C to F: (v. 4. Cor.)
but A is equal to E, (constr.)
therefore C is equal to F: (v. A.)
and F is the same multiple of D, that A is of B; (coı
therefore C is the same multiple of D, that A is of B

Next, let A the first be a part of B the second.
Then C the third shall be the same part of D the fou
Because A is to B, as C is to D; (hyp.)
then, inversely, B is to A, as D to C: (v. B.)
but A is a part of B, therefore B is a multiple of A: (

therefore, by the preceding case, D is the same multiple of C;
that is, C is the same part of D, that A is of B.
Therefore, if the first, &c. Q.E.D.

PROPOSITION VII. THEOREM.

Equal magnitudes have the same ratio to the same magnitude: and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other.
Then A and B shall each of them have the same ratio to C:
and C shall have the same ratio to each of the magnitudes A and B.

Take of A and B any equimultiples whatever D and E,
and of C any multiple whatever F.
Then, because D is the same multiple of A, that E is of B, (constr.)
and that A is equal to B: (hyp.)
therefore D is equal to E: (v. ax. 1.)
therefore, if D be greater than F, E is greater than F;
and if equal, equal; if less, less;
but D, E are any equimultiples of A, B, (constr.)
and F is any multiple of C;
therefore, as A is to C, so is B to C. (v. def. 5.)
Likewise C shall have the same ratio to A, that it has to B.
For having made the same construction,
D may in like manner be shewn to be equal to E:
therefore, if F be greater than D,
it is likewise greater than E;
and if equal, equal; if less, less:
but F is any multiple whatever of C,
and D, E are any equimultiples whatever of A, B;
therefore, C is to A as C is to B. (v. def. 5.)
Therefore, equal magnitudes, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

Of two unequal magnitudes, the greater has a greater ratio to any other magnitude than the less has: and the same magnitude has a greater ratio to the less of two other magnitudes, than it has to the greater.

Let AB, BC be two unequal magnitudes, of which AB is the greater,
and let D be any other magnitude.
Then AB shall have a greater ratio to D than BC has to D:
and D shall have a greater ratio to BC than it has to AB.

Fig. 1.

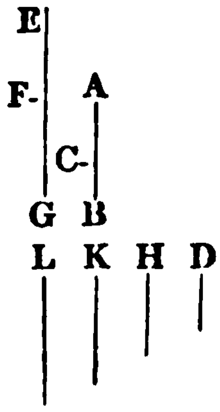


Fig. 2.

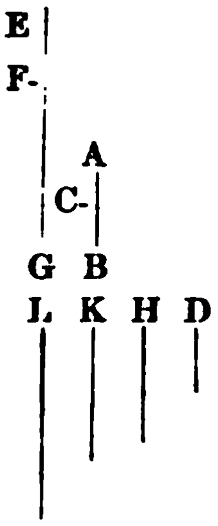
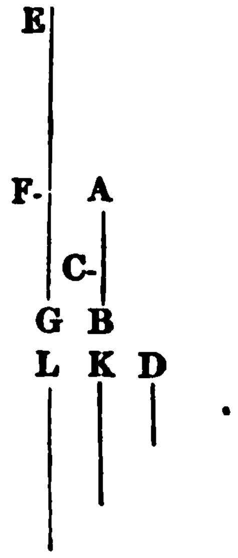


Fig. 3.



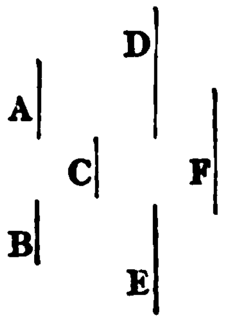
If the magnitude which is not the greater of the two AC, CB, be not less than D,
take EF, FG, the doubles of AC, CB, (as in fig. 1.)
But if that which is not the greater of the two AC, CB, be less than D,
(as in fig. 2 and 3.) this magnitude can be multiplied, so as to become greater than D, whether it be AC or CB.
Let it be multiplied until it become greater than D,
and let the other be multiplied as often;
and let EF be the multiple thus taken of AC,
and FG the same multiple of CB:
therefore EF and FG are each of them greater than D:
and in every one of the cases,
take H the double of D, K its triple, and so on,
till the multiple of D be that which first becomes greater than FG:
let L be that multiple of D which is first greater than FG,
and K the multiple of D which is next less than L.
Then because L is the multiple of D, which is the first that becomes greater than FG,
the next preceding multiple K is not greater than FG:
that is, FG is not less than K:
and since EF is the same multiple of AC, that FG is of CB; (constr.)
therefore FG is the same multiple of CB, that EG is of AB; (v. 1.)
that is, EG and FG are equimultiples of AB and CB:
and since it was shewn, that FG is not less than K,
and, by the construction, EF is greater than D;
therefore the whole EG is greater than K and D together:
but K together with D is equal to L; (constr.)
therefore EG is greater than L:
but FG is not greater than L: (constr.)
and EG, FG were proved to be equimultiples of AB, BC;
and L is a multiple of D; (constr.)
therefore AB has to D a greater ratio than BC has to D. (v. def. 7.)
Also D shall have to BC a greater ratio than it has to AB.
For having made the same construction,
it may be shewn, in like manner, that L is greater than FG,
but that it is not greater than EG:
and L is a multiple of D; (constr.)

and FG, EG were proved to be equimultiples of CB, AB:
erefore D has to CB a greater ratio than it has to AB. (v. def. 7.)
Wherefore, of two unequal magnitudes, &c. Q.E.D.

PROPOSITION IX. THEOREM.

Magnitudes which have the same ratio to the same magnitude are equal ne another: and those to which the same magnitude has the same ratio equal to one another.

Let A, B have each of them the same ratio to C.
Then A shall be equal to B.

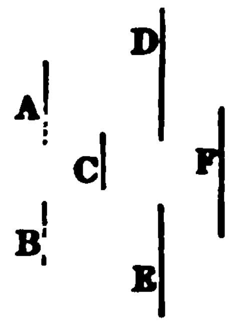


r, if they are not equal, one of them must be greater than the other:
let A be the greater:
then, by what was shewn in the preceding proposition,
ere are some equimultiples of A and B, and some multiple of C, such,
that the multiple of A is greater than the multiple of C,
but the multiple of B is not greater than that of C.
Let these multiples be taken;
and let D, E be the equimultiples of A, B,
and F the multiple of C,
such that D may be greater than F, but E not greater than F.
Then, because A is to C as B is to C, (hyp.)
and of A, B, are taken equimultiples D, E,
and of C is taken a multiple F;
and that D is greater than F;
therefore E is also greater than F: (v. def. 5.)
but E is not greater than F; (constr.) which is impossible:
therefore A and B are not unequal; that is, they are equal.
ext, let C have the same ratio to each of the magnitudes A and B.
Then A shall be equal to B.
, if they are not equal, one of them must be greater than the other:
let A be the greater:
therefore, as was shewn in Prop. VIII.
there is some multiple F of C,
and some equimultiples E and D of B and A such,
that F is greater than E, but not greater than D:
and because C is to B, as C is to A, (hyp.)
and that F the multiple of the first, is greater than E the multiple of the second;
therefore F the multiple of the third, is greater than D the multiple of the fourth: (v. def. 5.)
but F is not greater than D (hyp.); which is impossible:
therefore A is equal to B.
Wherefore, magnitudes which, &c. Q.E.D.

PROPOSITION X. THEOREM.

That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the lesser of the two.

Let A have to C a greater ratio than B has to C.
Then A shall be greater than B.

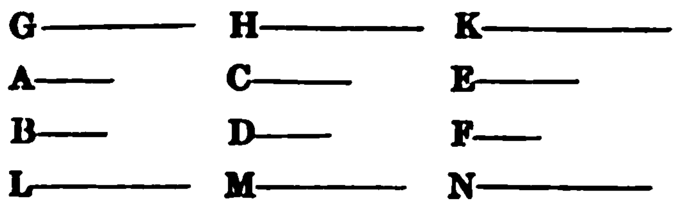


For, because A has a greater ratio to C, than B has to C,
there are some equimultiples of A and B, and some multiple of C such, (v. def. 7.)
that the multiple of A is greater than the multiple of C,
but the multiple of B is not greater than it:
let them be taken;
and let D, E be the equimultiples of A, B, and F the multiple of C;
such, that D is greater than F, but E is not greater than F:
therefore D is greater than E:
and, because D and E are equimultiples of A and B,
and that D is greater than E;
therefore A is greater than B. (v. ax. 4.)
Next, let C have a greater ratio to B than it has to A.
Then B shall be less than A.
For there is some multiple F of C, (v. def. 7.)
and some equimultiples E and D of B and A such,
that F is greater than E, but not greater than D:
therefore E is less than D:
and because E and D are equimultiples of B and A,
and that E is less than D,
therefore B is less than A. (v. ax. 4.)
Therefore, that magnitude, &c. Q.E.D.

PROPOSITION XI. THEOREM.

Ratios that are the same to the same ratio, are the same to one another.

Let A be to B as C is to D;
and as C to D, so let E be to F.



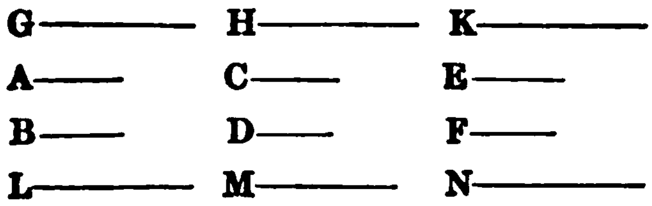
Then A shall be to B, as E to F.
Take of A, C, E, any equimultiples whatever G, H, K;

and of *B, D, F,* any equimultiples whatever *L, M, N.*
Therefore, since *A* is to *B* as *C* to *D,*
and *G, H* are taken equimultiples of *A, C,*
and *L, M,* of *B, D;*
if *G* be greater than *L, H* is greater than *M;*
and if equal, equal; and if less, less. (v. def. 5.)
Again, because *C* is to *D,* as *E* is to *F,*
and *H, K* are taken equimultiples of *C, E;*
and *M, N,* of *D, F;*
if *H* be greater than *M, K* is greater than *N;*
and if equal, equal; and if less, less:
but if *G* be greater than *L,*
it has been shewn that *H* is greater than *M;*
and if equal, equal; and if less, less:
therefore, if *G* be greater than *L,*
K is greater than *N;* and if equal, equal; and if less, less:
and *G, K* are any equimultiples whatever of *A, E;*
and *L, N* any whatever of *B, F:*
therefore, as *A* is to *B,* so is *E* to *F.* (v. def. 5.)
Wherefore, ratios that, &c. Q.E.D.

PROPOSITION XII. THEOREM.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let any number of magnitudes *A, B, C, D, E, F,* be proportionals;
that is, as *A* is to *B,* so *C* to *D,* and *E* to *F.*
Then as *A* is to *B,* so shall *A, C, E* together, be to *B, D, F* together.



Take of *A, C, E* any equimultiples whatever *G, H, K;*
and of *B, D, F* any equimultiples whatever, *L, M, N.*
Then, because *A* is to *B,* as *C* is to *D,* and as *E* to *F;*
and that *G, H, K* are equimultiples of *A, C, E,*
and *L, M, N,* equimultiples of *B, D, F;*
therefore, if *G* be greater than *L,*
H is greater than *M,* and *K* greater than *N;*
and if equal, equal; and if less, less: (v. def. 5.)
wherefore if *G* be greater than *L,*
then *G, H, K* together, are greater than *L, M, N* together;
and if equal, equal; and if less, less:
but *G,* and *G, H, K* together, are any equimultiples of *A,* and *A, C, E* together;
because if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the same multiple is the whole of the whole: (v. 1.)

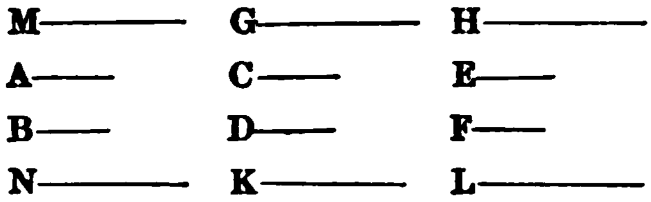
for the same reason L, and L, M, N are any equimultiples of B, and B, D, F:
therefore as A is to B, so are A, C, E together to B, D, F together. (v. def. 5.)
Wherefore, if any number, &c. Q. E. D.

PROPOSITION XIII. THEOREM.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

Let A the first have the same ratio to B the second, which C the third has to D the fourth, but C the third a greater ratio to D the fourth, than E the fifth has to F the sixth.

Then also the first A shall have to the second B, a greater ratio than the fifth E has to the sixth F.



Because C has a greater ratio to D, than E to F,
there are some equimultiples of C and E, and some of D and F such, that the multiple of C is greater than the multiple of D, but the multiple of E is not greater than the multiple of F: (v. def. 7.)
let these be taken,
and let G, H be equimultiples of C, E,
and K, L equimultiples of D, F, such that D may be greater than K, but H not greater than L:
and whatever multiple G is of C, take M the same multiple of A;
and whatever multiple K is of D, take N the same multiple of B:
then, because A is to B, as C to D, (hyp.)
and of A and C, M and G are equimultiples;
and of B and D, N and K are equimultiples;
therefore, if M be greater than N, G is greater than K;
and if equal, equal; and if less, less: (v. def. 5.)
but G is greater than K; (constr.)
therefore M is greater than N:
but H is not greater than L: (constr.)
and M, H are equimultiples of A, E;
and N, L equimultiples of B, F;
therefore A has a greater ratio to B, than E has to F. (v. def. 7.)
Wherefore, if the first, &c. Q. E. D.

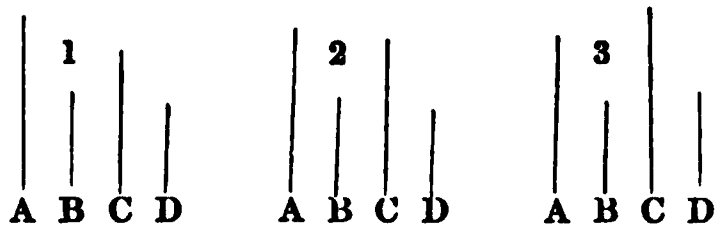
COR. And if the first have a greater ratio to the second, than the third has to the fourth, but the third the same ratio to the fourth, which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has a greater ratio to the second, than the fifth has to the sixth.

PROPOSITION XIV. THEOREM.

If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let the first A have the same ratio to the second B which the third C has to the fourth D.

If A be greater than C, B shall be greater than D.

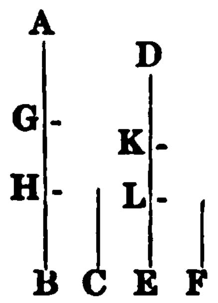


Because A is greater than C, and B is any other magnitude,
A has to B a greater ratio than C has to B: (v. 8.)
but, as A is to B, so is C to D; (hyp.)
therefore also C has to D a greater ratio than C has to B: (v. 13.)
but of two magnitudes, that to which the same has the greater ratio is the less: (v. 10.)
therefore D is less than B;
that is, B is greater than D.
Secondly, if A be equal to C,
then B shall be equal to D.
For A is to B, as C, that is, A to D:
therefore B is equal to D. (v. 9.)
Thirdly, if A be less than C,
then B shall be less than D.
For C is greater than A;
and because C is to D, as A is to B,
therefore D is greater than B, by the first case;
that is, B is less than D.
Therefore, if the first, &c. Q.E.D.

PROPOSITION XV. THEOREM.

Magnitudes have the same ratio to one another which their equimultiples have.

Let AB be the same multiple of C, that DE is of F.
Then C shall be to F, as AB to DE.



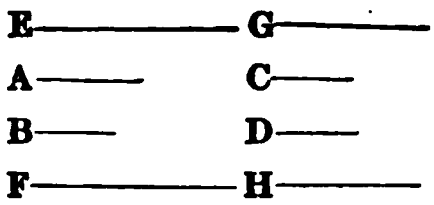
Because AB is the same multiple of C, that DE is of F;
there are as many magnitudes in AB equal to C, as there are in DE equal to F:
let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB;

and DE into magnitudes, each equal to F, viz. DK, KL, LE:
then the number of the first AG, GH, HB, is equal to the number of the last DK, KL, LE:
and because AG, GH, HB are all equal,
and that DK, KL, LE, are also equal to one another;
therefore AG is to DK as GH to KL, and as HB to LE: (v. 7.)
but as one of the antecedents is to its consequent, so are all the antecedents together to all the consequents together, (v. 12.)
wherefore, as AG is to DK, so is AB to DE:
but AG is equal to C, and DK to F;
therefore, as C is to F, so is AB to DE.
Therefore, magnitudes, &c. Q.E.D.

PROPOSITION XVI. THEOREM.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

Let A, B, C, D be four magnitudes of the same kind, which are proportionals, viz. as A to B, so C to D.
They shall also be proportionals when taken alternately;
that is, A shall be to C, as B to D.

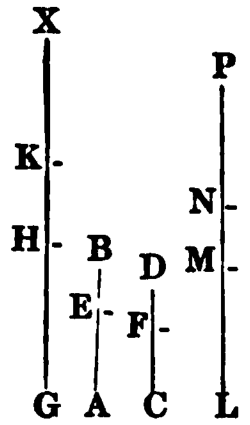


Take of A and B any equimultiples whatever E and F;
and of C and D take any equimultiples whatever G and H:
and because E is the same multiple of A, that F is of B,
and that magnitudes have the same ratio to one another which their equimultiples have; (v. 15.)
therefore A is to B, as E is to F:
but as A is to B so is C to D; (hyp.)
wherefore as C is to D, so is E to F: (v. 11.)
again, because G, H are equimultiples of C, D,
therefore as C is to D, so is G to H: (v. 15.)
but it was proved that as C is to D, so is E to F;
therefore, as E is to F, so is G to H. (v. 11.)
But when four magnitudes are proportionals, if the first be greater than the third, the second is greater than the fourth;
and if equal, equal; if less, less; (v. 14.)
therefore, if E be greater than G, F likewise is greater than H;
and if equal, equal; if less, less:
and E, F are any equimultiples whatever of A, B; (constr.)
and G, H any whatever of C, D:
therefore A is to C as B to D. (v. def. 5.)
If, then, four magnitudes, &c. Q.E.D.

PROPOSITION XVII. THEOREM.

If magnitudes, taken jointly, be proportionals, they shall also be pro-
ortionals when taken separately: that is, if two magnitudes together have
one of them, the same ratio which two others have to one of these, the
maining one of the first two shall have to the other the same ratio which
e remaining one of the last two has to the other of these.

Let AB, BE, CD, DF be the magnitudes, taken jointly which are roportionals;
that is, as AB to BE, so let CD be to DF.
Then they shall also be proportionals taken separately,
viz. as AE to EB, so shall CF be to FD.



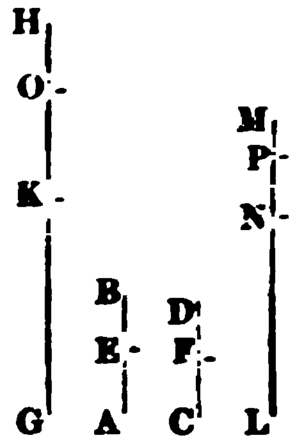
Take of AE, EB, CF, FD any equimultiples whatever GH, HK, LM, MN;
and again, of EB, FD take any equimultiples whatever KX, NP.
Then because GH is the same multiple of AE, that HK is of EB,
therefore GH is the same multiple of AE, that GK is of AB: (v. 1.)
but GH is the same multiple of AE, that LM is of CF;
therefore GK is the same multiple of AB, that LM is of CF.
Again, because LM is the same multiple of CF, that MN is of FD;
therefore LM is the same multiple of CF, that LN is of CD: (v. 1.)
ut LM was shewn to be the same multiple of CF, that GK is of AB;
therefore GK is the same multiple of AB, that LN is of CD;
that is, GK, LN are equimultiples of AB, CD.
Next, because HK is the same multiple of EB, that MN is of FD;
and that KX is also the same multiple of EB, that NP is of FD;
therefore HX is the same multiple of EB, that MP is of FD. (v. 2.)
And because AB is to BE as CD is to DF, (hyp.)
and that of AB and CD, GK and LN are equimultiples,
and of EB and FD, HX and MP are equimultiples;
therefore if GK be greater than HX, then LN is greater than MP;
and if equal, equal; and if less, less: (v. def. 5.)
but if GH be greater than KX,
then, by adding the common part HK to both,
GK is greater than HX; (I. ax. 4.)
wherefore also LN is greater than MP;
and by taking away MN from both,
LM is greater than NP; (I. ax. 5.)
therefore, if GH be greater than KX,
LM is greater than NP.
In like manner it may be demonstrated,
that if GH be equal to KX,

LM is equal to *NP*; and if less, less:
but *GH, LM* are any equimultiples whatever of *AE, CF*, (const
and *KX, NP* are any whatever of *EB, FD*:
therefore, as *AE* is to *EB*, so is *CF* to *FD*. (v. def. 5.)
If then, magnitudes, &c. Q. E. D.

PROPOSITION XVIII. THEOREM.

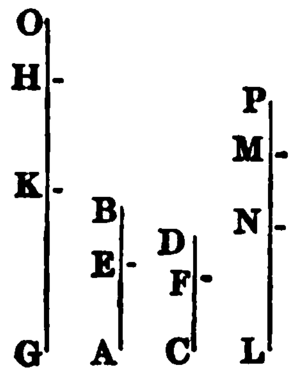
If magnitudes, taken separately, be proportionals, they shall als proportionals when taken jointly: that is, if the first be to the secon the third to the fourth, the first and second together shall be to the sec as the third and fourth together to the fourth.

Let *AE, EB, CF, FD* be proportionals;
that is, as *AE* to *EB*, so let *CF* be to *FD*.
Then they shall also be proportionals when taken jointly;
that is, as *AB* to *BE*, so shall *CD* be to *DF*.

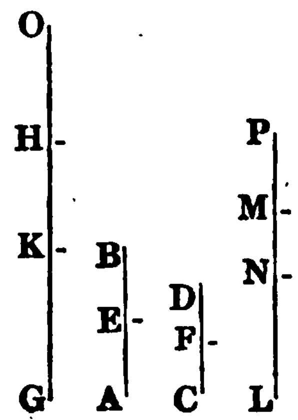


Take of *AB, BE, CD, DF* any equimultiples whatever *GH, I LM, MN*;
and again, of *BE, DF*, take any equimultiples whatever *KO, NP*
and because *KO, NP* are equimultiples of *BE, DF*,
and that *KH, NM* are likewise equimultiples of *BE, DF*;
therefore if *KO*, the multiple of *BE*, be greater than *KH*, wh is a multiple of the same *BE*,
then *NP*, the multiple of *DF*, is also greater than *NM*, the multi of the same *DF*;
and if *KO* be equal to *KH*,
NP is equal to *NM*; and if less, less.
First, let *KO* be not greater than *KH*;
therefore *NP* is not greater than *NM*:
and because *GH, HK*, are equimultiples of *AB, BE*, and that is greater than *BE*,
therefore *GH* is greater than *HK*; (v. ax. 3.)
but *KO* is not greater than *KH*;
therefore *GH* is greater than *KO*.
In like manner it may be shewn, that *LM* is greater than *NP*.
Therefore, if *KO* be not greater than *KH*,
then *GH*, the multiple of *AB*, is always greater than *KO*, the mult of *BE*;
and likewise *LM*, the multiple of *CD*, is greater than *NP*, multiple of *DF*.

Next, let KO be greater than KH;
therefore, as has been shewn, NP is greater than NM.



And because the whole GH is the same multiple of the whole AB, that HK is of BE,
therefore the remainder GK is the same multiple of the remainder AE that GH is of AB. (v. 5.)
which is the same that LM is of CD.
In like manner, because LM is the same multiple of CD, that MN is of DE,
therefore the remainder LN is the same multiple of the remainder CF, that the whole LM is of the whole CD: (v. 5.)
but it was shewn that LM is the same multiple of CD, that GK is of AE;
therefore GK is the same multiple of AE, that LN is of CF;
that is, GK, LN are equimultiples of AE, CF.
And because KO, NP are equimultiples of BE, DF,
therefore if from KO, NP there be taken KH, NM, which are likewise equimultiples of BE, DF,
the remainders HO, MP are either equal to BE, DF, or equimultiples of them. (v. 6.)
First, let HO, MP be equal to BE, DF:
then because AE is to EB, as CF to FD, (hyp.)
and that GK, LN are equimultiples of AE, CF;
therefore GK is to EB, as LN to FD: (v. 4. Cor.)
but HO is equal to EB, and MP to FD;
wherefore GK is to HO, as LN to MP:
therefore if GK be greater than HO, LN is greater than MP; (5. A.)
and if equal, equal; and if less, less.
But let HO, MP be equimultiples of EB, FD.
Then because AE is to EB, as CF to FD, (hyp.)



and that of AE, CF are taken equimultiples GK, LN;
and of EB, FD, the equimultiples HO, MP;
if GK be greater than HO, LN is greater than MP;
and if equal, equal; and if less, less; (v. def. 5.)
which was likewise shewn in the preceding case.
But if GH be greater than KO,

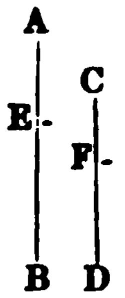
taking KH from both, GK is greater than HO: (I. ax. 5.)
wherefore also LN is greater than MP;
and consequently adding NM to both,
LM is greater than NP: (I. ax. 4.)
therefore, if GH be greater than KO,
LM is greater than NP.
In like manner it may be shewn, that if GH be equal to KO,
LM is equal to NP; and if less, less.
And in the case in which KO is not greater than KH,
it has been shewn that GH is always greater than KO,
and likewise LM greater than NP:
but GH, LM are any equimultiples whatever of AB, CD, (constr.)
and KO, NP are any whatever of BE, DF;
therefore, as AB is to BE, so is CD to DF. (v. def. 5.)
If then magnitudes, &c. Q.E.D.

PROPOSITION XIX. THEOREM.

If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall be to the remainder as the whole to the whole.

Let the whole AB be to the whole CD, as AE a magnitude taken from AB is to CF a magnitude taken from CD.

Then the remainder EB shall be to the remainder FD, as the whole AB to the whole CD.



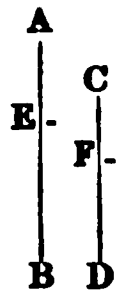
Because AB is to CD, as AE to CF:
therefore alternately, BA is to AE, as DC to CF: (v. 16.)
and because if magnitudes taken jointly be proportionals, they are also proportionals, when taken separately; (v. 17.)
therefore, as BE is to EA, so is DF to FC;
and alternately, as BE is to DF, so is EA to FC:
but, as AE to CF, so, by the hypothesis, is AB to CD;
therefore also BE the remainder is to the remainder DF, as the whole AB to the whole CD. (v. 11.)
Wherefore, if the whole, &c. Q.E.D.

COR.—If the whole be to the whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall likewise be to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

PROPOSITION E. THEOREM.

If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let AB be to BE, as CD to DF.
Then BA shall be to AE, as DC to CF.

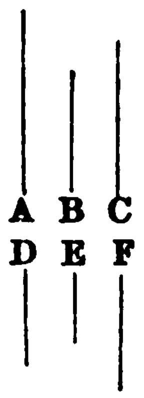


Because AB is to BE, as CD to DF,
therefore by division, AE is to EB, as CF to FD; (v. 17.)
and by inversion, BE is to EA, as DF to FC; (v. B.)
wherefore, by composition, BA is to AE, as DC is to CF. (v. 18.)
If therefore four, &c. Q.E.D.

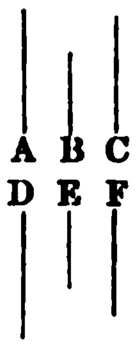
PROPOSITION XX. THEOREM.

If there be three magnitudes, and other three, which, taken two and vo, have the same ratio; then if the first be greater than the third, the urth shall be greater than the sixth; and if equal, equal; and if less, ss.

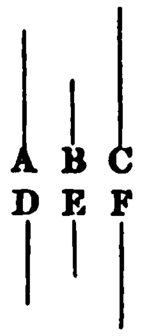
Let A, B, C be three magnitudes, and D, E, F other three, which taken two and two have the same ratio,
viz. as A is to B, so is D to E;
and as B to C, so is E to F.
If A be greater than C, D shall be greater than F;
and if equal, equal; and if less, less.



Because A is greater than C, and B is any other magnitude,
and that the greater has to the same magnitude a greater ratio than the less has to it; (v. 8.)
therefore A has to B a greater ratio than C has to B:
but as D is to E, so is A to B; (hyp.)
therefore D has to E a greater ratio than C to B: (v. 13.)
and because B is to C, as E to F,
by inversion, C is to B, as F is to E: (v. B.)
and D was shewn to have to E a greater ratio than C to B:
therefore D has to E a greater ratio than F to E: (v. 13. cor.)
but the magnitude which has a greater ratio than another to the same magnitude, is the greater of the two; (v. 10.)
therefore D is greater than F.
Secondly, let A be equal to C.
Then D shall be equal to F.



Because A and C are equal to one another,
A is to B, as C is to B: (v. 7.)
but A is to B, as D to E; (hyp.)
and C is to B, as F to E; (hyp.)
wherefore D is to E, as F to E; (v. 11. and v. B.)
and therefore D is equal to F. (v. 9.)
Next, let A be less than C.
Then D shall be less than F.

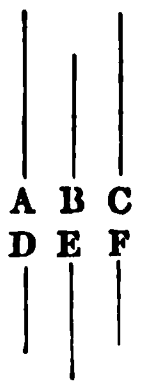


For C is greater than A;
and as was shewn in the first case, C is to B, as F to E,
and in like manner, B is to A, as E to D;
therefore F is greater than D, by the first case;
that is, D is less than F.
Therefore, if there be three, &c. Q.E.D.

PROPOSITION XXI. THEOREM.

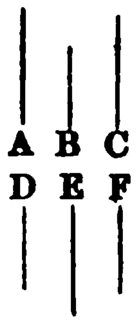
If there be three magnitudes, and other three, which have the s ratio taken two and two, but in a cross order; then if the first magni be greater than the third, the fourth shall be greater than the sixth; if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, w have the same ratio, taken two and two, but in a cross order,
viz. as A is to B, so is E to F,
and as B is to C, so is D to E.
If A be greater than C, D shall be greater than F;
and if equal, equal; and if less, less.



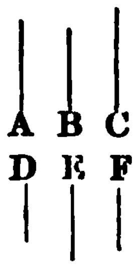
Because A is greater than C, and B is any other magnitude,
A has to B a greater ratio than C has to B: (v. 8.)
but as E to F, so is A to B; (hyp.)
therefore E has to F a greater ratio than C to B: (v. 13.)
and because B is to C, as D to E; (hyp.)
by inversion, C is to B, as E to D:
and E was shewn to have to F a greater ratio than C has to B;
therefore E has to F a greater ratio than E has to D: (v. 13. Cor.)
but the magnitude to which the same has a greater ratio than it has to another, is the less of the two: (v. 10.)
therefore F is less than D;
that is, D is greater than F.

Secondly, Let A be equal to C;
D shall be equal to F.



Because A and C are equal,
A is to B, as C is to B: (v. 7.)
but A is to B, as E to F; (hyp.)
and C is to B, as E to D;
wherefore E is to F, as E to D; (v. 11.)
and therefore D is equal to F. (v. 9.)

Next, let A be less than C:
D shall be less than F.



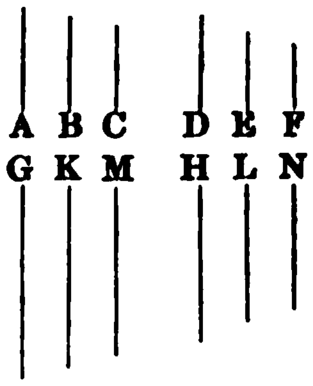
For C is greater than A;
and, as was shewn, C is to B, as E to D,
and in like manner B is to A, as F to E;
therefore F is greater than D, by case first;
that is, D is less than F.
Therefore, if there be three, &c. Q.E.D.

PROPOSITION XXII. THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes, the same ratio which the first has to the last of the others. *N. B.* This is usually cited by the words "ex æquali," or "ex æquo."

First, let there be three magnitudes A, B, C, and as many others D, E, F, which taken two and two in order, have the same ratio,

that is, such that *A* is to *B* as *D* to *E*;
and as *B* is to *C*, so is *E* to *F*.
Then *A* shall be to *C*, as *D* to *F*.



Take of *A* and *D* any equimultiples whatever *G* and *H*;
and of *B* and *E* any equimultiples whatever *K* and *L*;
and of *C* and *F* any whatever *M* and *N*:
then because *A* is to *B*, as *D* to *E*,
and that *G*, *H* are equimultiples of *A*, *D*,
and *K*, *L* equimultiples of *B*, *E*;
therefore as *G* is to *K*, so is *H* to *L*: (v. 4.)
for the same reason, *K* is to *M* as *L* to *N*:
and because there are three magnitudes *G*, *K*, *M*, and other three
H, *L*, *N*, which two and two, have the same ratio;
therefore if *G* be greater than *M*, *H* is greater than *N*;
and if equal, equal; and if less, less; (v. 20.)
but *G*, *H* are any equimultiples whatever of *A*, *D*,
and *M*, *N* are any equimultiples whatever of *C*, *F*; (constr.)
therefore, as *A* is to *C*, so is *D* to *F*. (v. def. 5.)
Next, let there be four magnitudes, *A*, *B*, *C*, *D*,
and other four *E*, *F*, *G*, *H*, which two and two have the same ratio,
viz. as *A* is to *B*, so is *E* to *F*;
and as *B* to *C*, so *F* to *G*;
and as *C* to *D*, so *G* to *H*:
Then *A* shall be to *D*, as *E* to *H*.

A.B.C.D
E.F.G.H

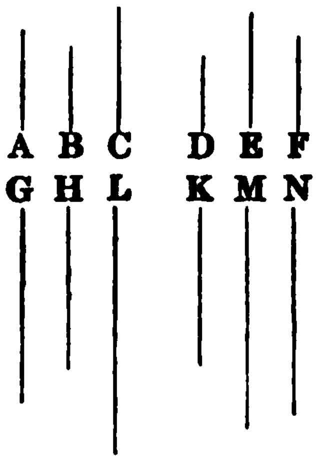
Because *A*, *B*, *C* are three magnitudes, and *E*, *F*, *G* other three, which taken two and two, have the same ratio;
therefore by the foregoing case, *A* is to *C*, as *E* to *G*:
but *C* is to *D*, as *G* is to *H*;
wherefore again, by the first case, *A* is to *D*, as *E* to *H*:
and so on, whatever be the number of magnitudes.
Therefore, if there be any number, &c. Q.E.D.

PROPOSITION XXIII. THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first has to the last of the others. *N.B.* This is usually cited by the words "ex æquali in proportione perturbatâ;" or "ex æquo perturbato."

First, let there be three magnitudes A, B, C, and other three D, E, F, which taken two and two in a cross order have the same ratio,

that is, such that A is to B, as E to F;
and as B is to C, so is D to E.
Then A shall be to C, as D to F.



Take of A, B, D any equimultiples whatever G, H, K;
and of C, E, F any equimultiples whatever L, M, N:
and because G, H are equimultiples of A, B,
and that magnitudes have the same ratio which their equimultiples have; (v. 15.)
therefore as A is to B, so is G to H:
and for the same reason, as E is to F, so is M to N:
but as A is to B, so is E to F; (hyp.)
therefore as G is to H, so is M to N: (v. 11.)
and because as B is to C, so is D to E, (hyp.)
and that H, K are equimultiples of B, D, and L, M of C, E;
therefore as H is to L, so is K to M: (v. 4.)
and it has been shewn that G is to H, as M to N:
therefore, because there are three magnitudes G, H, L, and other three K, M, N, which have the same ratio taken two and two in a cross order;
if G be greater than L, K is greater than N:
and if equal, equal; and if less, less: (v. 21.)
but G, K are any equimultiples whatever of A, D; (constr.)
and L, N any whatever of C, F;
therefore as A is to C, so is D to F. (v. def. 5.)

Next, let there be four magnitudes A, B, C, D, and other four E, F, G, H, which taken two and two in a cross order have the same ratio,

viz. A to B, as G to H;
B to C, as F to G;
and C to D, as E to F.
Then A shall be to D, as E to H.

A.B.C.D
E.F.G.H

Because A, B, C are three magnitudes, and F, G, H other three, which taken two and two in a cross order have the same ratio;
by the first case, A is to C, as F to H;
but C is to D, as E is to F;
wherefore again, by the first case, A is to D, as E to H;
and so on, whatever be the number of magnitudes.
Therefore, if there be any number, &c. Q.E.D.

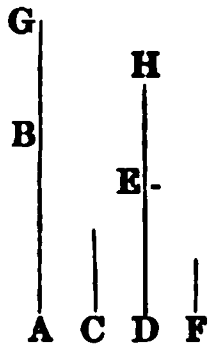
PROPOSITION XXIV. THEOREM.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Let AB the first have to C the second the same ratio which DE the third has to F the fourth;

and let BG the fifth have to C the second the same ratio which EH the sixth has to F the fourth.

Then AG, the first and fifth together, shall have to C the second, the same ratio which DH, the third and sixth together, has to F the fourth.



Because BG is to C, as EH to F;
by inversion, C is to BG, as F to EH: (v. B.)
and because, as AB is to C, so is DE to F; (hyp.)
and as C to BG, so F to EH;
ex æquali, AB is to BG, as DE to EH: (v. 22.)
and because these magnitudes are proportionals when taken separately, they are likewise proportionals when taken jointly; (v. 18.)
therefore as AG is to GB, so is DH to HE:
but as GB to C, so is HE to F: (hyp.)
therefore, ex æquali, as AG is to C, so is DH to F. (v. 22.)
Wherefore, if the first, &c. Q.E.D.

COR. 1.—If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second, as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

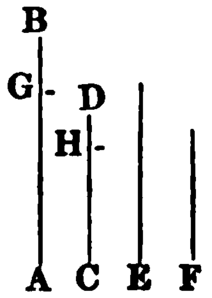
COR. 2.—The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude: as is manifest.

PROPOSITION XXV. THEOREM.

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

Let the four magnitudes AB, CD, E, F be proportionals,
viz. AB to CD, as E to F;
and let AB be the greatest of them, and consequently F the least. (v. 14. and A.)

Then AB together with F shall be greater than CD together with E.



Take AG equal to E, and CH equal to F.
Then because as AB is to CD, so is E to F,
and that AG is equal to E, and CH equal to F,
therefore AB is to CD, as AG to CH: (v. 11, and 7.)
and because AB the whole, is to the whole CD, as AG is to CH,
likewise the remainder GB is to the remainder HD, as the whole AB is to the whole CD: (v. 19.)
but AB is greater than CD; (hyp.)
therefore GB is greater than HD: (v. A.)
and because AG is equal to E, and CH to F;
AG and F together are equal to CH and E together: (I. ax. 2.)
therefore if to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and CH and E to HD;
AB and F together are greater than CD and E. (I. ax. 4.)
Therefore, if four magnitudes, &c. Q.E.D.

PROPOSITION F. THEOREM.

Ratios which are compounded of the same ratios, are the same to one another.

Let A be to B, as D to E; and B to C, as E to F.
Then the ratio which is compounded of the ratios of A to B, and B to C, which, by the definition of compound ratio, is the ratio of A to C, shall be the same with the ratio of D to F, which, by the same definition, is compounded of the ratios of D to E, and E to F.

A . B . C
D . E . F

Because there are three magnitudes A, B, C, and three others D, E, F, which, taken two and two, in order, have the same ratio;
ex æquali A is to C, as D to F. (v. 22.)
Next, let A be to B, as E to F, and B to C, as D to E:

A . B . C
D . E . F

therefore, *ex æquali in proportione perturbatâ*, (v. 23.)
A is to C, as D to F;
that is, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the same with the ratio of D to F, which is compounded of the ratios of D to E, and E to F.

And in like manner the proposition may be demonstrated, whatever be the number of ratios in either case.

PROPOSITION G. THEOREM.

If several ratios be the same to several ratios, each to each; the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.

Let A be to B, as E to F; and C to D, as G to H:
and let A be to B, as K to L; and C to D, as L to M.
Then the ratio of K to M,
by the definition of compound ratio, is compounded of the ratios of K to L, and L to M, which are the same with the ratios of A to B, and C to D.

Again, as E to F, so let N be to O; and as G to H, so let O be to P.
Then the ratio of N to P is compounded of the ratios of N to O, and O to P, which are the same with the ratios of E to F, and G to H:
and it is to be shewn that the ratio of K to M, is the same with the ratio of N to P;
or that K is to M, as N to P.

A.B.C.D.	K.L.M.
E.F.G.H.	N.O.P.

Because K is to L, as (A to B, that is, as E to F, that is, as) N to O:
and as L to M, so is (C to D, and so is G to H, and so is) O to P:
ex æquali K is to M, as N to P. (v. 22.)
Therefore, if several ratios, &c. Q.E.D.

PROPOSITION H. THEOREM.

If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them; then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.

Let the first ratios be those of A to B, B to C, C to D, D to E, and E to F;
and let the other ratios be those of G to H, H to K, K to L, and L to M:
also, let the ratio of A to F, which is compounded of the first ratios, be the same with the ratio of G to M, which is compounded of the other ratios;
and besides, let the ratio of A to D, which is compounded of the ratios of A to B, B to C, C to D, be the same with the ratio of G to K, which is compounded of the ratios of G to H, and H to K.

Then the ratio compounded of the remaining first ratios, to wit, of the ratios of D to E, and E to F, which compounded ratio is the ratio of D to F, shall be the same with the ratio of K to M, which is compounded of the remaining ratios of K to L, and L to M of the other ratios.

A.B.C.D.E.F.
G.H.K.L.M.

Because, by the hypothesis, A is to D, as G to K,
by inversion, D is to A, as K to G; (V. B.)
and as A is to F, so is G to M; (hyp.)
therefore, ex æquali, D is to F, as K to M. (V. 22.)
If, therefore, a ratio which is, &c. Q. E. D.

PROPOSITION K. THEOREM.

If there be any number of ratios, and any number of other ratios such, that the ratio which is compounded of ratios which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios which are the same, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same to several of the first ratios, each to each, be the same to one of the last ratios, or to the ratio which is compounded of ratios which are the same, each to each, to several of the last ratios; then the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are the same each to each to the remaining ratios of the first, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio which is compounded of ratios which are the same each to each to these remaining ratios.

Let the ratios of A to B, C to D, E to F, be the first ratios:
and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios:
and let A be to B, as S to T; and C to D, as T to V; and E to F, as V to X:
therefore, by the definition of compound ratio, the ratio of S to X is compounded of the ratios of S to T, T to V, and V to X, which are the same to the ratios of A to B, C to D, E to F: each to each.

Also, as G to H, so let Y be to Z; and K to L, as Z to a; M to N, as a to b; O to P, as b to c; and Q to R, as c to d:
therefore, by the same definition, the ratio of Y to d is compounded of the ratios of Y to Z, Z to a, a to b, b to c, and c to d, which are the same, each to each, to the ratios of G to H, K to L, M to N, O to P, and Q to R:

therefore, by the hypothesis, S is to X, as Y to d.

Also, let the ratio of A to B, that is, the ratio of S to T, which is one of the first ratios, be the same to the ratio of e to g, which is compounded of the ratios of e to f, and f to g, which, by the hypothesis, are the same to the ratios of G to H, and K to L, two of the other ratios;
and let the ratio of h to l be that which is compounded of the ratios of h to k, and k to l, which are the same to the remaining first ratios, viz. of C to D, and E to F;
also, let the ratio of m to p, be that which is compounded of the ratios of m to n, n to o, and o to p, which are the same, each to each, to the remaining other ratios, viz. of M to N, O to P, and Q to R.

Then the ratio of h to l shall be the same to the ratio of m to p; or h shall be to l, as m to p.

h, k, l
A, B: C, D; E, F. S, T, V, X.
G, H; K, L; M, N: O, P; Q, R. Y, Z, a, b, c, d.
e, f, g. m, n, o, p.

Because *e* is to *f*, as (*G* to *H*, that is, as) *Y* to *Z*;
and *f* is to *g*, as (*K* to *L*, that is, as) *Z* to *a*;
therefore, ex æquali, *e* is to *g*, as *Y* to *a*: (v. 22.)
and by the hypothesis, *A* is to *B*, that is, *S* to *T*, as *e* to *g*;
wherefore *S* is to *T*, as *Y* to *a*; (v. 11.)
and by inversion *T* is to *S*, as *a* to *Y*: (v. B.)
but *S* is to *X*, as *Y* to *D*; (hyp.)
therefore, ex æquali, *T* is to *X*, as *a* to *d*:
also, because *h* is to *k* as (*C* to *D*, that is, as) *T* to *V*; (hyp.)
and *k* is to *l* as (*E* to *F*, that is, as) *V* to *X*;
therefore, ex æquali, *h* is to *l*, as *T* to *x*:
in like manner, it may be demonstrated, that *m* is to *p*, as *a* to *d*;
and it has been shewn, that *T* is to *X*, as *a* to *d*;
therefore *h* is to *l*, as *m* to *p*. (v. 11.) Q.E.D.

The propositions *G* and *K* are usually, for the sake of brevity, expressed in the same terms with propositions *F* and *H*: and therefore it was proper to shew the true meaning of them when they are so expressed; especially since they are very frequently made use of by geometers.

NOTES TO BOOK V.

In the first four books of the Elements are considered, only the absolute equality and quality of Geometrical magnitudes. The fifth book contains an exposition of the nciples whereby a more definite comparison may be instituted of the relation of magni-les, besides their simple equality or inequality.

Def. I, II. In the first four books the word *part* is used in the same sense as we find in the ninth axiom, "The whole is greater than its part:" where the word *part* means y portion whatever of any whole magnitude: but in the fifth book, the word *part* is tricted to mean that portion of magnitude which is contained an exact number of times the whole. For instance, if any straight line be taken two, three, four, or any number times another straight line, by Euc. I. 3; the less line is called a part, or rather a bmultiple of the greater line; and the greater, a multiple of the less line. The ultiple is composed of a repetition of the same magnitude, and these definitions ppose that the multiple may be divided into its parts, any one of which is a measure of e multiple. And it is also obvious that when there are two magnitudes, one of which a multiple of the other, the two magnitudes must be of the same kind, that is, they ust be two lines, two angles, two surfaces, or two solids.

Euclid does not seem to consider one triangle a part or multiple of another, except hen both are between the same parallels; and a triangle is doubled, trebled, &c., by oubling, trebling, &c. the base. The same may be said of parallelograms. Also the rcs of two circles are not considered as part and multiple, unless the circles have equal adii. Angles, arcs, and sectors of equal circles may be doubled, trebled, or any multi-les found by Prop. XXVI—XXXIX, Book III.

Two magnitudes are said to be *commensurable* when a third magnitude of the same ind can be found which will measure both of them; and this third magnitude is called heir *common measure:* and when it is the greatest magnitude which will measure both of hem, it is called the *greatest common measure* of the two magnitudes: also when two agnitudes of the same kind have no common measure, they are said to be *incommen-urable*. The same terms are also applied to numbers.

Unity has no magnitude, properly so called, but may represent that portion of every ind of magnitude which is assumed as the measure of all magnitudes of the same kind. The composition of unities cannot produce Geometrical magnitude; three units are more *number* than one unit, but still as much different from magnitude as unity itself. Numbers may be considered as quantities, for we consider every thing that can be xactly measured, as a quantity.

Unity is a common measure of all rational numbers, and all numerical reasonings roceed upon the hypothesis that the unit is the same throughout the whole of any par-ular process. Euclid has not fixed the magnitude of any unit of length, nor made ference to any unit of measure of angles, surfaces, or volumes. Hence arises an essen-l difference between number and magnitude; unity, being invariable, measures all rational mbers; but though any quantity be assumed as the unit of magnitude, it is impossible assert that this assumed unit will measure all other magnitudes of the same kind.

All whole numbers therefore are commensurable; for unity is their common measure: o all rational fractions proper or improper, are commensurable; for any such frac-

tions may be reduced to other equivalent fractions having one common denominator, and that fraction whose denominator is the common denominator, and whose numerator is unity, will measure any one of the fractions. Two magnitudes having a common measure can be represented by two numbers which express the number of times the common measure is contained in both the magnitudes.

But two incommensurable magnitudes cannot be exactly represented by any two whole numbers or fractions whatever; as, for instance, the side of a square is incommensurable to the diagonal of the square. For, it may be shewn numerically, that if the side of the square contain one unity of length, the diagonal contains more than one, but less than 2 units of length. If the side be divided into 10 units, the diagonal contains more than 14, but less than 15 such units. Also if the side contain 100 units, the diagonal contains more than 141, but less than 142 of such units. It is also obvious, that as the side is successively divided into a greater number of equal parts, the error in the magnitude of the diagonal will be diminished continually, but never can be entirely exhausted; and therefore into whatever number of equal parts the side of a square be divided, the diagonal will never contain an exact number of such parts. Thus the diagonal and side of a square having no common measure, cannot be exactly represented by any two numbers.

The term *equimultiple* in Geometry is to be understood of magnitudes of the same kind, or of different kinds, taken an equal number of times, and implies only a division of the magnitudes into the same number of equal parts. Thus, if two given lines are trebled, the trebles of the lines are *equimultiples* of the two lines: and if a given line and a given angle or triangle be trebled, the trebles of the line and angle or triangle are equimultiples of the line and angle or triangle: as (fig. vi. 1.) the straight line HC and the triangle AHC are equimultiples of the line BC and the triangle ABC: and (fig. vi. 33.) the arc EN and the angle EHN are equimultiples of the arc EF and the angle EHF.

Def. III. *Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα πρὸς ἄλληλα ποιὰ σχέσις.* By this definition of *ratio* is to be understood the conception of the mutual relation of two magnitudes of the same kind, as two straight lines, two angles, two surfaces, or two solids. To prevent any misconception, Def. IV. lays down the criterion, whereby it may be known what kinds of magnitudes can have a ratio to one another; namely, *Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.* "Magnitudes are said to have a ratio to one another, which, when they are multiplied, can exceed one another;" in other words, the magnitudes which are capable of mutual comparison must be of the same kind. The former of the two terms is called the *antecedent;* and the latter, the *consequent* of the ratio. If the antecedent and consequent are equal, the ratio is called a ratio of equality; but if the antecedent be greater or less than the consequent, the ratio is called a ratio of greater or less inequality. Care must be taken not to confound the expressions "ratio of quality" and "equality of ratio:" the former is applied to the terms of a ratio when they, the antecedent and consequent, are equal to one another, but the latter, to the ratios, when they are equal.

Arithmetical ratio has been defined to be the relation which one number bears to another with respect to *quotity;* the comparison being made by considering what multiple, part or parts, one number is of the other.

An arithmetical ratio, therefore, is represented by the quotient which arises from dividing the antecedent by the consequent of the ratio; or by the fraction which has the antecedent for its numerator and the consequent for its denominator. Hence it will at once be obvious that the properties of arithmetical ratios will be made to depend on the properties of fractions.

It must ever be borne in mind that the subject of Geometry is not number, but the magnitude of lines, angles, surfaces, and solids; and its object is to demonstrate their properties by a comparison of their absolute and relative magnitudes.

Also, in Geometry, *multiplication* is only a repeated addition of the same magnitude; and *division* is only a repeated subtraction, or the taking of a less magnitude successively from a greater, until there be either no remainder, or a remainder less than the magnitude which is successively subtracted.

The Geometrical ratio of any two given magnitudes of the same kind will obviously be represented by the magnitudes themselves; thus, the ratio of two lines is represented by the lengths of the lines themselves; and, in the same manner, the ratio of two angles, two surfaces, or two solids, will be properly represented by the magnitudes themselves.

In measuring any magnitude, it is obvious that a magnitude of the same kind must be used; but the ratio of two magnitudes may be measured by every thing which has the property of quantity. Two straight lines will measure the ratio of two triangles, or two parallelograms (fig. vi. 1.): and two triangles, or two parallelograms will measure the ratio of two straight lines. It would manifestly be absurd to speak of the line as measuring the triangle, or the triangle measuring the line. (See Notes on Book II. p. 68.)

The ratio of any two quantities depends on their *relative* and not their *absolute* magnitudes; and it is possible for the *absolute* magnitude of two quantities to be changed, and their *relative* magnitude to continue the same as before; and thus, the *same ratio* may subsist between two given magnitudes, and any other two of the same kind.

In this method of measuring Geometrical ratios, the measures of the ratios are the same in number as the magnitudes themselves. It has however two advantages; first, it enables us to pass from one kind of magnitude to another, and thus, independently of any numerical measure, to institute a comparison between such magnitudes as cannot be directly compared with one another: and secondly, the ratio of two magnitudes of the same kind may be measured by two straight lines, which form a simpler measure of ratios than any other kind of magnitude.

But the simplest method of all would be, to express the measure of the *ratio of two magnitudes* by *one;* but this cannot be done, unless the two magnitudes are commensurable. If two lines AB, CD, one of which AB contains 12 units of any length, and the other CD contains 4 units; then the ratio of the line AB to the line CD, is the same as the ratio of the number 12 to 4. Thus, two numbers may represent the ratio of two lines when the lines are commensurable. In the same manner, two numbers may represent the ratio of two angles, two surfaces, or two solids.

Thus, the ratio of any two magnitudes of the same kind may be expressed by two numbers, when the magnitudes are commensurable. By this means, the consideration of the ratio of two magnitudes is changed to the consideration of the ratio of two numbers, and when one number is divided by the other, the quotient will be a *single number, or a fraction,* which will be a *measure of the ratio* of the two numbers, and therefore of the two quantities. If 12 be divided by 4, the quotient is 3, which measures the ratio of the two numbers 12 and 4. Again, if besides the ratio of the lines AB and CD which contain 12 and 4 units respectively, we consider two other lines EF and GH which contain 9 and 3 units respectively; it is obvious that the ratio of the line EF to GH is the same as the ratio of the number 9 to the number 3. And the measure of the ratio of 9 to 3 is 3. That is, the numbers 9 and 3 have the same ratio as the numbers 12 and 4.

But this is a numerical measure of ratio, and can only be applied strictly when the antecedent and consequent are to one another as one number to another.

And generally, if the two lines AB, CD contain a and b units respectively, and q be

the quotient which indicates the number of times the number b is contained in a, then q is the measure of the ratio of the two numbers a and b: and if EF and GH contain c and d units, and the number d be contained q times in c. The number a has to b the same ratio as the number c has to d.

This is the numerical definition of proportion, which is thus expressed in Euclid's Elements, Book VII, definition 20. "Four numbers are proportionals when the first is the same multiple of the second, or the same part or parts of it, as the third is of the fourth." This definition of the proportion of four numbers, leads at once to an equation:

$$\text{for, since } a \text{ contains } b,\ q \text{ times; } \frac{a}{b} = q:$$

$$\text{and since } c \text{ contains } d,\ q \text{ times; } \frac{c}{d} = q:$$

therefore $\frac{a}{b} = \frac{c}{d}$, the fundamental equation upon which all reasonings on the proportion of numbers depend.

Sometimes a proportion is defined to be the *equality* of two ratios.

But we are anticipating the subject of the sixth and eighth definitions.

Def. VIII declares the meaning of the term analogy or proportion. The ratio of two lines, two angles, two surfaces or two solids, means nothing more than their relative magnitude in contradistinction to their absolute magnitudes; and a similitude or likeness of ratios implies, at least, the two ratios of the four magnitudes which constitute the analogy or proportion.

Def. IX states that a proportion consists in three terms at least; the meaning of which is, that the second magnitude is repeated, being made the consequent of the first, and the antecedent of the second ratio. It is also obvious that when a proportion consists of three magnitudes, all three are of the same kind. Def. VI appears only to be a further explanation of what is implied in Def. VIII.

Def. V. Proportion having been defined to be the *similitude of ratios*, the fifth definition lays down a criterion by which two ratios may be known to be similar, or four magnitudes proportionals, without involving any enquiry respecting the four quantities, whether the antecedents of the ratios contain or are contained in their consequents exactly; or whether there are any magnitudes which measure the terms of the two ratios. The criterion only requires, that the relation of the equimultiples expressed should hold good, not merely for any particular multiples, as the doubles or trebles, but for any multiples whatever, whether large or small.

This criterion of proportion may be applied to all Geometrical magnitudes which can be *multiplied*, that is, to all which can be doubled, trebled, quadrupled, &c. But it must be borne in mind, that this criterion does not exhibit a definite measure for either of the two ratios which constitute the proportion, but only, an undetermined measure for the sameness or similarity of the two ratios. The nature of the proportion of Geometrical magnitudes neither requires nor admits of a definite measure of either of the two ratios, for this would be to suppose that all magnitudes are commensurable. Though we know not the definite measure of either of the ratios, further than that they are both similar, and one may be taken as the measure of the other, yet particular conclusions may be arrived at by this method: for by the test of proportionality here laid down, it can be proved that one magnitude is greater than, equal to, or less than another: that a third proportional can be found to two, and a fourth proportional to three straight lines, also that a mean proportional can be found between two straight lines: and further, that which is here stated of straight lines may be extended to other Geometrical magnitudes.

With respect to this test or criterion of the proportionality of four magnitudes, it

has been objected, that it is utterly impossible to make trial of *all* the possible equimultiples of the first and third magnitudes, and also of the second and fourth. It may be replied, that the point in question is not determined by making such trials, but by shewing from the nature of the magnitudes, that whatever be the multipliers, if the multiple of the first exceeds the multiple of the second magnitude, the multiple of the third *will* exceed the multiple of the fourth magnitude, and if equal, *will* be equal; and if less, *will* be less, in any case which may be taken.

The Arithmetical definition of proportion in Book VII, Def. 20, even if it were equally general with the Geometrical definition in Book v, Def. 5, is by means universally applicable to the subject of Geometrical magnitudes. The Geometrical criterion is founded on multiplication which is always possible. When the magnitudes are commensurable, the multiples of the first and second *may* be equal or unequal; but when the magnitudes are incommensurable, any multiples whatever of the first and second *must* be unequal; but the Arithmetical criterion of proportion is founded on division, which is not always possible. Euclid has not shewn how to take *any part* of a line or other magnitude, or that the two terms of a ratio have a common measure, and therefore the numerical definition could not be strictly applied, even in the limited way in which it may be applied.

Number and *Magnitude* do not correspond in all their relations; and hence the distinction between Geometrical ratio and Arithmetical ratio; the former is a comparison κατὰ πηλικότητα, according to quantity, but the latter, according to quotity. The former gives an undetermined, though definite measure, in magnitudes; but the latter attempts to give the exact value in numbers.

The fifth book exhibits no method whereby two magnitudes may be determined to be commensurable, and the Geometrical conclusions deduced from the multiples of magnitudes are too general to furnish a numerical measure of ratios, being all independent of the commensurability or incommensurability of the magnitudes themselves.

It is the numerical ratio of two magnitudes which will more certainly discover whether they are commensurable or incommensurable, and hence, recourse must be had to the forms and properties of numbers. All numbers and fractions are either rational or irrational. It has been seen that rational numbers and fractions *can express* the ratios of Geometrical magnitudes, when they are commensurable. Similar relations of incommensurable magnitudes *may be expressed* by irrational numbers, if the Algebraical expressions for such numbers may be assumed and employed in the same manner as rational numbers. The irrational expressions being considered the exact and definite, though undetermined, values of the ratios, to which a series of rational numbers may successively approximate.

Though two incommensurable magnitudes have not an assignable numerical ratio to one another, yet they have a certain definite ratio to one another, and two other magnitudes may have the same ratio as the first two: and it will be found, that, when reference is made to the numerical value of the ratios of four incommensurable magnitudes, that the same irrational number appears in the two ratios.

The sides and diagonals of squares can be shewn to be proportionals, and though the ratio of the side to the diagonal is represented Geometrically by the two lines which form the side and the diagonal, there is no rational number or fraction which will measure exactly their ratio.

If the side of a square contain a units, the ratio of the diagonal to the side is numerically as $\sqrt{2}$ to 1; and if the side of another square contain b units, the ratio of the diagonal to the side will be found to be in the ratio of $\sqrt{2}$ to 1. Again, the two parts of any number of lines which may be divided in extreme and mean ratio will be found to be respectively in the ratio of the irrational number $\sqrt{5}-1$ to

$3-\sqrt{5}$. Also, the ratios of the diagonals of cubes to the diagonals of one of the faces will be found to be in the irrational or incommensurate ratio of $\sqrt{3}$ to $\sqrt{2}$.

Thus it will be found that the ratios of all incommensurable magnitudes which are proportionals do involve the same irrational numbers, and these may be used as the numerical measures of ratios in the same manner as rational numbers and fractions.

Def. VII is analogous to Def. V, and lays down the criterion whereby the ratio of two magnitudes of the same kind may be known to be *greater* or *less* than the ratio of two other magnitudes of the same kind.

Def. XI includes Def. X, as three magnitudes may be continual proportionals, as well as four or more than four. In continual proportionals, all the terms except the first and last, are made successively the consequent of one ratio, and the antecedent of the next; whereas in other proportionals, this is not the case.

A series of numbers or Algebraical quantities in continued proportion, is called a *Geometrical progression*, from the analogy they bear to a series of Geometrical magnitudes in continued proportion.

Def. A. The term *compound ratio* was devised for the purpose of avoiding circumlocution, and no difficulty can arise in the use of it, if its exact meaning be strictly attended to.

With respect to the Geometrical measures of compound ratios, three straight lines may measure the ratio of four, as in Prop. 23, Book VI. For K to L measures the ratio of BC to CG, and L to M measures the ratio of DC to CE; and the ratio of K to M is that which is said to be compounded of the ratios of K to L, and L to M, which is the same as the ratio which is compounded of the ratios of the sides of the parallelograms.

Both duplicate and triplicate ratio are species of compound ratio.

Duplicate ratio is a ratio compounded of two equal ratios; and in the case of three magnitudes which are continual proportionals, means the ratio of the first to a third proportional to the first and second.

It will be hereafter seen, that the ratio of two triangles, when they have the same altitude, are to one another as their bases; (Prop. 1, Book VI.) and, when their bases are equal, they are to one another as their altitudes; but when neither their bases nor altitudes are equal, they are to each other in a ratio compounded of their bases and altitudes. And if the ratio of their altitudes be the same as the ratio of their bases, or the triangles be similar, they are to each other in the duplicate ratio of their altitudes or bases. (Prop. 19, Book VI.) In the same manner, the ratio of two squares will be to each other in the ratio compounded of the ratios of their sides, and therefore in the duplicate ratio of their sides. And generally, that all similar figures are in the duplicate ratio of their homologous sides. (Prop. 20, Book VI.)

Triplicate ratio, in the same manner, is a ratio compounded of three equal ratios; and in the case of four magnitudes which are continual proportionals, the triplicate ratio of the first to the second means the ratio of the first to a fourth proportional to the first, second, and third magnitudes. Instances of the composition of three ratios, and of triplicate ratio will be found in the eleventh and twelfth books.

The product of the fractions which represent or measure the ratios of numbers corresponds to the composition of Geometrical ratios of magnitudes.

It has been shewn that the ratio of two numbers is represented by a fraction, where the numerator is the antecedent, and the denominator the consequent of the ratio; and if the antecedents of two ratios be multiplied together, as also the consequents, the new ratio thus formed is said to be compounded of these two ratios; and in the same manner if there be more than two. It is also obvious, that the ratio compounded of two equal ratios is equal to the ratio of the squares of one of the antecedents to its consequent; also when there are three equal ratios, the ratio compounded of the three ratios is equal

the ratio of the cubes of any one of the antecedents to its consequent. And further, it y be observed, that when several numbers are continual proportionals, the ratio of first to the last is equal to the ratio of the product of all the antecedents to the duct of all the consequents.

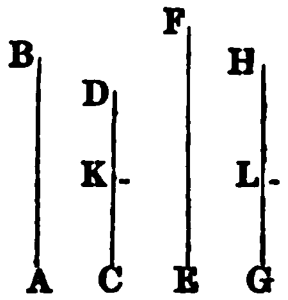
It may be here remarked, that, though the constructions of the propositions in ok v are exhibited by straight lines, the enunciations are expressed of magnitude general, and are equally true of angles, triangles, parallelograms, arcs, sectors, &c.

Prop. c. This is frequently made use of by geometers, and is necessary to the and 6th Propositions of the 10th Book. Clavius, in his notes subjoined to 8th def. of Book 5, demonstrates it only in numbers, by help of some of the positions of the 7th Book; in order to demonstrate the property contained in the definition of the 5th Book, when applied to numbers, from the property of pro- tionals contained in the 20th def. of the 7th Book: and most of the commentators lge it difficult to prove that four magnitudes which are proportionals according to 20th def. of the 7th Book, are also proportionals according to the 5th def. of the Book. But this is easily made out as follows:

First, if A, B, C, D, be four magnitudes, such that A is the same multiple, or same part of B, which C is of D:

Then A, B, C, D, are proportionals:
this is demonstrated in proposition (c).

Secondly, if AB contain the same parts of CD that EF does of GH;
in this case likewise AB is to CD, as EF to GH.

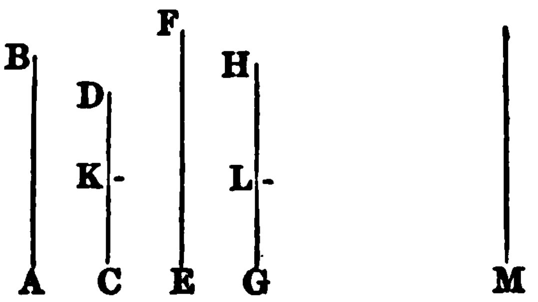


Let CK be a part of CD, and GL the same part of GH:
and let AB be the same multiple of CK, that EF is of GL:
therefore, by Prop. c, of Book v, AB is to CK, as EF to GL:
and CD, GH, are equimultiples of CK, GL, the second and fourth;
wherefore, by Cor. Prop. 4, Book v, AB is to CD, as EF to GH.

And if four magnitudes be proportionals according to the 5th def. of Book v,
they are also proportionals according to the 20th def. of Book vii.

First, if A be to B, as C to D;
then if A be any multiple or part of B, C is the same multiple or part of D, by Prop. d, Book v.

Next, if AB be to CD, as EF to GH:
then if AB contains any part of CD, EF contains the same part of GH:



for let CK be a part of CD, and GL the same part of GH, and let AB be a multiple of CK:

EF is the same multiple of GL:
take M the same multiple of GL that AB is of CK;
therefore, by Prop. c, Book v, AB is to CK, as M to GL:

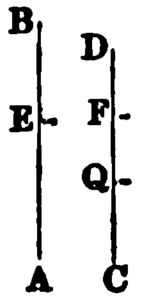
and CD, GH, are equimultiples of CK, GL;
wherefore, by Cor. Prop. 4, Book v, AB is to CD, as M to GH.
And, by the hypothesis, AB is to CD, as EF to GH;
therefore M is equal to EF by Prop. 9, Book v.
and consequently, EF is the same multiple of GL that AB is of CK.

This is the method by which Simson shews that the Geometrical definition of proportion is a consequence of the Arithmetical definition, and conversely.

It may however be shewn by employing the equation $\frac{a}{b} = \frac{c}{d}$, and taking ma, mc any equimultiples of a and c the first and third, and nb, nd any equimultiples of b and d the second and fourth.

Prop. XVIII, being the converse of Prop. XVII, has been demonstrated indirectly.

Let AE, EB, CF, FD be proportionals, that is, as AE to EB, so let CF be to FD.
Then these shall be proportionals also when taken jointly;
that is, as AB to BE, so shall CD be to DF.



For if the ratio of AB to BE be not the same as the ratio of CD to DF;
the ratio of AB to BE is either greater than, or less than the ratio of CD to DF.
First, let AB have to BE a greater ratio than CD has to DF;
and let DQ be taken so that AB has to BE the same ratio as CD to DQ.
And since magnitudes when taken jointly are proportionals,
are also proportionals when taken separately; (v. 17.)
therefore AE has to EB the same ratio as CQ to QD;
but, by the hypothesis, AE has to EB the same ratio as CF to FD;
therefore the ratio of CQ to QD is the same as the ratio of CF to FD. (v. 11.)
And when four magnitudes are proportionals, if the first be greater than the second, the third is greater than the fourth; and if equal, equal; and if less, less; (v. 14.)
but CQ is less than CF,
therefore QD is less than FD; which is absurd.
Wherefore the ratio of AB to BE is not greater than the ratio of CD to DF;
that is, AB has the same ratio to BE as CD has to DF.

Secondly. By a similar mode of reasoning, it may likewise be shewn, that AB has the same ratio to BE as CD has to DF, if AB be assumed to have to BE a less ratio than CD has to DF.

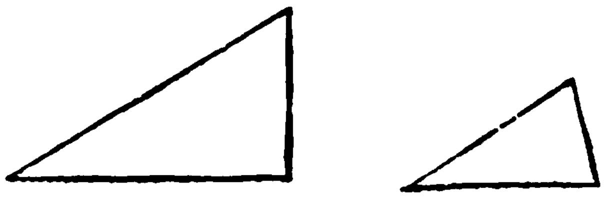
For further information on the very important subject of Ratio and Proportion, reference may be made to Dr. Barrow's Mathematical Lectures; Professor De Morgan's Connexion of Number and Magnitude; and the fourth chapter of Professor Peacock's Algebra.

BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.



II.

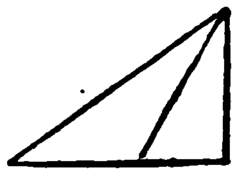
"Reciprocal figures, viz. triangles and parallelograms, are such as have their sides about two of their angles proportionals in such a manner, that a side of the first figure is to a side of the other, as the remaining side of this other is to the remaining side of the first."

III.

A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

IV.

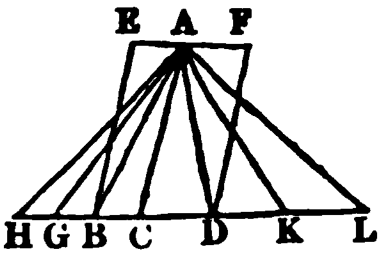
The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.



PROPOSITION I. THEOREM.

Triangles and parallelograms of the same altitude are one to another as their bases.

Let the triangles ABC, ACD, and the parallelograms EC, CF, have the same altitude,
viz. the perpendicular drawn from the point A to BD or BD produced.
As the base BC is to the base CD, so shall the triangle ABC be to the triangle ACD,
and the parallelogram EC to the parallelogram CF.



Produce BD both ways to the points H, L,
and take any number of straight lines BG, GH, each equal to the base BC; (I. 3.)
and DK, KL, any number of them, each equal to the base CD;
and join AG, AH, AK, AL.
Then, because CB, BG, GH, are all equal,
the triangles AHG, AGB, ABC, are all equal: (I. 38.)
therefore, whatever multiple the base HC is of the base BC,
the same multiple is the triangle AHC of the triangle ABC:
for the same reason, whatever multiple the base LC is of the base CD,
the same multiple is the triangle ALC of the triangle ADC:
and if the base HC be equal to the base CL,
the triangle AHC is also equal to the triangle ALC: (I. 38.)
and if the base HC be greater than the base CL,
likewise the triangle AHC is greater than the triangle ALC;
and if less, less:
therefore, since there are four magnitudes,
viz. the two bases BC, CD, and the two triangles ABC, ACD;
and of the base BC, and the triangle ABC the first and third, any equimultiples whatever have been taken,
viz. the base HC and the triangle AHC;
and of the base CD and the triangle ACD, the second and fourth, have been taken any equimultiples whatever,
viz. the base CL and the triangle ALC;
and since it has been shewn, that, if the base HC be greater than the base CL,
the triangle AHC is greater than the triangle ALC;
and if equal, equal; and if less, less:
therefore, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD. (V. def. 5.)
And because the parallelogram CE is double of the triangle ABC, (I. 41.)
and the parallelogram CF double of the triangle ACD,
and that magnitudes have the same ratio which their equimultiples have; (V. 15.)
as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF;

and because it has been shewn, that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD;
and as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF;
therefore, as the base BC is to the base CD, so is the parallelogram EC to the parallelogram CF. (v. 11.)
Wherefore, triangles, &c. Q.E.D.

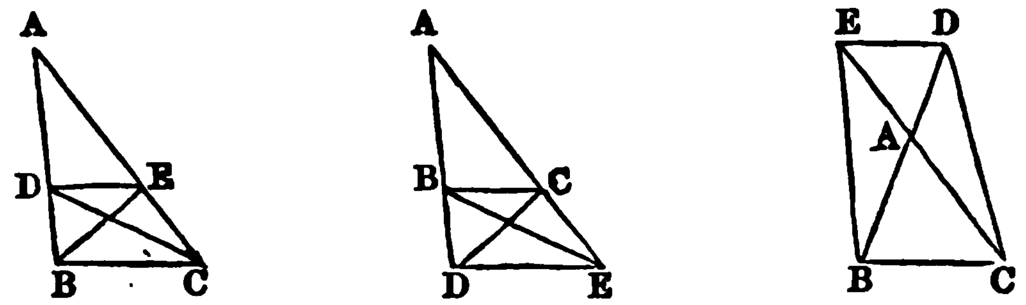
COR. From this it is plain, that triangles and parallelograms that ave equal altitudes, are one to another as their bases.

Let the figures be placed so as to have their bases in the same raight line; and having drawn perpendiculars from the vertices of ie triangles to the bases, the straight line which joins the vertices is arallel to that in which their bases are, (I. 33.) because the perpeniculars are both equal and parallel to one another. (I. 28.) Then, if ie same construction be made as in the proposition, the demonstration rill be the same.

PROPOSITION II. THEOREM.

If a straight line be drawn parallel to one of the sides of a triangle, t shall cut the other sides, or these produced, proportionally: and conversely, f the sides, or the sides produced, be cut proportionally, the straight line vhich joins the points of section shall be parallel to the remaining side of he triangle.

Let DE be drawn parallel to BC, one of the sides of the triangle ABC.
Then BD shall be to DA, as CE to EA.



Join BE, CD.
Then the triangle BDE is equal to the triangle CDE, (I. 37.) because they are on the same base DE, and between the same parallels DE, BC:
but ADE is another triangle;
nd equal magnitudes have the same ratio to the same magnitude; (v. 7.) therefore, as the triangle BDE is to the triangle ADE, so is the triangle CDE to the triangle ADE:
ut as the triangle BDE to the triangle ADE, so is BD to DA, (VI. 1.) because, having the same altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their bases;
and for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA:
therefore, as BD to DA, so is CE to EA. (v. 11.)

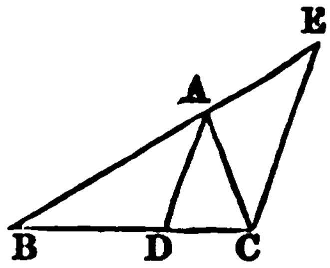
Next, let the sides AB, AC of the triangle ABC, or these sides roduced, be cut proportionally in the points D, E, that is, so that BD ay be to DA as CE to EA, and join DE.
Then DE shall be parallel to BC.

The same construction being made,
because as BD to DA, so is CE to EA;
and as BD to DA, so is the triangle BDE to the triangle ADE; (VI. 1.)
and as CE to EA, so is the triangle CDE to the triangle ADE;
therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; (V. 11)
that is, the triangles BDE, CDE have the same ratio to the triangle ADE:
therefore the triangle BDE is equal to the triangle CDE: (V. 9.)
and they are on the same base DE:
but equal triangles on the same base and on the same side of it, are between the same parallels; (I. 39.)
therefore DE is parallel to BC.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION III. THEOREM.

If the angle of a triangle be divided into two equal angles, by a straight line which also cuts the base; the segments of the base shall have the same ratio which the other sides of the triangle have to one another: and conversely, if the segments of the base have the same ratio which the other sides of the triangle have to one another; the straight line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.

Let ABC be a triangle, and let the angle BAC be divided into two equal angles by the straight line AD.
Then BD shall be to DC, as BA to AC.



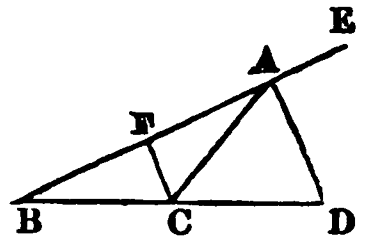
Through the point C draw CE parallel to DA, (I. 31.)
and let BA produced meet CE in E.
Because the straight line AC meets the parallels AD, EC,
the angle ACE is equal to the alternate angle CAD: (I. 29.)
but CAD, by the hypothesis, is equal to the angle BAD;
wherefore BAD is equal to the angle ACE. (ax. 1.)
Again, because the straight line BAE meets the parallels AD, EC,
the outward angle BAD is equal to the inward and opposite angle AEC: (I. 29.)
but the angle ACE has been proved equal to the angle BAD;
therefore also ACE is equal to the angle AEC, (ax. 1.)
and consequently the side AE is equal to the side AC: (I. 6.)
and because AD is drawn parallel to EC, one of the sides of the triangle BCE,
therefore BD is to DC, as BA to AE: (VI. 2.)
but AE is equal to AC;
therefore, as BD to DC, so is BA to AC. (V. 7.)
Next, let BD be to DC, as BA to AC, and join AD.
Then the angle BAC shall be divided into two equal angles by the straight line AD.

The same construction being made;
because, as BD to DC, so is BA to AC;
nd as BD to DC, so is BA to AE, because AD is parallel to EC; (VI. 2.)
therefore BA is to AC, as BA to AE: (V. 11.)
consequently AC is equal to AE, (V. 9.)
and therefore the angle AEC is equal to the angle ACE: (I. 5.)
but the angle AEC is equal to the outward and opposite angle BAD;
and the angle ACE is equal to the alternate angle CAD: (I. 29.)
wherefore also the angle BAD is equal to the angle CAD; (ax. 1.)
that is, the angle BAC is cut into two equal angles by the straight line AD.
Therefore, if the angle, &c. Q. E. D.

PROPOSITION A. THEOREM.

If the outward angle of a triangle made by producing one of its sides, be divided into two equal angles, by a straight line, which also cuts the base produced; the segments between the dividing line and the extremities of the base, have the same ratio which the other sides of the triangle have to one another: and conversely, if the segments of the base produced have the same ratio which the other sides of the triangle have; the straight line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.

Let ABC be a triangle, and let one of its sides BA be produced to E;
and let the outward angle CAE be divided into two equal angles
by the straight line AD which meets the base produced in D.
Then BD shall be to DC, as BA to AC.



Through C draw CF parallel to AD. (I. 31.)
And because the straight line AC meets the parallels AD, FC,
the angle ACF is equal to the alternate angle CAD: (I. 29.)
but CAD is equal to the angle DAE; (hyp.)
therefore also DAE is equal to the angle ACF. (ax. 1.)
Again, because the straight line FAE meets the parallels AD, FC,
the outward angle DAE is equal to the inward and opposite angle CFA: (I. 29.)
but the angle ACF has been proved equal to the angle DAE;
therefore also the angle ACF is equal to the angle CFA; (ax. I.)
and consequently the side AF is equal to the side AC. (I. 6.)
and because AD is parallel to FC, a side of the triangle BCF,
therefore BD is to DC, as BA to AF: (VI. 2.)
but AF is equal to AC;
therefore, as BD is to DC, so is BA to AC. (V. 7.)
Next, let BD be to DC, as BA to AC, and join AD.
The angle CAD shall be equal to the angle DAE.
The same construction being made,
because BD is to DC, as BA to AC;

and that BD is also to DC, as BA to AF; (VI. 2.)
therefore BA is to AC, as BA to AF: (V. 11.)
wherefore AC is equal to AF, (V. 9.)
and the angle AFC equal to the angle ACF: (I. 5.)
but the angle AFC is equal to the outward angle EAD, (I. 29.)
and the angle ACF to the alternate angle CAD;
therefore also EAD is equal to the angle CAD. (ax. 1.)
Wherefore, if the outward, &c. Q.E.D.

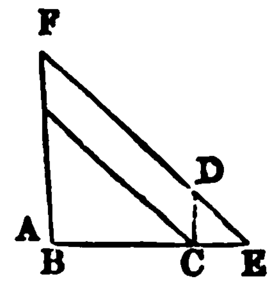
PROPOSITION IV. THEOREM.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC; and consequently the angle BAC equal to the angle CDE. (I. 32.)

The sides about the equal angles of the triangles ABC, DCE shall be proportionals;

and those shall be the homologous sides which are opposite to the equal angles.



Let the triangle DCE be placed, so that its side CE may be contiguous to BC, and in the same straight line with it. (I. 22.)

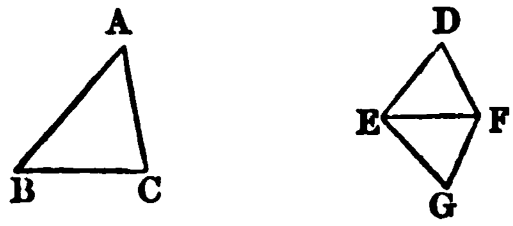
Then, because the angle BCA is equal to the angle CED, (hyp.)
add to each the angle ABC;
therefore the two angles ABC, BCA are equal to the two angles ABC, CED: (ax. 2.)
but the angles ABC, BCA are together less than two right angles; (I. 17.)
therefore the angles ABC, CED are also less than two right angles:
wherefore BA, ED if produced will meet: (I. ax. 12.)
let them be produced and meet in the point F:
then because the angle ABC is equal to the angle DCE, (hyp.)
BF is parallel to CD; (I. 28.)
and because the angle ACB is equal to the angle DEC,
AC is parallel to FE: (I. 28.)
therefore $FACD$ is a parallelogram;
and consequently AF is equal to CD, and AC to FD: (I. 34.)
and because AC is parallel to FE, one of the sides of the triangle FBE,
BA is to AF, as BC to CE: (VI. 2.)
but AF is equal to CD;
therefore, as BA to CD, so is BC to CE: (V. 7.)
and alternately, as AB to BC, so is DC to CE; (V. 16.)
again, because CD is parallel to BF,
as BC to CE, so is FD to DE: (VI. 2.)
but FD is equal to AC;
therefore, as BC to CE, so is AC to DE; (V. 7.)

and alternately, as BC to CA, so CE to ED: (v. 16.)
erefore, because it has been proved that AB is to BC, as DC to CE,
and as BC to CA, so CE to ED,
ex æquali, BA is to AC, as CD to DE. (v. 22.)
Therefore the sides, &c. Q.E.D.

PROPOSITION V. THEOREM.

If the sides of two triangles, about each of their angles, be proportionals, triangles shall be equiangular; and the equal angles shall be those ich are opposite to the homologous sides.

Let the triangles ABC, DEF have their sides proportionals,
so that AB is to BC, as DE to EF;
and BC to CA, as EF to FD;
and consequently, ex æquali, BA to AC, as ED to DF.
Then the triangle ABC shall be equiangular to the triangle DEF, d the angles which are opposite to the homologous sides shall be equal, :. the angle ABC equal to the angle DEF, and BCA to EFD, and also AC to EDF.



At the points E, F, in the straight line EF, make the angle FEG ual to the angle ABC, and the angle EFG equal to BCA: (I. 23.)
wherefore the remaining angle EGF, is equal to the remaining angle BAC, (I. 32.)
and the triangle GEF is therefore equiangular to the triangle ABC:
consequently they have their sides opposite to the equal angles proportionals: (VI. 4.)
wherefore, as AB to BC, so is GE to EF;
but as AB to BC, so is DE to EF; (hyp.)
therefore as DE to EF, so GE to EF; (v. 11.)
that is, DE and GE have the same ratio to EF,
and consequently are equal; (v. 9.)
for the same reason, DF is equal to FG:
and because, in the triangles DEF, GEF, DE is equal to EG, and EF is common,
the two sides DE, EF are equal to the two GE, EF, each to each;
and the base DF is equal to the base GF;
therefore the angle DEF is equal to the angle GEF, (I. 8.)
and the other angles to the other angles which are subtended by the equal sides; (I. 4.)
ierefore the angle DFE is equal to the angle GFE, and EDF to EGF:
and because the angle DEF is equal to the angle GEF,
and GEF equal to the angle ABC; (constr.)
therefore the angle ABC is equal to the angle DEF: (ax. I.)
for the same reason, the angle ACB is equal to the angle DFE,
and the angle at A equal to the angle at D:
therefore the triangle ABC is equiangular to the triangle DEF.
Wherefore, if the sides, &c. Q.E.D.

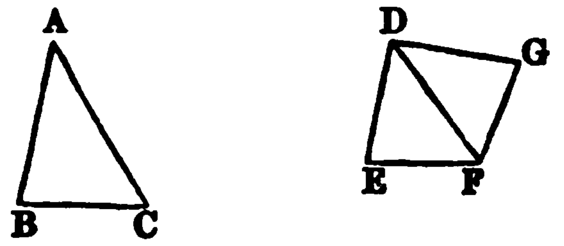
PROPOSITION VI. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let the triangles *ABC, DEF* have the angle *BAC* in the one equal to the angle *EDF* in the other, and the sides about those angles proportionals;

that is, *BA* to *AC*, as *ED* to *DF*.

Then the triangles *ABC, DEF* shall be equiangular, and shall have the angle *ABC* equal to the angle *DEF*, and *ACB* to *DFE*.



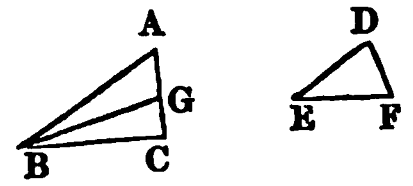
At the points *D, F*, in the straight line *DF*, make the angle *FDG* equal to either of the angles *BAC, EDF*; (I. 23.)
and the angle *DFG* equal to the angle *ACB*:
wherefore the remaining angle at *B* is equal to the remaining angle at *G*: (I. 32.)
and consequently the triangle *DGF* is equiangular to the triangle *ABC*;
therefore as *BA* to *AC*, so is *GD* to *DF*: (VI. 4.)
but, by the hypothesis, as *BA* to *AC*, so is *ED* to *DF*;
therefore as *ED* to *DF*, so is *GD* to *DF*; (V. 11.)
wherefore *ED* is equal to *DG*; (V. 9.)
and *DF* is common to the two triangles *EDF, GDF*:
therefore the two sides *ED, DF* are equal to the two sides *GD, DF*, each to each;
and the angle *EDF* is equal to the angle *GDF*; (constr.)
wherefore the base *EF* is equal to the base *FG*, (I. 4.)
and the triangle *EDF* to the triangle *GDF*,
and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides:
therefore the angle *DFG* is equal to the angle *DFE*,
and the angle at *G* to the angle at *E*;
but the angle *DFG* is equal to the angle *ACB*; (constr.)
therefore the angle *ACB* is equal to the angle *DFE*; (ax. 1.)
and the angle *BAC* is equal to the angle *EDF*: (hyp.)
wherefore also the remaining angle at *B* is equal to the remaining angle at *E*; (I. 32.)
therefore the triangle *ABC* is equiangular to the triangle *DEF*.
Wherefore, if two triangles, &c. Q.E.D.

PROPOSITION VII. THEOREM.

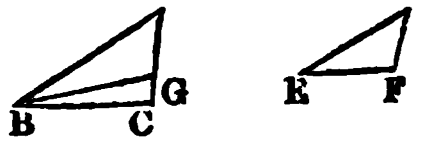
If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less, or not less, than a right angle, or if

one of them be a right angle; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other,
viz. the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF proportionals,
so that AB is to BC, as DE to EF;
and in the first case, let each of the remaining angles at C, F be less than a right angle.
The triangle ABC shall be equiangular to the triangle DEF,
viz. the angle ABC shall be equal to the angle DEF,
and the remaining angle at C equal to the remaining angle at F.

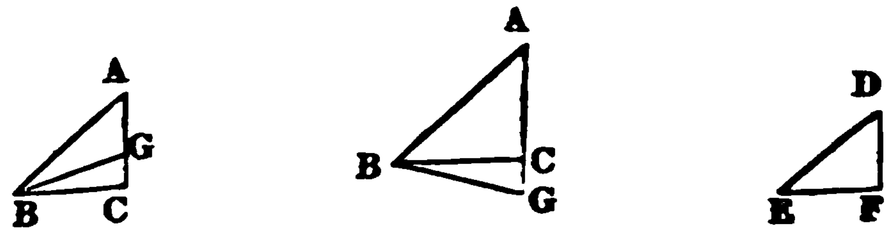


For if the angles ABC, DEF be not equal,
one of them must be greater than the other:
let ABC be the greater,
and at the point B, in the straight line AB,
make the angle ABG equal to the angle DEF; (I. 23.)
and because the angle at A is equal to the angle at D, (hyp.)
and the angle ABG to the angle DEF;
the remaining angle AGB is equal to the remaining angle DFE: (I. 32.)
therefore the triangle ABG is equiangular to the triangle DEF:
wherefore as AB is to BG, so is DE to EF: (VI. 4.)
but as DE to EF, so, by hypothesis, is AB to BC;
therefore as AB to BC, so is AB to BG: (V. 11.)
and because AB has the same ratio to each of the lines BC, BG,
BC is equal to BG; (V. 9.)
and therefore the angle BGC is equal to the angle BCG: (I. 5.)
but the angle BCG is, by hypothesis, less than a right angle;
therefore also the angle BGC is less than a right angle;
and therefore the adjacent angle AGB must be greater than a right angle; (I. 13.)
but it was proved that the angle AGB is equal to the angle at F;
therefore the angle at F is greater than a right angle:
but, by the hypothesis, it is less than a right angle; which is absurd.
Therefore the angles ABC, DEF are not unequal,
that is, they are equal:
and the angle at A is equal to the angle at D: (hyp.)
wherefore the remaining angle at C is equal to the remaining angle at F: (I. 32.)
therefore the triangle ABC is equiangular to the triangle DEF.
Next, let each of the angles at C, F be not less than a right angle.
Then the triangle ABC shall also in this case be equiangular to the triangle DEF.



The same construction being made,
it may be proved in like manner that BC is equal to BG,
and therefore the angle at C equal to the angle BGC:
but the angle at C is not less than a right angle; (hyp.)
therefore the angle BGC is not less than a right angle:
wherefore two angles of the triangle BGC are together not less than two right angles:
which is impossible; (I. 17.)
and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the first case.

Lastly, let one of the angles at C, F, viz. the angle at C, be a right angle: in this case likewise the triangle ABC shall be equiangular to the triangle DEF.



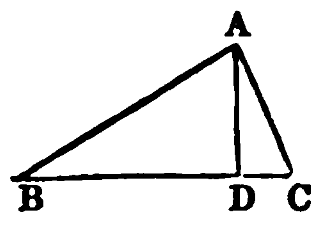
For, if they be not equiangular,
at the point B in the straight line AB make the angle ABG equal to the angle DEF;
then it may be proved, as in the first case, that BG is equal to BC:
and therefore the angle BCG equal to the angle BGC: (I. 5.)
but the angle BCG is a right angle, (hyp.)
therefore the angle BGC is also a right angle; (ax. 1.)
whence two of the angles of the triangle BGC are together not less than two right angles;
which is impossible: (I. 17.)
therefore the triangle ABC is equiangular to the triangle DEF.
Wherefore, if two triangles, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the base BC.

Then the triangles ABD, ADC shall be similar to the whole triangle ABC, and to one another.



Because the angle BAC is equal to the angle ADB, each of them being a right angle, (ax. 11.)
and that the angle at B is common to the two triangles ABC, ABD;
the remaining angle ACB is equal to the remaining angle BAD: (I. 32.)

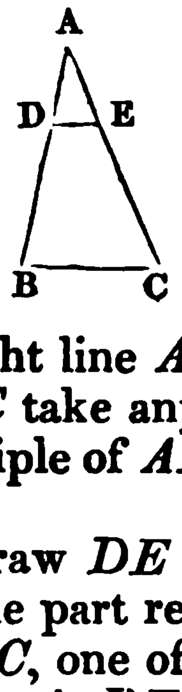
therefore the triangle *ABC* is equiangular to the triangle *ABD*, and the sides about their equal angles are proportionals; (VI. 4.)
wherefore the triangles are similar: (VI. def. 1.)
in the like manner it may be demonstrated, that the triangle *ADC* is equiangular and similar to the triangle *ABC*.
And the triangles *ABD*, *ACD*, being both equiangular and similar to *ABC*, are equiangular and similar to each other.
Therefore, in a right-angled, &c. Q.E.D.

COR. From this it is manifest, that the perpendicular drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base; and also that each of the sides is a mean proportional between the base, and the segment of it adjacent to that side: because in the triangles *BDA*, *ADC*; *BD* is to *DA*, as *DA* to *DC*; (VI. 4.)
and in the triangles *ABC*, *DBA*; *BC* is to *BA*, as *BA* to *BD*; (VI. 4.)
and in the triangles *ABC*, *ACD*; *BC* is to *CA*, as *CA* to *CD*. (VI. 4.)

PROPOSITION IX. PROBLEM.

From a given straight line to cut off any part required.

Let *AB* be the given straight line.
It is required to cut off any part from it.

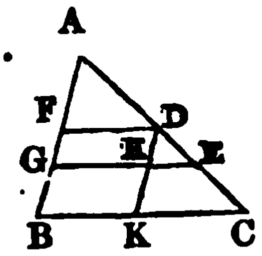


From the point *A* draw a straight line *AC*, making any angle with *AB*;
and in *AC* take any point *D*,
and take *AC* the same multiple of *AD*, that *AB* is of the part which is to be cut off from it;
join *BC*, and draw *DE* parallel to *CB*.
Then *AE* shall be the part required to be cut off.
Because *ED* is parallel to *BC*, one of the sides of the triangle *ABC*,
as *CD* is to *DA*, so is *BE* to *EA*; (VI. 2.)
and by composition, *CA* is to *AD*, as *BA* to *AE*: (V. 18.)
but *CA* is a multiple of *AD*; (constr.)
therefore *BA* is the same multiple of *AE*: (V. D.)
whatever part therefore *AD* is of *AC*, *AE* is the same part of *AB*:
wherefore, from the straight line *AB* the part required is cut off. Q.E.F.

PROPOSITION X. PROBLEM.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Let AB be the straight line given to be divided, and AC the divided line.
It is required to divide AB similarly to AC.

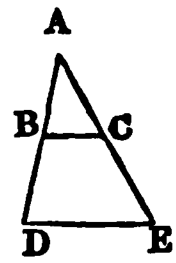


Let AC be divided in the points D, E;
and let AB, AC be placed so as to contain any angle, and join BC,
and through the points D, E draw DF, EG parallels to BC. (I. 31.)
Then AB shall be divided in the points F, G similarly to AC.
Through D draw DHK parallel to AB:
therefore each of the figures, FH, HB is a parallelogram;
wherefore DH is equal to FG, and HK to GB: (I. 34.)
and because HE is parallel to KC, one of the sides of the triangle DKC,
as CE to ED, so is KH to HD: (VI. 2.)
but KH is equal to BG, and HD to GF;
therefore, as CE to ED, so is BG to GF: (V. 7.)
again, because FD is parallel to GE, one of the sides of the triangle AGE,
as ED to DA, so is GF to FA: (VI. 2.)
therefore, as has been proved, CE is to ED, so is BG to GF,
and as ED to DA, so GF to FA:
therefore the given straight line AB is divided similarly to AC. Q.E.F.

PROPOSITION XI. PROBLEM.

To find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines.
It is required to find a third proportional to AB, AC.



Let AB, AC be placed so as to contain any angle:
produce AB, AC to the points D, E;
and make BD equal to AC;
join BC, and through D, draw DE parallel to BC. (I. 31.)
Then CE shall be a third proportional to AB and AC.
Because BC is parallel to DE, a side of the triangle ADE,
AB is to BD, as AC to CE: (VI. 2.)
but BD is equal to AC;
therefore as AB is to AC, so is AC to CE. (V. 7.)
Wherefore, to the two given straight lines AB, AC, a third proportional CE is found. Q.E.F.

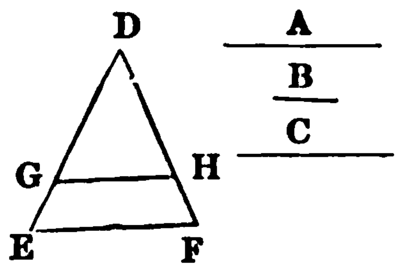
PROPOSITION XII. PROBLEM.

o find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines.

It is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF, containing any angle EDF: and upon these make DG equal to A, GE equal to B, and DH equal to C; (I. 3.)



join GH, and through E draw EF parallel to it. (I. 31.)

Then HF shall be the fourth proportional to A, B, C.

ecause GH is parallel to EF, one of the sides of the triangle DEF, DG is to GE, as DH to HF; (VI. 2.)

but DG is equal to A, GE to B, and DH to C;

therefore, as A is to B, so is C to HF. (V. 7.)

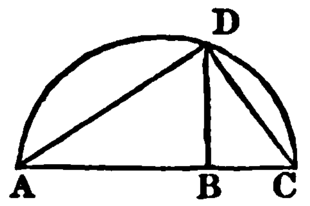
Wherefore to the three given straight lines A, B, C, a fourth proportional HF is found. Q.E.F.

PROPOSITION XIII. PROBLEM.

To find a mean proportional between two given straight lines.

Let AB, BC be the two given straight lines.

It is required to find a mean proportional between them.



Place AB, BC in a straight line, and upon AC describe the semicle ADC,

and from the point B draw BD at right angles to AC. (I. 11.)

Then BD shall be a mean proportional between AB and BC.

Join AD, DC.

ld because the angle ADC in a semicircle is a right angle, (III. 31.)

and because in the right-angled triangle ADC, BD is drawn from the right angle perpendicular to the base,

DB is a mean proportional between AB, BC the segments of the base: (VI. 8. Cor.)

therefore between the two given straight lines AB, BC, a mean proportional DB is found. Q.E.F.

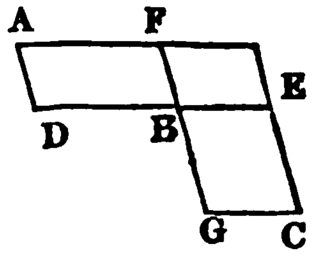
PROPOSITION XIV. THEOREM.

Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and conversely, parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let *AB, BC* be equal parallelograms, which have the angles at *B* equal.

The sides of the parallelograms *AB, BC* about the equal angles, shall be reciprocally proportional;

that is, *DB* shall be to *BE*, as *GB* to *BF*.



Let the sides *DB, BE* be placed in the same straight line;
wherefore also *FB, BG* are in one straight line: (I. 14.)
complete the parallelogram *FE*.

And because the parallelogram *AB* is equal to *BC*, and that *FE* is another parallelogram,
AB is to *FE*, as *BC* to *FE*: (v. 7.)
but as *AB* to *FE*, so is the base *DB* to *BE*, (vi. 1.)
and as *BC* to *FE*, so is the base *GB* to *BF*;
therefore, as *DB* to *BE*, so is *GB* to *BF*. (v. 11.)

Wherefore, the sides of the parallelograms *AB, BC* about their equal angles are reciprocally proportional.

Next, let the sides about the equal angles be reciprocally proportional,
viz. as *DB* to *BE*, so *GB* to *BF*:
the parallelogram *AB* shall be equal to the parallelogram *BC*.
Because, as *DB* to *BE*, so is *GB* to *BF*;
and as *DB* to *BE*, so is the parallelogram *AB* to the parallelogram *FE*; (vi. 1.)
and as *GB* to *BF*, so is the parallelogram *BC* to the parallelogram *FE*;
therefore as *AB* to *FE*, so *BC* to *FE*: (v. 11.)
therefore the parallelogram *AB* is equal to the parallelogram *BC*. (v. 9.)
Therefore equal parallelograms, &c. Q. E. D.

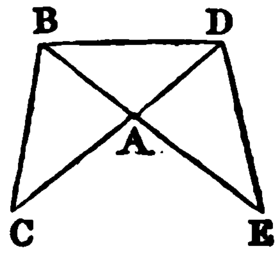
PROPOSITION XV. THEOREM.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and conversely, triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let *ABC, ADE* be equal triangles, which have the angle *BAC* equal to the angle *DAE*.

Then the sides about the equal angles of the triangles shall be reciprocally proportional;

that is, *CA* shall be to *AD*, as *EA* to *AB*.



Let the triangles be placed so that their sides *CA*, *AD* be in one straight line;

wherefore also *EA* and *AB* are in one straight line; (I. 14.)

and join *BD*.

Because the triangle *ABC* is equal to the triangle *ADE*,

and that *ABD* is another triangle;

therefore as the triangle *CAB*, is to the triangle *BAD*, so is the triangle *AED* to the triangle *DAB*; (V. 7.)

but as the triangle *CAB* to the triangle *BAD*, so is the base *CA* to the base *AD*, (VI. 1.)

and as the triangle *EAD* to the triangle *DAB*, so is the base *EA* to the base *AB*; (VI. 1.)

therefore as *CA* to *AD*, so is *EA* to *AB*: (V. 11.)

wherefore the sides of the triangles *ABC*, *ADE*, about the equal angles are reciprocally proportional.

Next, let the sides of the triangles *ABC*, *ADE* about the equal angles be reciprocally proportional,

viz. *CA* to *AD*, as *EA* to *AB*.

Then the triangle *ABC* shall be equal to the triangle *ADE*.

Join *BD* as before.

Then because, as *CA* to *AD*, so is *EA* to *AB*; (hyp.)

and as *CA* to *AD*, so is the triangle *ABC* to the triangle *BAD*; (VI. 1.)

and as *EA* to *AB*, so is the triangle *EAD* to the triangle *BAD*; (VI. 1.)

therefore as the triangle *BAC* to the triangle *BAD*, so is the triangle *EAD* to the triangle *BAD*; (V. 11.)

that is, the triangles *BAC*, *EAD* have the same ratio to the triangle *BAD*:

wherefore the triangle *ABC* is equal to the triangle *ADE*. (V. 9.)

Therefore, equal triangles, &c. Q.E.D.

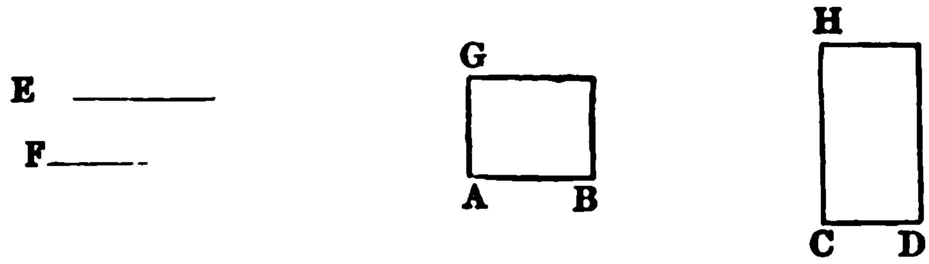
PROPOSITION XVI. THEOREM.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and conversely, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Let the four straight lines *AB*, *CD*, *E*, *F* be proportionals,

viz. as *AB* to *CD*, so *E* to *F*.

The rectangle contained by *AB*, *F*, shall be equal to the rectangle contained by *CD*, *E*.



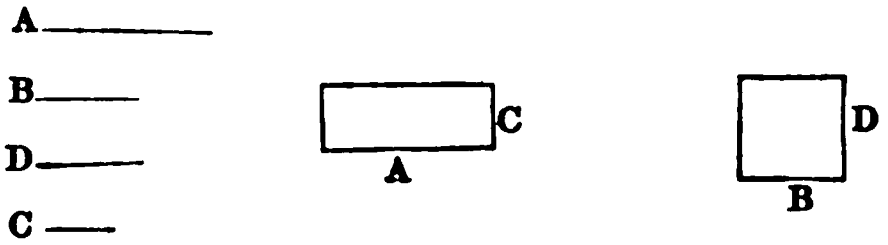
From the points A, C draw AG, CH at right angles to AB, CD; (I. 11.)
and make AG equal to F, and CH equal to E: (I. 3.)
and complete the parallelograms BG, DH. (I. 31.)
Because, as AB to CD, so is E to F;
and that E is equal to CH, and F to AG;
AB is to CD as CH to AG: (V. 7.)
therefore the sides of the parallelograms BG, DH about the equal angles are reciprocally proportional;
but parallelograms which have their sides about equal angles reciprocally proportional, are equal to one another; (VI. 14.)
therefore the parallelogram BG is equal to the parallelogram DH:
but the parallelogram BG is contained by the straight lines AB, F;
because AG is equal to F;
and the parallelogram DH is contained by CD and E;
because CH is equal to E;
therefore the rectangle contained by the straight lines AB, F, is equal to that which is contained by CD and E.

And if the rectangle contained by the straight lines AB, F be equal to that which is contained by CD, E;
these four lines shall be proportional,
viz, AB shall be to CD, as E to F.
The same construction being made,
because the rectangle contained by the straight lines AB, F, is equal to that which is contained by CD, E,
and that the rectangle BG is contained by AB, F;
because AG is equal to F;
and the rectangle DH by CD, E; because CH is equal to E;
therefore the parallelogram BG is equal to the parallelogram DH; (ax. 1.)
and they are equiangular:
but the sides about the equal angles of equal parallelograms are reciprocally proportional: (VI. 14.)
wherefore, as AB to CD, so is CH to AG:
but CH is equal to E, and AG to F;
therefore as AB is to CD, so is E to F. (V. 7.)
Wherefore if four, &c. Q.E.D.

PROPOSITION XVII. THEOREM.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean; and conversely, if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

Let the three straight lines A, B, C be proportionals,
viz. as A to B, so B to C.
The rectangle contained by A, C shall be equal to the square of B.



Take D equal to B.

And because as A to B, so B to C, and that B is equal to D;

A is to B, as D to C: (v. 7.)

but if four straight lines be proportionals, the rectangle contained ʏ the extremes is equal to that which is contained by the means; ɪ. 16.)

therefore the rectangle contained by A, C is equal to that contained by B, D:

but the rectangle contained by B, D, is the square of B, because B is equal to D;

herefore the rectangle contained by A, C, is equal to the square of B.

And if the rectangle contained by A, C, be equal to the square of B.

Then A shall be to B as B to C.

The same construction being made,

because the rectangle contained by A, C is equal to the square of B, and the square of B is equal to the rectangle contained by B, D, because B is equal to D;

therefore the rectangle contained by A, C, is equal to that contained by B, D:

but if the rectangle contained by the extremes be equal to that ɔntained by the means, the four straight lines are proportionals: ˈɪ. 16.)

therefore A is to B, as D to C:

but B is equal to D;

wherefore, as A to B, so B to C.

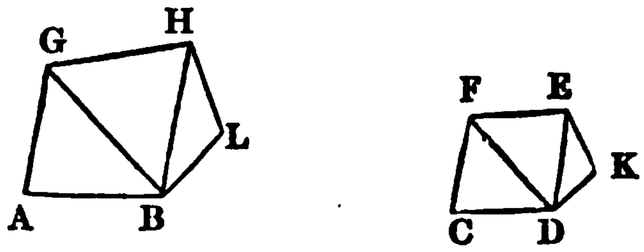
Therefore, if three straight lines, &c. Q.E.D.

PROPOSITION XVIII. PROBLEM.

Upon a given straight line to describe a rectilineal figure similar, and ʍilarly situated, to a given rectilineal figure.

Let AB be the given straight line, and $CDEF$ the given rectilineal ʒure of four sides.

It is required upon the given straight line AB to describe a rectilineal ʒure similar, and similarly situated, to $CDEF$.



Join DF, and at the points A, B in the straight line AB make the ɪgle BAG equal to the angle at C, (ɪ. 23.)

and the angle ABG equal to the angle CDF;

therefore the remaining angle AGB is equal to the remaining angle FD: (ɪ. 32. and ax. 3.)

therefore the triangle *FCD* is equiangular to the triangle *GAB*:
again, at the points *G*, *B*, in the straight line *GB*, make the angle *BGH* equal to the angle *DFE*, (I. 23.)
and the angle *GBH* equal to *FDE*;
therefore the remaining angle *GHB* is equal to the remaining angle *FED*,
and the triangle *FDE* equiangular to the triangle *GBH*:
then, because the angle *AGB* is equal to the angle *CFD*, and *BGH* to *DFE*,
the whole angle *AGH* is equal to the whole *CFE*; (ax. 2.)
for the same reason, the angle *ABH* is equal to the angle *CDE*:
also the angle at *A* is equal to the angle at *C*, (constr.)
and the angle *GHB* to *FED*:
therefore the rectilineal figure *ABHG* is equiangular to *CDEF*:
likewise these figures have their sides about the equal angles proportionals;
because the triangles *GAB*, *FCD* being equiangular,
BA is to *AG*, as *DC* to *CF*; (VI. 4.)
and because *AG* is to *GB*, as *CF* to *FD*;
and as *GB* is to *GH*, so is *FD* to *FE*,
by reason of the equiangular triangles *BGH*, *DFE*,
therefore, ex æquali, *AG* is to *GH*, as *CF* to *FE*. (V. 22.)
In the same manner it may be proved that *AB* is to *BH*, as *CD* to *DE*:
and *GH* is to *HB*, as *FE* to *ED*. (VI. 4.)
Wherefore, because the rectilineal figures *ABHG*, *CDEF* are equiangular,
and have their sides about the equal angles proportionals,
they are similar to one another. (VI. def. 1.)

Next, let it be required to describe upon a given straight line *AB*, a rectilineal figure similar, and similarly situated, to the rectilineal figure *CDKEF* of five sides.

Join *DE*, and upon the given straight line *AB* describe the rectilineal figure *ABHG* similar, and similarly situated, to the quadrilateral figure *CDEF*, by the former case:
and at the points *B*, *H*, in the straight line *BH*, make the angle *HBL* equal to the angle *EDK*,
and the angle *BHL* equal to the angle *DEK*;
therefore the remaining angle at *L* is equal to the remaining angle at *K*. (I. 32. and ax. 3.)
And because the figures *ABHG*, *CDEF* are similar,
the angle *GHB* is equal to the angle *FED*: (VI. def. 1.)
and *BHL* is equal to *DEK*;
wherefore the whole angle *GHL* is equal to the whole angle *FEK*:
for the same reason the angle *ABL* is equal to the angle *CDK*:
therefore the five-sided figures *AGHLB*, *CFEKD* are equiangular:
and because the figures *AGHB*, *CFED* are similar,
GH is to *HB*, as *FE* to *ED*; (VI. def. 1.)
but as *HB* to *HL*, so is *ED* to *EK*; (VI. 4.)
therefore, ex æquali, *GH* is to *HL*, as *FE* to *EK*: (V. 22.)
for the same reason, *AB* is to *BL*, as *CD* to *DK*:
and *BL* is to *LH*, as *DK* to *KE*, (VI. 4.)
because the triangles *BLH*, *DKE* are equiangular:
therefore because the five-sided figures *AGHLB*, *CFEKD* are equiangular,

and have their sides about the equal angles proportionals,
they are similar to one another.

In the same manner a rectilineal figure of six sides may be described upon a given straight line similar to one given, and so on. Q.E.F.

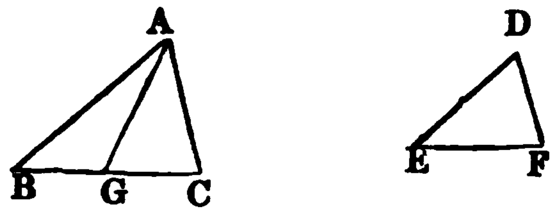
PROPOSITION XIX. THEOREM.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let *ABC, DEF* be similar triangles, having the angle *B* equal to the angle *E*,

and let *AB* be to *BC*, as *DE* to *EF*,
so that the side *BC* may be homologous to *EF*. (v. def. 12.)

Then the triangle *ABC* shall have to the triangle *DEF* the duplicate ratio of that which *BC* has to *EF*.



Take *BG* a third proportional to *BC, EF*, (VI. 11.)
so that *BC* may be to *EF*, as *EF* to *BG*, and join *GA*.
Then, because, as *AB* to *BC*, so *DE* to *EF*;
alternately, *AB* is to *DE*, as *BC* to *EF*; (v. 16.)
but as *BC* to *EF*, so is *EF* to *BG*; (constr.)
therefore, as *AB* to *DE*, so is *EF* to *BG*: (v. 11.)
therefore the sides of the triangles *ABG, DEF*, which are about the equal angles, are reciprocally proportional:
but triangles, which have the sides about two equal angles reciprocally proportional, are equal to one another; (VI. 15.)
therefore the triangle *ABG* is equal to the triangle *DEF*:
and because as *BC* is to *EF*, so *EF* to *BG*;
and that if three straight lines be proportional, the first is said to have to the third the duplicate ratio of that which it has to the second; (v. def. 10.)
therefore *BC* has to *BG* the duplicate ratio of that which *BC* has to *EF*:
but as *BC* to *BG*, so is the triangle *ABC* to the triangle *ABG*; (VI. 1.)
therefore the triangle *ABC* has to the triangle *ABG* the duplicate ratio of that which *BC* has to *EF*:
but the triangle *ABG* is equal to the triangle *DEF*;
therefore also the triangle *ABC* has to the triangle *DEF* the duplicate ratio of that which *BC* has to *EF*.
Therefore similar triangles, &c. Q.E.D.

COR. From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar and similarly described triangle upon the second.

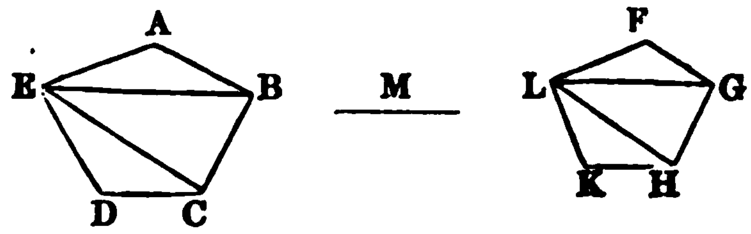
PROPOSITION XX. THEOREM.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let *ABCDE, FGHKL* be similar polygons, and let *AB* be the side homologous to *FG*:

the polygons *ABCDE, FGHKL* may be divided into the same number of similar triangles, whereof each shall have to each the same ratio which the polygons have;

and the polygon *ABCDE* shall have to the polygon *FGHKL* the duplicate ratio of that which the side *AB* has to the side *FG*.



Join *BE, EC, GL, LH*.

And because the polygon *ABCDE* is similar to the polygon *FGHKL*,
the angle *BAE* is equal to the angle *GFL*, (VI. def. 1.)
and *BA* is to *AE*, as *GF* to *FL*: (VI. def. 1.)
therefore, because the triangles *ABE, FGL* have an angle in one equal to an angle in the other, and their sides about these equal angles p oportionals,
the triangle *ABE* is equiangular to the triangle *FGL*; (VI. 6.)
and therefore similar to it; (VI. 4.)
wherefore the angle *ABE* is equal to the angle *FGL*:
and, because the polygons are similar,
the whole angle *ABC* is equal to the whole angle *FGH*; (VI. def. 1.)
therefore the remaining angle *EBC* is equal to the remaining angle *LGH*: (I. 32. and ax. 3.)
and because the triangles *ABE, FGL* are similar,
EB is to *BA*, as *LG* to *GF*; (VI. 4.)
and also, because the polygons are similar,
AB is to *BC*, as *FG* to *GH*; (VI. def. 1.)
therefore, ex æquali, *EB* is to *BC*, as *LG* to *GH*; (V. 22.)
that is, the sides about the equal angles *EBC, LGH* are proportionals;
therefore, the triangle *EBC* is equiangular to the triangle *LGH*, (VI. 6.) and similar to it; (VI. 4.)
for the same reason, the triangle *ECD* likewise is similar to the triangle *LHK*:
therefore the similar polygons *ABCDE, FGHKL* are divided into the same number of similar triangles.

Also these triangles shall have, each to each, the same ratio which the polygons have to one another,
the antecedents being *ABE, EBC, ECD*, and the consequents *FGL, LGH, LHK*:

and the polygon *ABCDE* shall have to the polygon *FGHKL* the duplicate ratio of that which the side *AB* has to the homologous side *FG*.

Because the triangle *ABE* is similar to the triangle *FGL*,
ABE has to *FGL*, the duplicate ratio of that which the side *BE* has to the side *GL*: (VI. 19.)
for the same reason, the triangle *BEC* has to *GLH* the duplicate ratio of that which *BE* has to *GL*:
therefore, as the triangle *ABE* is to the triangle *FGL*, so is the triangle *BEC* to the triangle *GLH*. (V. 11.)

Again, because the triangle EBC is similar to the triangle LGH,
EBC has to LGH, the duplicate ratio of that which the side EC has to the side LH:

for the same reason, the triangle ECD has to the triangle LHK, the duplicate ratio of that which EC has to LH:

therefore, as the triangle EBC to the triangle LGH, so is the triangle ECD to the triangle LHK: (v. 11.)

but it has been proved,

that the triangle EBC is likewise to the triangle LGH, as the triangle ABE to the triangle FGL:

therefore, as the triangle ABE to the triangle FGL, so is the triangle EBC to the triangle LGH, and the triangle ECD to the triangle LHK:

and therefore, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents: (v. 12.)

that is, as the triangle ABE to the triangle FGL, so is the polygon $ABCDE$ to the polygon $FGHKL$:

but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the side AB has to the homologous side FG; (vi. 19.)

therefore also the polygon $ABCDE$ has to the polygon $FGHKL$ the duplicate ratio of that which AB has to the homologous side FG.

Wherefore similar polygons, &c. Q.E.D.

Cor. 1. In like manner it may be proved, that similar four-sided figures, or of any number of sides, are one to another in the duplicate ratio of their homologous sides: and it has already been proved in triangles: (vi. 19.) therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

Cor. 2. And if to AB, FG, two of the homologous sides, a third proportional M be taken, (vi. 11.)

AB has to M the duplicate ratio of that which AB has to FG: (v. def. 10.)

but the four-sided figure or polygon upon AB, has to the four-sided figure or polygon upon FG likewise the duplicate ratio of that which AB has to FG; (1. Cor.)

therefore, as AB is to M, so is the figure upon AB to the figure upon FG: (v. 11.)

which was also proved in triangles: (vi. 19. Cor.)

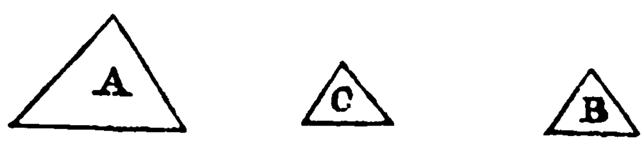
therefore, universally, it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

PROPOSITION XXI. THEOREM.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rectilineal figures A, B be similar to the rectilineal figure C.

The figure A shall be similar to the figure B.

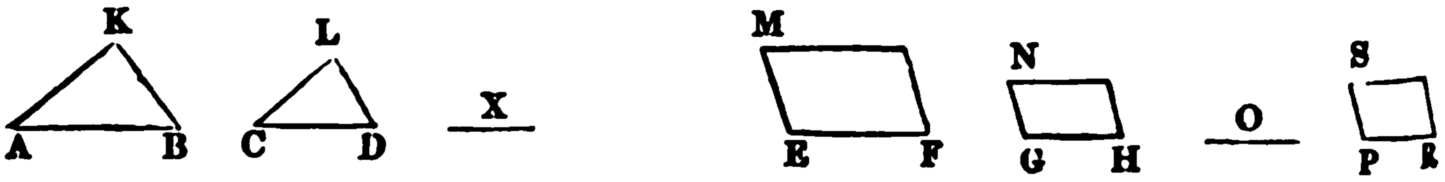


Because A is similar to C,
they are equiangular, and also have their sides about the equal angles proportional: (VI. def. 1.)
again, because B is similar to C,
they are equiangular, and have their sides about the equal angles proportionals: (VI. def. 1.)
therefore the figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals.
Wherefore the rectilineal figures A and C are equiangular, (I. ax. 1.) and have their sides about the equal angles proportionals: (V. 11.)
therefore A is similar to B. (VI. def. 1.)
Therefore rectilineal figures, &c. Q.E.D.

PROPOSITION XXII. THEOREM.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals: and conversely, if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals,
viz. AB to CD, as EF to GH;
and upon AB, CD let the similar rectilineal figures KAB, LCD be similarly described;
and upon EF, GH the similar rectilineal figures MF, NH, in like manner:
the rectilineal figure KAB shall be to LCD, as MF to NH.



To AB, CD take a third proportional X; (VI. 11.)
and to EF, GH a third proportional O:
and because AB is to CD as EF to GH,
therefore CD is to X, as GH to O; (V. 11.)
wherefore, ex æquali, as AB to X, so EF to O: (V. 22.)
but as AB to X, so is the rectilineal figure KAB to the rectilineal figure LCD,
and as EF to O, so is the rectilineal figure MF to the rectilineal figure NH: (VI. 20. Cor. 2.)
therefore, as KAB to LCD, so is MF to NH. (V. 11.)
And if the rectilineal figure KAB be to LCD, as MF to NH;
the straight line AB shall be to CD, as EF to GH.
Make as AB to CD, so EF to PR, (VI. 12.)
and upon PR describe the rectilineal figure SR similar and similarly situated to either of the figures MF, NH: (VI. 18.)
then, because as AB to CD, so is EF to PR,
and that upon AB, CD are described the similar and similarly situated rectilineals KAB, LCD,
and upon EF, PR, in like manner, the similar rectilineals MF, SR;
therefore KAB is to LCD, as MF to SR:
but by the hypothesis KAB is to LCD, as MF to NH;

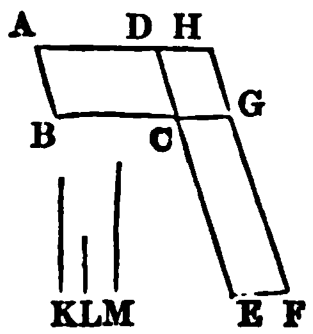
and therefore the rectilineal *MF* having the same ratio to each of the two *NH, SR,*
these are equal to one another; (v. 9.)
they are also similar, and similarly situated;
therefore *GH* is equal to *PR*:
and because as *AB* to *CD*, so is *EF* to *PR*,
and that *PR* is equal to *GH*;
AB is to *CD*, as *EF* to *GH*. (v. 7.)
If, therefore, four straight lines, &c. Q.E.D.

PROPOSITION XXIII. THEOREM.

Equiangular parallelograms have to one another the ratio which is ompounded of the ratios of their sides.

Let *AC, CF* be equiangular parallelograms, having the angle *BCD* qual to the angle *ECG*.

Then the ratio of the parallelogram *AC* to the parallelogram *CF*, hall be the same with the ratio which is compounded of the ratios of heir sides.



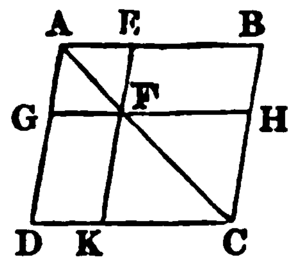
Let *BC, CG* be placed in a straight line;
therefore *DC* and *CE* are also in a straight line; (I. 14.)
and complete the parallelogram *DG*;
and taking any straight line *K*,
make as *BC* to *CG*, so *K* to *L*; (VI. 12.)
and as *DC* to *CE*, so make *L* to *M*: (VI. 12.)
therefore, the ratios of *K* to *L*, and *L* to *M*, are the same with the ratios of the sides,
viz. of *BC* to *CG*, and *DC* to *CE*:
but the ratio of *K* to *M* is that which is said to be compounded of the ratios of *K* to *L*, and *L* to *M*; (v. def. A.)
therefore *K* has to *M* the ratio compounded of the ratios of the sides:
and because as *BC* to *CG*, so is the parallelogram *AC* to the parallelogram *CH*; (VI. 1.)
but as *BC* to *CG*, so is *K* to *L*;
therefore *K* is to *L*, as the parallelogram *AC* to the parallelogram *CH*: (v. 11.)
again, because as *DC* to *CE*, so is the parallelogram *CH* to the parallelogram *CF*;
but as *DC* to *CE*, so is *L* to *M*;
wherefore *L* is to *M*, as the parallelogram *CH* to the parallelogram *CF*; (v. 11.)
therefore since it has been proved,
that as *K* to *L*, so is the parallelogram *AC* to the parallelogram *CH*;

and as L to M, so is the parallelogram CH to the parallelogram CF;
ex æquali, K is to M, as the parallelogram AC to the parallelogram CF: (v. 22.)
but K has to M the ratio which is compounded of the ratios of the sides;
therefore also the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides.
Wherefore equiangular parallelograms, &c. Q. E. D.

PROPOSITION XXIV. THEOREM.

Parallelograms about the diameter of any parallelogram, are similar to the whole, and to one another.

Let $ABCD$ be a parallelogram, of which the diameter is AC;
and EG, HK parallelograms about the diameter.
The parallelograms EG, HK shall be similar both to the whole parallelogram $ABCD$, and to one another.



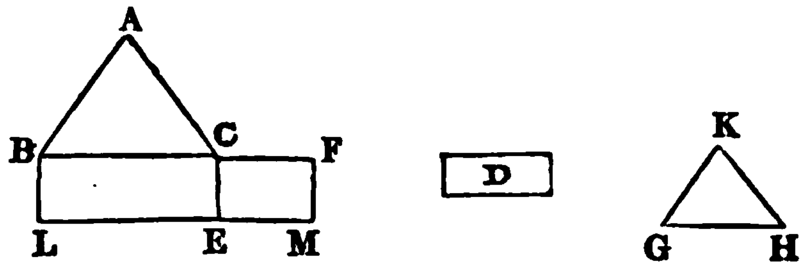
Because DC, GF are parallels,
the angle ADC is equal to the angle AGF: (I. 29.)
for the same reason, because BC, EF are parallels,
the angle ABC is equal to the angle AEF:
and each of the angles BCD, EFG is equal to the opposite angle DAB, (I. 34.)
and therefore they are equal to one another:
wherefore the parallelograms $ABCD$, $AEFG$, are equiangular:
and because the angle ABC is equal to the angle AEF,
and the angle BAC common to the two triangles BAC, EAF,
they are equiangular to one another;
therefore as AB to BC, so is AE to EF: (VI. 4.)
and because the opposite sides of parallelograms are equal to one another, (I. 34.)
AB is to AD as AE to AG; (v. 7.)
and DC to CB, as GF to FE;
and also CD to DA, as FG to GA:
therefore the sides of the parallelograms $ABCD$, $AEFG$ about the equal angles are proportionals;
and they are therefore similar to one another; (VI. def. 1.)
for the same reason, the parallelogram $ABCD$ is similar to the parallelogram $FHCK$:
wherefore each of the parallelograms GE, KH is similar to DB:
but rectilineal figures which are similar to the same rectilineal figure, are also similar to one another: (VI. 21.)
therefore the parallelogram GE is similar to KH.
Wherefore parallelograms, &c. (Q. E. D.)

PROPOSITION XXV. PROBLEM.

To describe a rectilineal figure which shall be similar to one, and equal o another given rectilineal figure.

Let *ABC* be the given rectilineal figure, to which the figure to be lescribed is required to be similar, and *D* that to which it must be equal.

It is required to describe a rectilineal figure similar to *ABC*, and equal to *D*.



Upon the straight line *BC* describe the parallelogram *BE* equal to the figure *ABC*; (I. 45. Cor.)
also upon *CE* describe the parallelogram *CM* equal to *D*, (I. 45. Cor.) and having the angle *FCE* equal to the angle *CBL*:
therefore *BC* and *CF* are in a straight line, as also *LE* and *EM*: (I. 29. and I. 14.)
between *BC* and *CF* find a mean proportional *GH*, (VI. 13.)
and upon *GH* describe the rectilineal figure *KGH* similar and similarly situated to the figure *ABC*. (VI. 18.)

Because *BC* is to *GH* as *GH* to *CF*,
and that if three straight lines be proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly described figure upon the second; (VI. 20. Cor. 2.)
therefore, as *BC* to *CF*, so is the rectilineal figure *ABC* to *KGH*:
but as *BC* to *CF*, so is the parallelogram *BE* to the parallelogram *EF*; (VI. 1.)
therefore as the rectilineal figure *ABC* is to *KGH*, so is the parallelogram *BE* to the parallelogram *EF*: (V. 11.)
and the rectilineal figure *ABC* is equal to the parallelogram *BE*; (constr.)
therefore the rectilineal figure *KGH* is equal to the parallelogram *EF*: (V. 14.)
but *EF* is equal to the figure *D*; (constr.)
wherefore also *KGH* is equal to *D*: and it is similar to *ABC*.

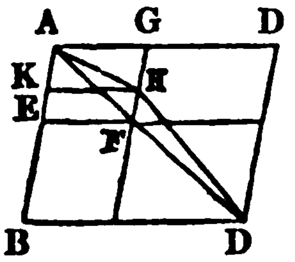
Therefore the rectilineal figure *KGH* has been described similar to the figure *ABC*, and equal to *D*. Q.E.F.

PROPOSITION XXVI. THEOREM.

If two similar parallelograms have a common angle, and be similarly situated; they are about the same diameter.

Let the parallelograms *ABCD*, *AEFG* be similar and similarly situated, and have the angle *DAB* common.

ABCD and *AEFG* shall be about the same diameter.



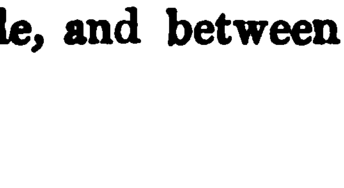
For, if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF, the diameter of the parallelogram EG,

and let GF meet AHC in H;
and through H draw HK parallel to AD or BC:
therefore the parallelograms $ABCD$, $AKHG$ being about the same diameter, they are similar to one another; (VI. 24.)
wherefore as DA to AB, so is GA to AK: (VI. def. 1.)
but because $ABCD$ and $AEFG$ are similar parallelograms, (hyp.)
as DA is to AB, so is GA to AE;
therefore as GA to AE, so GA to AK; (V. 11.)
that is, GA has the same ratio to each of the straight lines AE, AK;
and consequently AK is equal to AE, (V. 9.)
the less equal to the greater, which is impossible:
therefore $ABCD$ and $AKHG$ are not about the same diameter:
wherefore $ABCD$ and $AEFG$ must be about the same diameter.
Therefore, if two similar, &c. Q. E. D.

'To understand the three following propositions more easily, it is to be observed:

1. 'That a parallelogram is said to be applied to a straight line, when it is described upon it as one of its sides. Ex. gr. the parallelogram AC is said to be applied to the straight line AB.

2. 'But a parallelogram AE is said to be applied to a straight line AB, deficient by a parallelogram, when AD the base of AE is less than AB, and therefore AE is less than the parallelogram AC described upon AB in the same angle, and between the same parallels, by the parallelogram DC; and DC is therefore called the defect of AE.



3. 'And a parallelogram AG is said to be applied to a straight line AB, exceeding by a parallelogram, when AF the base of AG is greater than AB, and therefore AG exceeds AC the parallelogram described upon AB in the same angle, and between the same parallels, by the parallelogram BG.'

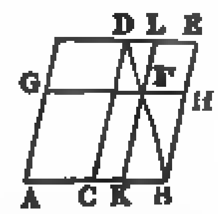
PROPOSITION XXVII. THEOREM.

Of all parallelograms applied to the same straight line, and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half, and is similar to its defect, is the greatest.

Let AB be a straight line divided into two equal parts in C;
and let the parallelogram AD be applied to the half AC, which is therefore deficient from the parallelogram upon the whole line AB by the parallelogram CE upon the other half CB:

of all the parallelograms applied to any other parts of AB, and deficient by parallelograms that are similar and similarly situated to CE, AD shall be the greatest.

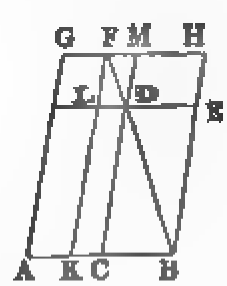
Let AF be any parallelogram applied to AK, any other part of AB than the half, so as to be deficient from the parallelogram upon the whole line AB by the parallelogram KH similar and similarly situated to CE:



AD shall be greater than AF.

First, let AK the base of AF, be greater than AC the half of AB:
and because CE is similar to the parallelogram HK, (hyp.)
they are about the same diameter: (VI. 26.)
draw their diameter DB, and complete the scheme:
then, because the parallelogram CF is equal to FE, (I. 43.)
add KH to both;
therefore the whole CH is equal to the whole KE:
but CH is equal to CG, (I. 36.)
because the base AC is equal to the base CB;
therefore CG is equal to KE: (ax. 1.)
to each of these equals add CF;
then the whole AF is equal to the gnomon CHL: (ax. 2.)
therefore CE, or the parallelogram AD is greater than the parallelogram AF.

Next, let AK the base of AF be less than AC:



then, the same construction being made, because BC is equal to CA,
therefore HM is equal to MG; (I. 34.)
therefore, the parallelogram DH is equal to the parallelogram DG; (I. 36.)
wherefore DH is greater than LG:
but DH is equal to DK; (I. 43.)
therefore DK is greater than LG:
to each of these add AL;
then the whole AD is greater than the whole AF.
Therefore, of all parallelograms applied, &c. Q.E.D.

PROPOSITION XXVIII. PROBLEM.

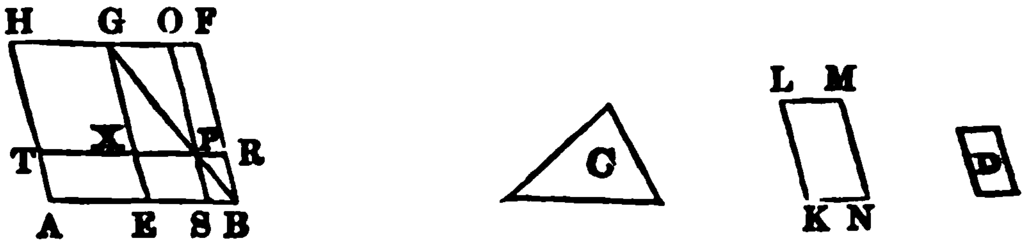
To a given straight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram: but the given rectilineal figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram

applied to half of the given line, having its defect similar to the defect of that which is to be applied; that is, to the given parallelogram.

Let AB be the given straight line, and C the given rectilineal figure, to which the parallelogram to be applied is required to be equal, which figure must not be greater (VI. 27.) than the parallelogram applied to the half of the line, having its defect from that upon the whole line similar to the defect of that which is to be applied;

and let D be the parallelogram to which this defect is required to be similar.

It is required to apply a parallelogram to the straight line AB, which shall be equal to the figure C, and be deficient from the parallelogram upon the whole line by a parallelogram similar to D.



Divide AB into two equal parts in the point E, (I. 10.)
and upon EB describe the parallelogram $EBFG$ similar and similarly situated to D, (VI. 18.)
and complete the parallelogram AG, which must either be equal to C, or greater than it, by the determination.

If AG be equal to C, then what was required is already done:
for, upon the straight line AB, the parallelogram AG is applied equal to the figure C, and deficient by the parallelogram EF similar to D.

But, if AG be not equal to C, it is greater than it:
and EF is equal to AG; (I. 36.)
therefore EF also is greater than C.

Make the parallelogram $KLMN$ equal to the excess of EF above C, and similar and similarly situated to D: (VI. 25.)
then, since D is similar to EF, (constr.)
therefore also KM is similar to EF: (VI. 21.)
let KL be the homologous side to EG, and LM to GF:
and because EF is equal to C and KM together,
EF is greater than KM;
therefore the straight line EG is greater than KL, and GF than LM:
make GX equal to LK, and GO equal to LM, (I. 3.)
and complete the parallelogram $XGOP$: (I. 31.)
therefore XO is equal and similar to KM:
but KM is similar to EF;
wherefore also XO is similar to EF;
and therefore XO and EF are about the same diameter: (VI. 26.)
let GPB be their diameter, and complete the scheme.

Then, because EF is equal to C and KM together,
and XO a part of the one is equal to KM a part of the other,
the remainder, viz. the gnomon ERO, is equal to the remainder C: (ax. 3.)
and because OR is equal to XS, by adding SR to each, (I. 43.)
the whole OB is equal to the whole XB:
but XB is equal to TE, because the base AE is equal to the base EB; (I. 36.)

wherefore also TE is equal to OB: (ax. 1.)
add XS to each, then the whole TS is equal to the whole, viz. to the gnomon ERO:
but it has been proved that the gnomon ERO is equal to C;
and therefore also TS is equal to C.

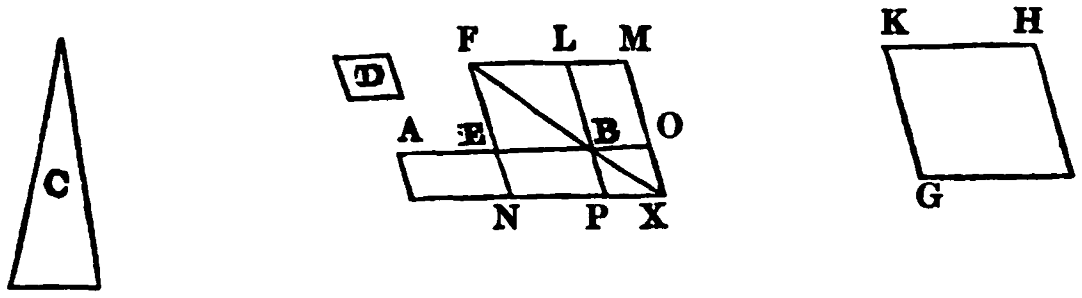
Wherefore the parallelogram TS, equal to the given rectilineal figure is applied to the given straight line AB, deficient by the parallelogram R, similar to the given one D, because SR is similar to EF. (VI. 24.) E. F.

PROPOSITION XXIX. PROBLEM.

To a given straight line to apply a parallelogram equal to a given :ctilineal figure, exceeding by a parallelogram similar to another given.

Let AB be the given straight line, and C the given rectilineal figure) which the parallelogram to be applied is required to be equal, and D he parallelogram to which the excess of the one to be applied above hat upon the given line is required to be similar.

It is required to apply a parallelogram to the given straight line lB which shall be equal to the figure C, exceeding by a parallelogram imilar to D.



Divide AB into two equal parts in the point E, (I. 10.) and upon EB describe the parallelogram EL similar and similarly situated to D: (VI. 18.)
and make the parallelogram GH equal to EL and C together, and similar and similarly situated to D: (VI. 25.)
wherefore GH is similar to EL: (VI. 21.)
let KH be the side homologous to FL, and KG to FE:
and because the parallelogram GH is greater than EL,
therefore the side KH is greater than FL,
and KG than FE:
produce FL and FE, and make FLM equal to KH, and FEN to KG,
and complete the parallelogram MN:
MN is therefore equal and similar to GH:
but GH is similar to EL;
wherefore MN is similar to EL;
and consequently EL and MN are about the same diameter: (VI. 26.)
draw their diameter FX, and complete the scheme.
Therefore, since GH is equal to EL and C together,
and that GH is equal to MN;
MN is equal to EL and C:
take away the common part EL;
then the remainder, viz. the gnomon NOL, is equal to C.
And because AE is equal to EB,
the parallelogram AN is equal to the parallelogram NB, (I. 36.) that is, to BM: (I. 43.)

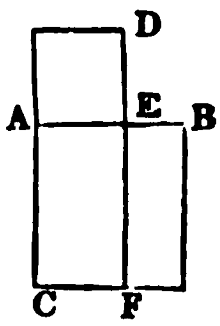
add NO to each;
therefore the whole, viz. the parallelogram AX, is equal to the gnomon NOL:
but the gnomon NOL is equal to C;
therefore also AX is equal to C.

Wherefore to the straight line AB there is applied the parallelogram AX equal to the given rectilineal figure C, exceeding by the parallelogram PO, which is similar to D, because PO is similar to EL. (VI. 24.) Q. E. F.

PROPOSITION XXX. PROBLEM.

To cut a given straight line in extreme and mean ratio.

Let AB be the given straight line.
It is required to cut it in extreme and mean ratio.



Upon AB describe the square BC, (I. 46.)
and to AC apply the parallelogram CD, equal to BC, exceeding by the figure AD similar to BC: (VI. 29.)
then, since BC is a square,
therefore also AD is a square:
and because BC is equal to CD,
by taking the common part CE from each,
the remainder BF is equal to the remainder AD:
and these figures are equiangular,
therefore their sides about the equal angles are reciprocally proportional: (VI. 14.)
therefore, as FE to ED, so AE to EB:
but FE is equal to AC, (I. 34.) that is, to AB; (def. 30.)
and ED is equal to AE;
therefore as BA to AE, so is AE to EB:
but AB is greater than AE;
wherefore AE is greater than EB: (V. 14.)
therefore the straight line AB is cut in extreme and mean ratio in E. (VI. def. 3.) Q. E. F.

Otherwise,

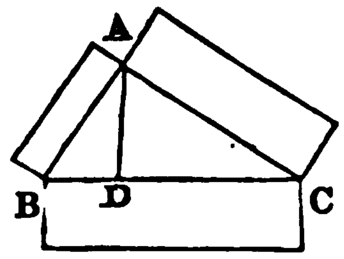
Let AB be the given straight line.
It is required to cut it in extreme and mean ratio.

Divide AB in the point C, so that the rectangle contained by AB, BC, may be equal to the square of AC. (II. 11.)
Then, because the rectangle AB, BC is equal to the square of AC;
as BA to AC, so is AC to CB: (VI. 17.)
therefore AB is cut in extreme and mean ratio in C. (VI. def. 3.) Q.E.F.

PROPOSITION XXXI. THEOREM.

In right-angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.

Let ABC be a right-angled triangle, having the right angle BAC.
The rectilineal figure described upon BC shall be equal to the similar and similarly described figures upon BA, AC.

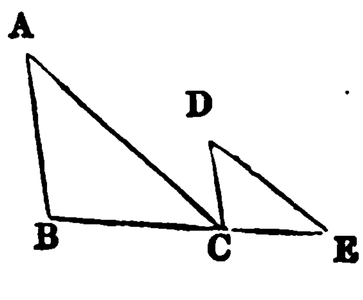


Draw the perpendicular AD: (I. 12.)
therefore, because in the right-angled triangle ABC,
AD is drawn from the right angle at A perpendicular to the base BC,
the triangles ABD, ADC are similar to the whole triangle ABC, and to one another: (VI. 8.)
and because the triangle ABC is similar to ADB,
as CB to BA, so is BA to BD: (VI. 4.)
and because these three straight lines are proportionals,
as the first is to the third, so is the figure upon the first to the similar and similarly described figure upon the second: (VI. 20. Cor. 2.)
therefore as CB to BD, so is the figure upon CB to the similar and similarly described figure upon BA:
and inversely, as DB to BC, so is the figure upon BA to that upon BC: (V. B.)
for the same reason, as DC to CB, so is the figure upon CA to that upon CB:
therefore as BD and DC together to BC, so are the figures upon BA, AC to that upon BC: (V. 24.)
but BD and DC together are equal to BC;
therefore the figure described on BC is equal to the similar and similarly described figures on BA, AC. (V. A.)
Wherefore, in right-angled triangles, &c. Q.E.D.

PROPOSITION XXXII. THEOREM.

If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another; the remaining sides shall be in a straight line.

Let ABC, DCE be two triangles which have the two sides BA, AC proportional to the two CD, DE,
viz. BA to AC, as CD to DE;
and let AB be parallel to DC, and AC to DE.



Then BC and CE shall be in a straight line.

Because AB is parallel to DC, and the straight line AC meets them, the alternate angles BAC, ACD are equal; (I. 29.)

for the same reason, the angle CDE is equal to the angle ACD;

wherefore also BAC is equal to CDE: (ax. 1.)

and because the triangles ABC, DCE have one angle at A equal to one at D, and the sides about these angles proportionals, viz. BA to AC, as CD to DE,

the triangle ABC is equiangular to DCE: (VI. 6.)

therefore the angle ABC is equal to the angle DCE:

and the angle BAC was proved to be equal to ACD;

therefore the whole angle ACE is equal to the two angles ABC, BAC: (ax. 2.)

add to each of these equals the common angle ACB,

then the angles ACE, ACB are equal to the angles ABC, BAC, ACB:

but ABC, BAC, ACB are equal to two right angles: (I. 32.)

therefore also the angles ACE, ACB are equal to two right angles:

and since at the point C, in the straight line AC, the two straight lines BC, CE, which are on the opposite sides of it, make the adjacent angles ACE, ACB equal to two right angles;

therefore BC and CE are in a straight line. (I. 14.)

Wherefore, if two triangles, &c. Q.E.D.

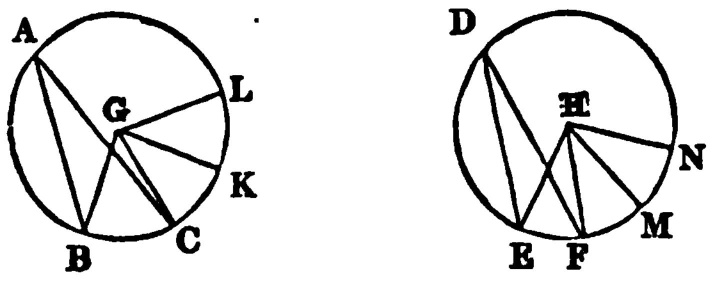
PROPOSITION XXXIII. THEOREM.

In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another: so also have the sectors.

Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF, at their circumferences.

As the circumference BC to the circumference EF, so shall the angle BGC be to the angle EHF, and the angle BAC to the angle EDF;

and also the sector BGC to the sector EHF.



Take any number of circumferences CK, KL, each equal to BC, and any number whatever FM, MN, each equal to EF:

and join GK, GL, HM, HN.

Because the circumferences BC, CK, KL are all equal,

the angles BGC, CGK, KGL are also all equal: (III. 27.)

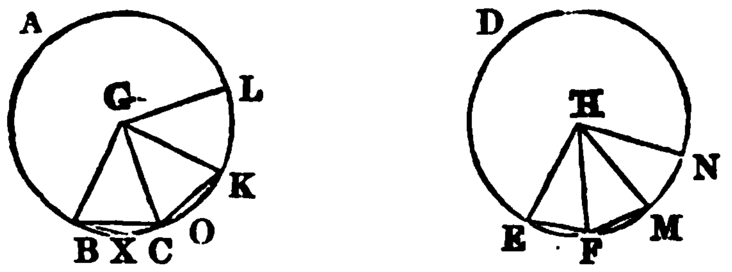
therefore what multiple soever the circumference BL is of the circumference BC, the same multiple is the angle BGL of the angle BGC:

for the same reason, whatever multiple the circumference EN is of the circumference EF, the same multiple is the angle EHN of the angle EHF:

and if the circumference BL be equal to the circumference EN,

the angle BGL is also equal to the angle EHN; (III. 27.)
and if the circumference BL be greater than EN,
likewise the angle BGL is greater than EHN; and if less, less:
therefore since there are four magnitudes, the two circumferences BC, EF, and the two angles BGC, EHF; and that of the circumference BC, and of the angle BGC, have been taken any equimultiples whatever, viz. the circumference BL, and the angle BGL; and of the circumference EF, and of the angle EHF, any equimultiples whatever, viz. the circumference EN, and the angle EHN;
and since it has been proved, that if the circumference BL be greater than EN;
the angle BGL is greater than EHN;
and if equal, equal; and if less, less;
therefore as the circumference BC to the circumference EF, so is the angle BGC to the angle EHF: (v. def. 5.)
but as the angle BGC is to the angle EHF, so is the angle BAC to the angle EDF: (v. 15.)
for each is double of each; (III. 20.)
therefore, as the circumference BC is to EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.

Also, as the circumference BC to EF, so shall the sector BGC be to the sector EHF.



Join BC, CK, and in the circumferences, BC, CK take any points X, O, and join BX, XC, CO, OK.
Then, because in the triangles GBC, GCK
the two sides BG, GC are equal to the two CG, GK each to each,
and that they contain equal angles;
the base BC is equal to the base CK, (I. 4.)
and the triangle GBC to the triangle GCK:
and because the circumference BC is equal to the circumference CK,
the remaining part of the whole circumference of the circle ABC, is equal to the remaining part of the whole circumference of the same circle: (ax. 3.)
therefore the angle BXC is equal to the angle COK; (III. 27.)
and the segment BXC is therefore similar to the segment COK; (III. def. 11.)
and they are upon equal straight lines, BC, CK:
but similar segment of circles upon equal straight lines, are equal to one another; (III. 24.)
therefore the segment BXC is equal to the segment COK:
and the triangle BGC was proved to be equal to the triangle CGK;
therefore the whole, the sector BGC, is equal to the whole, the sector CGK:
for the same reason, the sector KGL is equal to each of the sectors BGC, CGK:
in the same manner, the sectors EHF, FHM, MHN may be proved equal to one another:

therefore, what multiple soever the circumference BL is of the circumference BC, the same multiple is the sector BGL of the sector BGC;

and for the same reason, whatever multiple the circumference EN is of EF, the same multiple is the sector EHN of the sector EHF:

and if the circumference BL be equal to EN, the sector BGL is equal to the sector EHN;

and if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN;

and if less, less;

since, then, there are four magnitudes, the two circumferences BC, EF, and the two sectors BGC, EHF, and that of the circumference BC, and sector BGC, the circumference BL and sector BGL are any equimultiples whatever; and of the circumference EF, and sector EHF, the circumference EN, and sector EHN are any equimultiples whatever;

and since it has been proved, that if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN;

and if equal, equal; and if less, less:

therefore, as the circumference BC is to the circumference EF, so is the sector BGC to the sector EHF. (v. def. 5.)

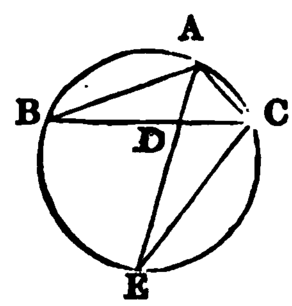
Wherefore, in equal circles, &c. Q. E. D.

PROPOSITION B. THEOREM.

If an angle of a triangle be bisected by a straight line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line which bisects the angle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD.

The rectangle BA, AC shall be equal to the rectangle BD, DC, together with the square of AD.



Describe the circle ACB about the triangle, (IV. 5.)

and produce AD to the circumference in E, and join EC.

Then because the angle BAD is equal to the angle CAE, (hyp.) and the angle ABD to the angle AEC, (III. 21.)

for they are in the same segment;

the triangles ABD, AEC are equiangular to one another: (I. 32.)

therefore as BA to AD, so is EA to AC; (VI. 4.)

and consequently the rectangle BA, AC is equal to the rectangle EA, AD, (VI. 16.)

that is, to the rectangle ED, DA, together with the square of AD; (II. 3.)

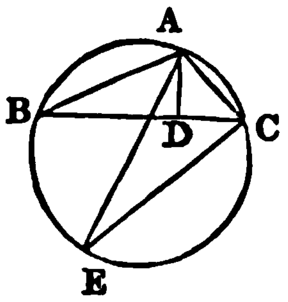
ut the rectangle ED, DA is equal to the rectangle BD, DC; (III. 35.)
therefore the rectangle BA, AC is equal to the rectangle BD, DC, together with the square of AD.
Wherefore, if an angle, &c. Q.E.D.

PROPOSITION C. THEOREM.

If from any angle of a triangle a straight line be drawn perpendicular) the base; the rectangle contained by the sides of the triangle is equal to he rectangle contained by the perpendicular and the diameter of the circle escribed about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A o the base BC.

The rectangle BA, AC shall be equal to the rectangle contained by AD and the diameter of the circle described about the triangle.



Describe the circle ACB about the triangle, (IV. 5.) and draw its diameter AE, and join EC.

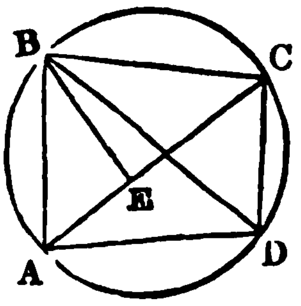
because the right angle BDA is equal to the angle ECA in a semicircle, (III. 31.)
and the angle ABD equal to the angle AEC in the same segment; (III. 21.)
the triangles ABD, AEC are equiangular:
therefore as BA to AD, so is EA to AC; (VI. 4.)
and consequently the rectangle BA, AC is equal to the rectangle EA, AD. (VI. 16.)
If therefore from an angle, &c. Q.E.D.

PROPOSITION D. THEOREM.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let $ABCD$ be any quadrilateral figure inscribed in a circle, and join AC, BD.

The rectangle contained by AC, BD shall be equal to the two rectangles contained by AB, CD, and by AD, BC.



Make the angle ABE equal to the angle DBC: (I. 23.)
add to each of these equals the common angle EBD,
then the angle ABD is equal to the angle EBC:
and the angle BDA is equal to the angle BCE, because they are in the same segment: (III. 21.)
therefore the triangle ABD is equiangular to the triangle BCE:
wherefore, as BC is to CE, so is BD to DA; (VI. 4.)
and consequently the rectangle BC, AD is equal to the rectangle BD, CE: (VI. 16.)
again, because the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC, (III. 21.)
the triangle ABE is equiangular to the triangle BCD:
therefore as BA to AE, so is BD to DC;
wherefore the rectangle BA, DC is equal to the rectangle BD, AE:
but the rectangle BC, AD has been shewn equal to the rectangle BD, CE;
therefore the whole rectangle AC, BD is equal to the rectangle AB, DC, together with the rectangle AD, BC. (II. 1.)
Therefore the rectangle, &c. Q.E.D.

This is a Lemma of Cl. Ptolemæus, in page 9 of his Μεγάλη Σύνταξις.

NOTES TO BOOK VI.

In this book, the theory of proportion exhibited in the fifth book, is applied to ıe comparison of the sides and areas of plane rectilineal figures, both to those which are ımilar, and those which are not similar.

Def. I. In defining similar triangles, one condition is sufficient, namely, that ımilar triangles are those which have their three angles respectively equal; as in Prop. 4, ook VI, it is proved that the sides about the equal angles of equiangular triangles are roportionals. But in defining similar figures of more than three sides, both of the onditions stated in Def. I, are requisite, as it is obvious, for instance, in the case of a uare and a rectangle, which have their angles respectively equal, but have not their des about their equal angles proportionals.

The following definition has been proposed: "Similar rectilineal figures of more ıan three sides, are those which may be divided into the same number of similar iangles." This definition would, if adopted, require the omission of a part of rop. 20, Book VI.

Def. III. To this definition may be added the following:

A straight line is said to be divided *harmonically*, when it is divided into three arts, such that the whole line is to one of the extreme segments, as the other extreme egment is to the middle part. Three lines are in *harmonical* proportion, when the first s to the third, as the difference between the first and second, is to the difference between he second and third; and the second is called a harmonic mean between the first and hird.

The expression 'harmonical proportion' is derived from the following fact in the Science of Acoustics, that three musical strings of the same material, thickness and tension, when divided in the manner stated in the definition, or numerically as 6, 4, and 3, produce a certain musical note, its fifth, and its octave.

Def. IV. The term *altitude*, as applied to the same triangles and parallelograms, will be different according to the sides which may be assumed as the base.

Prop. I. In the same manner may be proved, that triangles and parallelograms pon equal bases, are to one another as their altitudes.

Prop. A. When the triangle ABC is isosceles, the line which bisects the exterior ngle at the vertex is parallel to the base. In all other cases;

If the line which bisects the angle BAC cut the base BC in the point G.

Then the straight line BD is harmonically divided in the points G, C.

For BG is to GC as BA is to AC; (VI. 3.)

and BD is to DC as BA is to AC, (VI. A.)

therefore BD is to DC as BG is to GC,

but $BG = BD - DG$ and $GC = GD - DC$.

Wherefore BD is to DC as $BD - DG$ is to $GD - DC$.

Hence BD, DG, DC, are in harmonical proportion.

Prop. IV is the first case of similar triangles, and corresponds to the third case of ual triangles, Prop. 26, Book I.

Sometimes the sides opposite to the equal angles in two equiangular triangles, are lled the *corresponding sides*, and these are said to be proportional, which is simply king the proportion in Euclid alternately.

The term *homologous* (ὁμόλογος), has reference to the places the sides of the triangles ave in the ratios, and in one sense, homologous sides may be considered as corresponding

sides. The homologous sides of any two similar rectilineal figures will be found to be those which are adjacent to two equal angles in each figure.

Prop. V, the converse of Prop. IV, is the second case of similar triangles, and corresponds to Prop. 8, Book I, the second case of equal triangles.

Prop. VI is the third case of similar triangles, and corresponds to Prop. 4, Book I.

The property of similar triangles, and that contained in Prop. 47, Book I, are the most important theorems in Geometry.

Prop. VII is the fourth case of similar triangles, and corresponds to the fourth case of equal triangles pointed out in the note to Prop. 26, Book I, p. 49.

Prop. XIII may be compared with Prop. XVI, Book II.

It may be observed, that half the sum of AB and BC is called the *Arithmetic* mean between these lines; also that BD is called the *Geometric* mean between the same lines.

To find two mean proportionals between two given lines is impossible by the straight line and circle. Pappus has given several solutions of this problem in Book III, of his Mathematical Collections; and Eutocius has given, in his Commentary on the Sphere and Cylinder of Archimedes, ten different methods of solving this problem.

Prop. XIV, depends on the same principle as Prop. XV, and both may easily be demonstrated from one diagram. Join DF, FE, EG in the fig. to Prop. XIV, and the figure to Prop. XV is formed. We may add, that there does not appear any reason why the properties of the triangle and parallelogram should be here separated and not in the first proposition of the sixth book.

Prop. XV, holds good when one angle of one triangle is equal to the defect from what the corresponding angle in the other wants of two right angles.

Prop. XVII is only a particular case of Prop. XVI, and more properly, might appear as a corollary.

Algebraically, Let AB, CD, E, F, contain a, b, c, d units respectively.

Then, since a, b, c, d are proportionals, $\therefore\ \frac{a}{b} = \frac{c}{d}$.

Multiply these equals by bd, $\therefore\ ad = bc$,

or, the product of the extremes is equal to the product of the means.

And conversely, If the product of the extremes be equal to the product of the means,

or $ad = bc$,

then dividing these equals by bd, $\therefore\ \frac{a}{b} = \frac{c}{d}$,

or the ratio of the first to the second number, is equal to the ratio of the third to the fourth.

Similarly may be shewn, that if $\frac{a}{b} = \frac{b}{d}$; then $ad = b^2$.

And conversely, if $ad = b^2$; then $\frac{a}{b} = \frac{b}{d}$.

Prop. XVIII. Similar figures are said to be similarly situated when their homologous sides are parallel.

Prop. XX. It may easily be shewn, that the perimeters of similar polygons are proportional to their homologous sides.

Prop. XXXI. This proposition is an extension of Prop. 47, Book I, to similar rectilineal figures, and may be deduced from Prop. 22, Book VI, and Prop. 47, Book I.

Prop. B. The converse of this proposition does not hold good when the triangle is isosceles.

The seventh, eighth, ninth and tenth books of the Elements treat of numbers, and employ the Greek numerical notation.

BOOK XI.

DEFINITIONS.

I.

A SOLID is that which hath length, breadth, and thickness.

II.

That which bounds a solid is a superficies.

III.

A straight line is perpendicular, or at right angles, to a plane, when it makes right angles with every straight line meeting it in that plane.

IV.

A plane is perpendicular to a plane, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes, are perpendicular to the other plane.

V.

The inclination of a straight line to a plane is the acute angle contained by that straight line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.

VI.

The inclination of a plane to a plane is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one upon one plane, and the other upon the other plane.

VII.

Two planes are said to have the same, or a like inclination to one another, which two other planes have, when the said angles of inclination are equal to one another.

VIII.

Parallel planes are such as do not meet one another though produced.

IX.

A solid angle is that which is made by the meeting, in one point, of more than two plane angles, which are not in the same plane.

X.

Equal and similar solid figures are such as are contained by similar planes equal in number and magnitude.

XI.

Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

XII.

A pyramid is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet.

XIII.

A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others parallelograms.

XIV.

A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

XV.

The axis of a sphere is the fixed straight line about which the semicircle revolves.

XVI.

The centre of a sphere is the same with that of the semicircle.

XVII.

The diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

XVIII.

A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled; and if greater, an acute-angled cone.

XIX.

The axis of a cone is the fixed straight line about which the triangle revolves.

XX.

The base of a cone is the circle described by that side containing the right angle, which revolves.

XXI.

A cylinder is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed.

XXII.

The axis of a cylinder is the fixed straight line about which the parallelogram revolves.

XXIII.

The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

XXIV.

Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

XXV.

A cube is a solid figure contained by six equal squares.

XXVI.

A tetrahedron is a solid figure contained by four equal and equilateral triangles.

XXVII.

An octahedron is a solid figure contained by eight equal and equilateral triangles.

XXVIII.

A dodecahedron is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

Def. A.

A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

PROPOSITION I. THEOREM.

One part of a straight line cannot be in a plane, and another part above it.

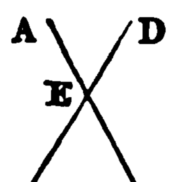
If it be possible, let AB, part of the straight line ABC, be in the plane, and the part BC above it:

and since the straight line AB is in the plane, it can be produced in that plane:
let it be produced to D;
and let any plane pass through the straight line AD, and be turned about it until it pass through the point C:
and because the points B, C are in this plane,
the straight line BC is in it: (I. def. 7.)
therefore there are two straight lines ABC, ABD in the same plane that have a common segment AB; (I. 11. Cor.)
which is impossible.
Therefore, one part, &c. Q.E.D.

PROPOSITION II. THEOREM.

Two straight lines which cut one another are in one plane, and three straight lines which meet one another are in one plane.

Let two straight lines AB, CD cut one another in E;
then AB, CD shall be in one plane:
and three straight lines EC, CB, BE, which meet one another, shall be in one plane.



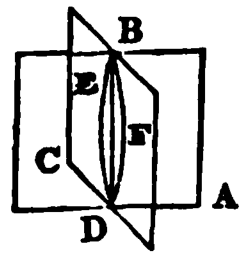
Let any plane pass through the straight line EB,
and let the plane be turned about EB, produced if necessary, until it pass through the point C.
Then, because the points E, C are in this plane,
the straight line EC is in it: (I. def. 7.)
for the same reason, the straight line BC is in the same:
and by the hypothesis, EB is in it:
therefore the three straight lines EC, CB, BE are in one plane;
but in the plane in which EC, EB are, in the same are CD, AB: (XI. 1.)
therefore, AB, CD are in one plane.
Wherefore two straight lines, &c. Q.E.D.

PROPOSITION III. THEOREM.

two planes cut one another, their common section is a straight line.

et two planes *AB*, *BC* cut one another, and let the line *DB* be common section.

Then *DB* shall be a straight line.



' it be not, from the point *D* to *B*, draw, in the plane *AB*, the ;ht line *DEB*, (post. 1.)

and in the plane *BC*, the straight line *DFB*:

:hen two straight lines *DEB*, *DFB* have the same extremities,

and therefore include a space betwixt them;

which is impossible: (I. ax. 10.)

:herefore *BD*, the common section of the planes *AB*, *BC*, cannot but be a straight line.

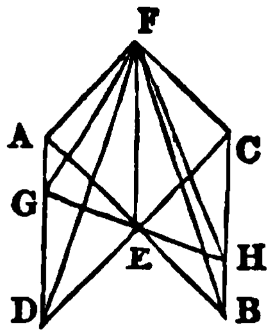
Wherefore, if two planes, &c. Q.E.D.

PROPOSITION IV. THEOREM.

ᶠ *a straight line stand at right angles to each of two straight lines e point of their intersection, it shall also be at right angles to the which passes through them, that is, to the plane in which they are.*

et the straight line *EF* stand at right angles to each of the straight *AB*, *CD*, in *E* the point of their intersection.

hen *EF* shall also be at right angles to the plane passing through *AB*, *CD*.



ke the straight lines *AE*, *EB*, *CE*, *ED* all equal to one another;

ınd through *E* draw, in the plane in which are *AB*, *CD*, any straight line *GEH*;

and join *AD*, *CB*;

en from any point *F*, in *EF*, draw *FA*, *FG*, *FD*, *FC*, *FH*, *FB*.

nd because the two straight lines *AE*, *ED* are equal to the two *BE*, *EC*, each to each,

and that they contain equal angles *AED*, *BEC*, (I. 15.)

the base *AD* is equal to the base *BC*, (I. 4.)

and the angle *DAE* to the angle *EBC*:

and the angle AEG is equal to the angle BEH: (I. 15.)
therefore the triangles AEG, BEH have two angles of the one equal to two angles of the other, each to each,
and the sides AE, EB, adjacent to the equal angles, equal to one another:
wherefore they have their other sides equal: (I. 26.)
therefore GE is equal to EH, and AG to BH:
and because AE is equal to EB, and FE common and at right angles to them,
the base AF is equal to the base FB; (I. 4.)
for the same reason, CF is equal to FD:
and because AD is equal to BC, and AF to FB,
the two sides FA, AD are equal to the two FB, BC, each to each;
and the base DF was proved equal to the base FC;
therefore the angle FAD is equal to the angle FBC: (I. 8.)
again, it was proved that GA is equal to BH, and also AF to FB;
therefore FA and AG are equal to FB and BH, each to each;
and the angle FAG has been proved equal to the angle FBH;
therefore the base GF is equal to the base FH: (I. 4.)
again, because it was proved that GE is equal to EH, and EF is common;
therefore GE, EF are equal to HE, EF, each to each;
and the base GF is equal to the base FH;
therefore the angle GEF is equal to the angle HEF; (I. 8.)
and consequently each of these angles is a right angle. (I. def. 10.)

Therefore FE makes right angles with GH, that is, with any straight line drawn through E in the plane passing through AB, CD.

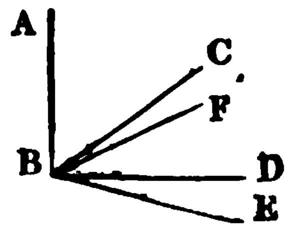
In like manner, it may be proved, that FE makes right angles with every straight line which meets it in that plane.

But a straight line is at right angles to a plane when it makes right angles with every straight line which meets it in that plane: (XI. def. 3.)
therefore EF is at right angles to the plane in which are AB, CD.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION V. THEOREM.

If three straight lines meet all in one point, and a straight line stands at right angles to each of them in that point; these three straight lines are in one and the same plane.

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE, in B the point where they meet:
Then BC, BD, BE shall be in one and the same plane.



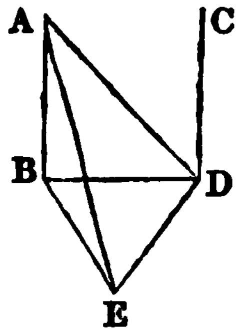
If not, let, if it be possible, BD and BE be in one plane, and BC be above it;
and let a plane pass through AB, BC, the common section of which, with the plane in which BD and BE are, is a straight line; (XI. 3.)

let this be BF:
therefore the three straight lines AB, BC, BF are all in one plane, viz. that which passes through AB, BC.
And because AB stands at right angles to each of the straight lines BD, BE,
it is also at right angles to the plane passing through them: (XI. 4.)
and therefore makes right angles with every straight line meeting it in that plane: (XI. def. 3.)
but BF, which is in that plane, meets it;
therefore the angle ABF is a right angle:
but the angle ABC, by the hypothesis, is also a right angle;
therefore the angle ABF is equal to the angle ABC,
and they are both in the same plane, which is impossible; (I. ax. 9.)
therefore the straight line BC is not above the plane in which are BD and BE:
wherefore the three straight lines BC, BD, BE are in one and the same plane.
Therefore, if three straight lines, &c. Q.E.D.

PROPOSITION VI. THEOREM.

If two straight lines be at right angles to the same plane, they shall be parallel to one another.

Let the straight lines AB, CD be at right angles to the same plane.
Then AB shall be parallel to CD.



Let them meet the plane in the points B, D,
and draw the straight line BD, to which draw DE at right angles, in the same plane; (I. 11.)
and make DE equal to AB, (I. 3.) and join BE, AE, AD.
Then, because AB is perpendicular to the plane,
it makes right angles with every straight line which meets it, and is in that plane: (XI. def. 3.)
but BD, BE, which are in that plane, do each of them meet AB;
therefore each of the angles ABD, ABE is a right angle;
for the same reason, each of the angles CDB, CDE is a right angle:
and because AB is equal to DE, and BD common,
the two sides AB, BD are equal to the two ED, DB, each to each;
and they contain right angles:
therefore the base AD is equal to the base BE: (I. 4.)
again, because AB is equal to DE, and BE to AD;
AB, BE are equal to ED, DA, each to each;
and, in the triangles ABE, EDA, the base AE is common:
therefore the angle ABE is equal to the angle EDA: (I. 8.)
but ABE is a right angle;

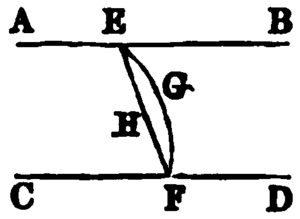
therefore EDA is also a right angle, and ED perpendicular to DA:
but it is also perpendicular to each of the two BD, DC;
wherefore ED is at right angles to each of the three straight lines BD, DA, DC in the point in which they meet:
therefore these three straight lines are all in the same plane: (XI. 5.)
but AB is in the plane in which are BD, DA, (XI. 2.)
because any three straight lines which meet one another are in one plane:
therefore AB, BD, DC are in one plane:
and each of the angles ABD, BDC is a right angle;
therefore AB is parallel to CD. (I. 28.)
Wherefore, if two straight lines, &c. Q.E.D.

PROPOSITION VII. THEOREM.

If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other, is in the same plane with the parallels.

Let AB, CD be parallel straight lines, and take any point E in the one, and the point F in the other.

Then the straight line which joins E and F shall be in the same plane with the parallels.



If not, let it be, if possible, above the plane, as EGF;
and in the plane $ABCD$ in which the parallels are,
draw the straight line EHF from E to F.

And since EGF also is a straight line, the two straight lines EHF, EGF include a space between them, which is impossible. (I. ax. 10.)

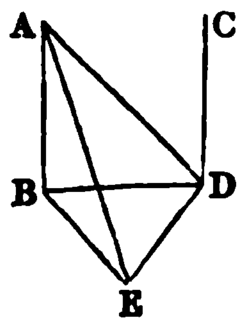
Therefore the straight line joining the points E, F is not above the plane in which the parallels AB, CD are,
and is therefore in that plane.
Wherefore, if two straight lines, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If two straight lines be parallel, and one of them is at right angles to a plane; the other also shall be at right angles to the same plane.

Let AB, CD be two parallel straight lines, and let one of them AB be at right angles to a plane.

Then the other CD shall be at right angles to the same plane.



Let AB, CD meet the plane in the points B, D, and join BD:
therefore AB, CD, BD are in one plane. (XI. 7.)

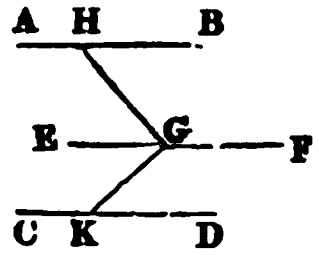
In the plane to which AB is at right angles, draw DE at right angles to BD, (I. 11.)
and make DE equal to AB, (I. 3.) and join BE, AE, AD.
And because AB is perpendicular to the plane,
it is perpendicular to every straight line which meets it, and is in that plane; (XI. def. 3.)
therefore each of the angles ABD, ABE is a right angle:
and because the straight line BD meets the parallel straight lines AB, CD,
the angles ABD, CDB are together equal to two right angles: (I. 29.)
and ABD is a right angle;
therefore also CDB is a right angle, and CD perpendicular to BD:
and because AB is equal to DE, and BD common,
the two AB, BD are equal to the two ED, DB, each to each;
and the angle ABD is equal to the angle EDB, because each of them is a right angle;
therefore the base AD is equal to the base BE: (I. 4.)
again, because AB is equal to DE, and BE to AD,
the two AB, BE are equal to the two ED, DA, each to each;
and the base AE is common to the triangles ABE, EDA;
wherefore the angle ABE is equal to the angle EDA: (I. 8.)
but ABE is a right angle;
and therefore EDA is a right angle, and ED perpendicular to DA:
but it is also perpendicular to BD; (constr.)
therefore ED is perpendicular to the plane which passes through BD, DA; (XI. 4.)
and therefore makes right angles with every straight line meeting it in that plane: (XI. def. 3.)
but DC is in the plane passing through BD, DA,
because all three are in the plane in which are the parallels AB, CD;
wherefore ED is at right angles to DC;
and therefore CD is at right angles to DE:
but CD is also at right angles to DB;
therefore CD is at right angles to the two straight lines DE, DB in the point of their intersection D;
and therefore is at right angles to the plane passing through DE, DB, (XI. 4.)
which is the same plane to which AB is at right angles.
Therefore, if two straight lines, &c. Q.E.D.

PROPOSITION IX. THEOREM.

Two straight lines which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the same plane with it.

Then AB shall be parallel to CD.



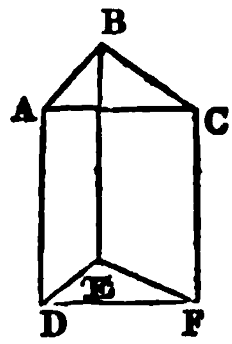
In EF take any point G, from which draw, in the plane passing through EF, AB, the straight line GH at right angles to EF; (I. 11.)
and in the plane passing through EF, CD draw GK at right angles to the same EF.

And because EF is perpendicular both to GH and GK,
EF is perpendicular to the plane HGK passing through them: (XI. 4.)
and EF is parallel to AB
therefore AB is at right angles to the plane HGK. (XI. 8.)
For the same reason, CD is likewise at right angles to the plane HGK.
Therefore AB, CD are each of them at right angles to the plane HGK.
But if two straight lines are at right angles to the same plane, they are parallel to one another: (XI. 6.)
therefore AB is parallel to CD.
Wherefore, two straight lines, &c. Q.E.D.

PROPOSITION X. THEOREM.

If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two; the first two and the other two shall contain equal angles.

Let the two straight lines AB, BC, which meet one another, be parallel to the two straight lines DE, EF, that meet one another, and are not in the same plane with AB, BC.
The angle ABC shall be equal to the angle DEF.



Take BA, BC, ED, EF all equal to one another;
and join AD, CF, BE, AC, DF.
Then, because BA is equal and parallel to ED,
therefore AD is both equal and parallel to BE. (I. 33.)
For the same reason, CF is equal and parallel to BE.
Therefore AD and CF are each of them equal and parallel to BE.
But straight lines that are parallel to the same straight line, and not in the same plane with it, are parallel to one another: (XI. 9.)
therefore AD is parallel to CF; and it is equal to it; (I. ax. 1.)
and AC, DF join them towards the same parts;
and therefore AC is equal and parallel to DF. (I. 33.)
And because AB, BC are equal to DE, EF, each to each,
and the base AC to the base DF;
the angle ABC is equal to the angle DEF. (I. 8.)
Therefore, if two straight lines, &c. Q.E.D.

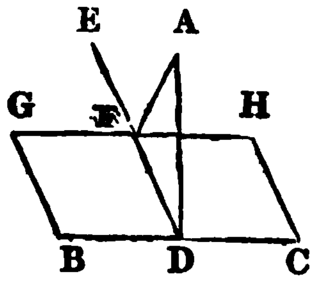
PROPOSITION XI. PROBLEM.

To draw a straight line perpendicular to a plane, from a given point ove it.

Let A be the given point above the plane BH.

It is required to draw from the point A a straight line perpendicular to the plane BH.

In the plane draw any straight line BC,
and from the point A draw AD perpendicular to BC. (I. 12.)
If then AD be also perpendicular to the plane BH, the thing required is already done;
but if it be not, from the point D draw, in the plane BH, the straight line DE at right angles to BC; (I. 11.)
and from the point A draw AF perpendicular to DE.
Then AF shall be perpendicular to the plane BH.



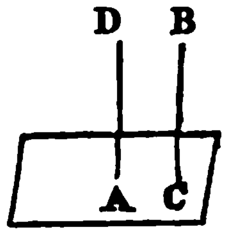
Through F draw GH parallel to BC. (I. 31.)
And because BC is at right angles to ED and DA,
BC is at right angles to the plane passing through ED, DA: (XI. 4.)
and GH is parallel to BC:
but, if two straight lines be parallel, one of which is at right angles to a plane,
the other is at right angles to the same plane; (XI. 8.)
wherefore GH is at right angles to the plane through ED, DA;
and is perpendicular to every straight line meeting it in that plane: (XI. def. 3.)
but AF, which is in the plane through ED, DA, meets it;
therefore GH is perpendicular to AF;
and consequently AF is perpendicular to GH;
and AF is perpendicular to DE;
therefore AF is perpendicular to each of the straight lines GH, DE.
But if a straight line stand at right angles to each of two straight nes in the point of their intersection, it is also at right angles to the lane passing through them: (XI. 4.)
but the plane passing through ED, GH is the plane BH;
therefore AF is perpendicular to the plane BH:
therefore, from the given point A, above the plane BH, the straight ne AF is drawn perpendicular to that plane. Q.E.F.

PROPOSITION XII. PROBLEM.

To erect a straight line at right angles to a given plane, from a point iven in the plane.

Let A be the point given in the plane.

It is required to erect a straight line from the point A at right ngles to the plane.



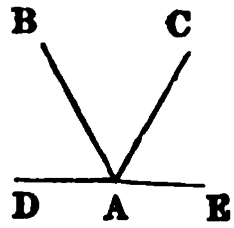
From any point B above the plane draw BC perpendicular to it; (XI. 11.)
and from A draw AD parallel to BC. (I. 31.)
Because, therefore, AD, CB are two parallel straight lines,
and one of them BC is at right angles to the given plane,
the other AD is also at right angles to it: (XI. 8.)
therefore a straight line has been erected at right angles to a given plane, from a point given in it. Q. E. F.

PROPOSITION XIII. THEOREM.

From the same point in a given plane, there cannot be two straight lines at right angles to the plane, upon the same side of it: and there can be but one perpendicular to a plane from a point above the plane.

For, if it be possible, let the two straight lines AB, AC be at right angles to a given plane from the same point A in the plane, and upon the same side of it.

Let a plane pass through BA, AC;
the common section of this with the given plane is a straight line passing through A: (XI. 3.)



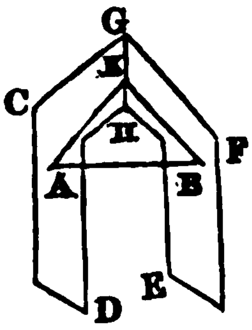
let DAE be their common section:
therefore the straight lines AB, AC, DAE are in one plane:
and because CA is at right angles to the given plane,
it makes right angles with every straight line meeting it in that plane: (XI. def. 3.)
but DAE, which is in that plane, meets CA;
therefore CAE is a right angle.
For the same reason, BAE is a right angle.
Wherefore the angle CAE is equal to the angle BAE; (ax. 11.)
and they are in one plane, which is impossible.
Also, from a point above a plane, there can be but one perpendicular to that plane:
for, if there could be two, they would be parallel to one another, which is absurd. (XI. 6.)
Therefore, from the same point, &c. Q.E.D.

PROPOSITION XIV. THEOREM.

Planes to which the same straight line is perpendicular, are parallel to one another.

Let the straight line AB be perpendicular to each of the planes *D, EF.*

These planes shall be parallel to one another.



If not, they shall meet one another when produced:

let them meet; their common section is a straight line GH, in which take any point K, and join AK, BK.

Then, because AB is perpendicular to the plane EF,

it is perpendicular to the straight line BK which is in that plane: (XI. def. 3.)

therefore ABK is a right angle.

For the same reason, BAK is a right angle:

wherefore the two angles ABK, BAK of the triangle ABK are equal to two right angles, which is impossible: (I. 17.)

therefore the planes CD, EF, though produced, do not meet one another;

that is, they are parallel. (XI. def. 8.)

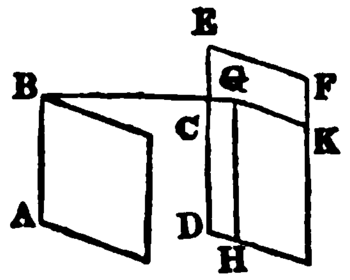
Therefore, planes, &c. Q.E.D.

PROPOSITION XV. THEOREM.

If two straight lines meeting one another be parallel to two other traight lines which meet one another, but are not in the same plane with he first two; the plane which passes through these is parallel to the plane assing through the others.

Let AB, BC, two straight lines meeting one another, be parallel to DE, EF, two other straight lines that meet one another, but are not in he same plane with AB, BC.

The planes through AB, BC, and DE, EF shall not meet, though produced.



From the point B draw BG perpendicular to the plane which passes through DE, EF, (XI. 11.)

and let it meet that plane in G;

and through G draw GH parallel to ED, and GK parallel to EF. (I. 31.)

And because BG is perpendicular to the plane through DE, EF,

it makes right angles with every straight line meeting it in that plane: (XI. def. 3.)

but the straight lines GH, GK in that plane meet it;

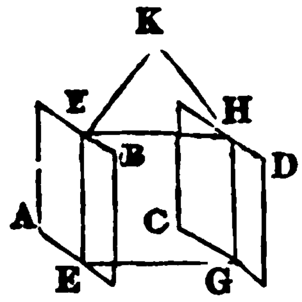
therefore each of the angles BGH, BGK is a right angle:

and because BA is parallel to GH (for each of them is parallel to DE, and they are not both in the same plane with it), (XI. 9.)
the angles GBA, BGH are together equal to two right angles: (I. 29.)
and BGH is a right angle;
therefore also GBA is a right angle, and GB perpendicular to BA.
For the same reason, GB is perpendicular to BC.
Since therefore the straight line GB stands at right angles to the two straight lines BA, BC that cut one another in B;
GB is perpendicular to the plane through BA, BC: (XI. 4.)
and it is perpendicular to the plane through DE, EF; (constr.)
therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF:
but planes to which the same straight line is perpendicular, are parallel to one another; (XI. 14.)
therefore the plane through AB, BC is parallel to the plane through DE, EF.
Wherefore, if two straight lines, &c. Q.E.D.

PROPOSITION XVI. THEOREM.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the parallel planes AB, CD be cut by the plane $EFHG$, and let their common sections with it be EF, GH.
Then EF shall be parallel to GH.



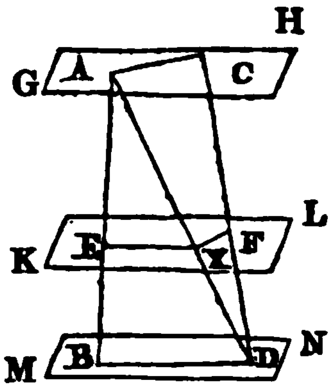
For, if it is not, EF, GH shall meet, if produced, either on the side of FH, or EG.
First, let them be produced on the side of FH, and meet in the point K.
Therefore, since EFK is in the plane AB,
every point in EFK is in that plane: (XI. 1.)
and K is a point in EFK;
therefore K is in the plane AB:
for the same reason, K is also in the plane CD:
wherefore the planes AB, CD produced, meet one another:
but they do not meet, since they are parallel by the hypothesis;
therefore the straight lines EF, GH, do not meet when produced on the side of FH.
In the same manner it may be proved, that EF, GH do not meet when produced on the side of EG.
But straight lines which are in the same plane, and do not meet, though produced either way, are parallel;
therefore EF is parallel to GH.
Wherefore, if two parallel planes, &c. Q.E.D.

PROPOSITION XVII. THEOREM.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let the straight lines *AB, CD* be cut by the parallel planes *GH, KL, MN,* in the points *A, E, B; C, F, D.*

As *AE* is to *EB,* so shall *CF* be to *FD.*



Join *AC, BD, AD,* and let *AD* meet the plane *KL* in the point *X;* and join *EX, XF.*

Because the two parallel planes *KL, MN* are cut by the plane *EBDX,* the common sections *EX, BD* are parallel: (XI. 16.)

for the same reason, because the two parallel planes *GH, KL* are cut by the plane *AXFC,*

the common sections *AC, XF* are parallel:

and because *EX* is parallel to *BD,* a side of the triangle *ABD;*

as *AE* to *EB,* so is *AX* to *XD:* (VI. 2.)

again, because *XF* is parallel to *AC,* a side of the triangle *ADC;*

as *AX* to *XD,* so is *CF* to *FD:*

and it was proved that *AX* is to *XD,* as *AE* to *EB;*

therefore, as *AE* to *EB,* so is *CF* to *FD.* (V. 11.)

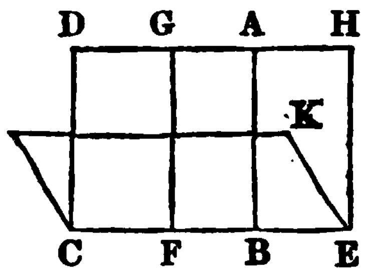
Wherefore, if two straight lines, &c. Q.E.D.

PROPOSITION XVIII. THEOREM.

If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let the straight line *AB* be at right angles to the plane *CK.*

Every plane which passes through *AB* shall be at right angles to the plane *CK.*



Let any plane *DE* pass through *AB,*

and let *CE* be the common section of the planes *DE, CK;*

take any point *F* in *CE,* from which draw *FG* in the plane *DE* at right angles to *CE.* (I. 11.)

And because *AB* is perpendicular to the plane *CK,*

therefore it is also perpendicular to every straight line in that plane meeting it; (XI. def. 3.)

and consequently it is perpendicular to CE:
wherefore ABF is a right angle:
but GFB is likewise a right angle; (constr.)
therefore AB is parallel to FG: (I. 28.)
and AB is at right angles to the plane CK;
therefore FG is also at right angles to the same plane. (XI. 8.)

But one plane is at right angles to another plane when the straight lines drawn in one of the planes, at right angles to their common section, are also at right angles to the other plane; (XI. def. 4.)

and any straight line FG in the plane DE, which is at right angles to CE, the common section of the planes, has been proved to be perpendicular to the other plane CK;

therefore the plane DE is at right angles to the plane CK.

In like manner, it may be proved that all planes which pass through AB are at right angles to the plane CK.

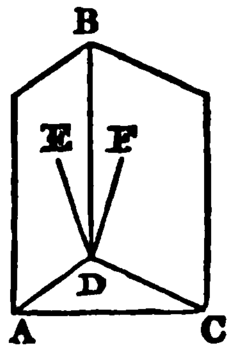
Therefore, if a straight line, &c. Q.E.D.

PROPOSITION XIX. THEOREM.

If two planes which cut one another be each of them perpendicular to a third plane; their common section shall be perpendicular to the same plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common section of the first two.

Then BD shall be perpendicular to the third plane.



If it be not, from the point D draw, in the plane AB, the straight line DE at right angles to AD the common section of the plane AB with the third plane; (I. 11.)

and in the plane BC draw DF at right angles to CD the common section of the plane BC with the third plane.

And because the plane AB is perpendicular to the third plane, and DE is drawn in the plane AB at right angles to AD, their common section,

DE is perpendicular to the third plane. (XI. def. 4.)

In the same manner, it may be proved, that DF is perpendicular to the third plane.

Wherefore, from the point D two straight lines stand at right angles to the third plane, upon the same side of it, which is impossible: (XI. 13.)

therefore, from the point D there cannot be any straight line at right angles to the third plane, except BD the common section of the planes AB, BC:

therefore BD is perpendicular to the third plane.

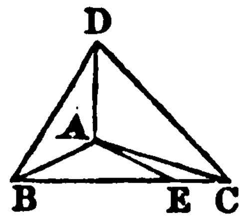
Wherefore, if two planes, &c. Q.E.D.

PROPOSITION XX. THEOREM.

If a solid angle be contained by three plane angles, any two of them are greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB.

Any two of them shall be greater than the third.



If the angles BAC, CAD, DAB be all equal, it is evident, that any two of them are greater than the third.

But if they are not, let BAC be that angle which is not less than either of the other two, and is greater than one of them DAB; and at the point A in the straight line AB, make, in the plane which passes through BA, AC, the angle BAE equal to the angle DAB; (I. 23.) and make AE equal to AD, and through E draw BEC cutting AB, AC in the points B, C, and join DB, DC.

And because DA is equal to AE, and AB is common, the two DA, AB are equal to the two EA, AB, each to each; and the angle DAB is equal to the angle EAB: therefore the base DB is equal to the base BE: (I. 4.) and because BD, DC are greater than CB, (I. 20.) and one of them BD has been proved equal to BE a part of CB, therefore the other DC is greater than the remaining part EC: (I. ax. 5.) and because DA is equal to AE, and AC common, but the base DC greater than the base EC; therefore the angle DAC is greater than the angle EAC; (I. 25.) and, by the construction, the angle DAB is equal to the angle BAE; wherefore the angles DAB, DAC are together greater than BAE, EAC, that is, than the angle BAC: (I. ax. 4.) but BAC is not less than either of the angles DAB, DAC: therefore BAC, with either of them, is greater than the other.

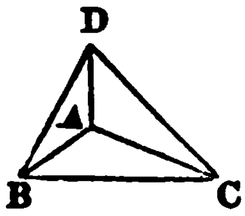
Wherefore, if a solid angle, &c. Q.E.D.

PROPOSITION XXI. THEOREM.

Every solid angle is contained by plane angles, which together are less than four right angles.

First, let the solid angle at A be contained by three plane angles BAC, CAD, DAB.

These three together shall be less than four right angles.



Take in each of the straight lines AB, AC, AD, any points B, C, D, and join BC, CD, DB.

Then, because the solid angle at B is contained by the three plane angles CBA, ABD, DBC,

any two of them are greater than the third; (XI. 20.)

therefore the angles CBA, ABD are greater than the angle DBC:

for the same reason, the angles BCA, ACD are greater than the angle DCB;

and the angles CDA, ADB greater than BDC:

wherefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than the three angles DBC, BCD, CDB:

but the three angles DBC, BCD, CDB are equal to two right angles; (I. 32.)

therefore the six angles CBA, ABD, BCA, ACD, CDA, ADB are greater than two right angles:

and because the three angles of each of the triangles ABC, ACD, ADB are equal to two right angles,

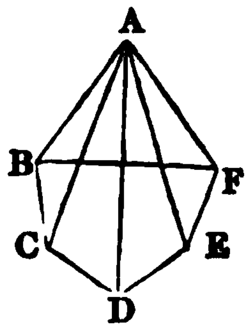
therefore the nine angles of these three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD are equal to six right angles;

of these the six angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles;

therefore the remaining three angles BAC, CAD, DAB, which contain the solid angle at A, are less than four right angles.

Next, let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB.

These shall together be less than four right angles.



Let the planes in which the angles are, be cut by a plane,

and let the common sections of it with those planes be BC, CD, DE, EF, FB.

And because the solid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater than the third, (XI. 20.)

the angles CBA, ABF, are greater than the angle FBC:

for the same reason, the two plane angles at each of the points C, D, E, F, viz. those angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the polygon $BCDEF$:

therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon:

and because all the angles of the triangles are together equal to twice as many right angles as there are triangles; (I. 32.)

that is, as there are sides in the polygon $BCDEF$;

and that all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles as there are sides in the polygon; (I. 32. Cor. 1.)

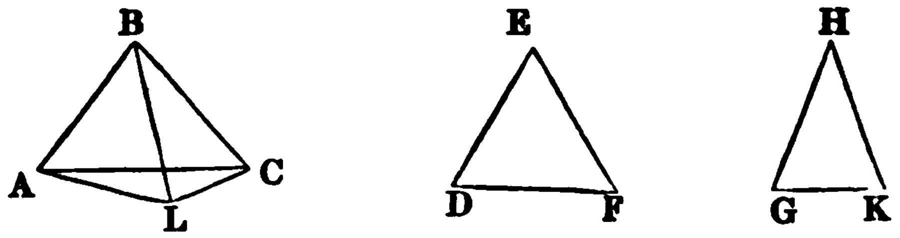
therefore all the angles of the triangles are equal to all the angles of the polygon together with four right angles: (I. ax. 1.)
but all the angles at the bases of the triangles are greater than all the angles of the polygon, as has been proved;
wherefore the remaining angles of the triangles, viz. those of the ertex, which contain the solid angle at A, are less than four right ngles.

Therefore, every solid angle, &c. Q.E.D.

PROPOSITION XXII. THEOREM.

If every two of three plane angles be greater than the third, and if he straight lines which contain them be all equal; a triangle may be made f the straight lines that join the extremities of those equal straight lines.

Let ABC, DEF, GHK be the three plane angles, whereof every wo are greater than the third, and let them be contained by the equal traight lines AB, BC, DE, EF, GH, HK:
if their extremities be joined by the straight lines AC, DF, GK, a triangle may be made of three straight lines equal to AC, DF, GK;
that is, every two of them shall together be greater than the third.



If the angles at B, E, H are equal,
AC, DF, GK are also equal, (I. 4.) and any two of them greater than the third:
but if the angles are not all equal,
let the angle ABC be not less than either of the two at E, H;
therefore the straight line AC is not less than either of the other two DF, GK; (I. 4. or 24.)
and therefore it is plain that AC, together with either of the other two, must be greater than the third.
Also, DF with GK shall be greater than AC.
For, at the point B in the straight line AB make the angle ABL equal to the angle GHK, (I. 23.)
and make BL equal to one of the straight lines AB, BC, DE, EF, GH, HK, and join AL, LC.
Then, because AB, BL are equal to GH, HK, each to each,
and the angle ABL to the angle GHK,
the base AL is equal to the base GK: (I. 4.)
and because the angles at E, H are greater than the angle ABC, (hyp.) of which the angle at H is equal to ABL,
therefore the remaining angle at E is greater than the angle LBC: (ax. 5.)
and because the two sides LB, BC are equal to the two DE, EF, each to each,
and that the angle DEF is greater than the angle LBC,
the base DF is greater than the base LC: (I. 24.)
and it has been proved that GK is equal to AL;
therefore DF and GK are greater than AL and LC: (I. ax. 4.)

but AL and LC are greater than AC; (I. 20.)
much more then are DF and GK greater than AC.
Wherefore every two of these straight lines AC, DF, GK are greater than the third;
and, therefore, a triangle may be made, the sides of which shall be equal to AC, DF, GK. (I. 22.) Q.E.D.

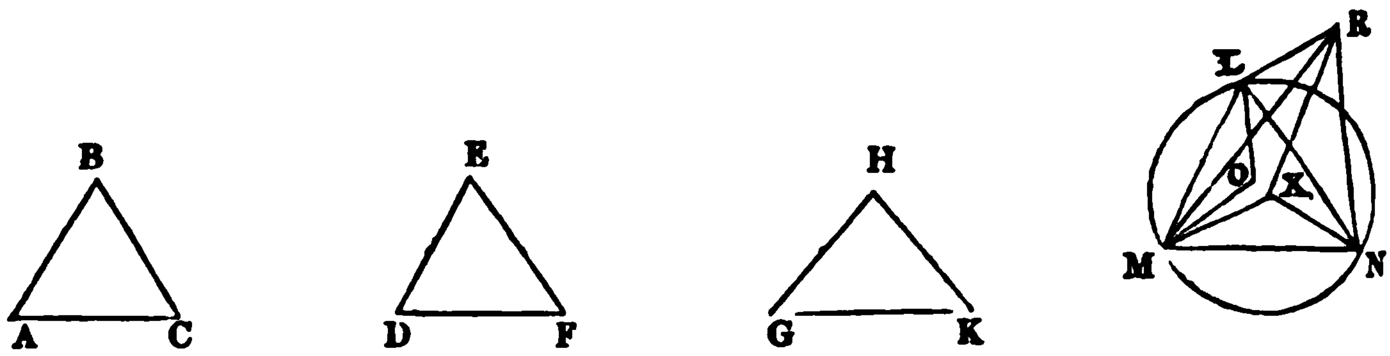
PROPOSITION XXIII. PROBLEM.

To make a solid angle which shall be contained by three given plane angles, any two of them being greater than the third, (XI. 20.) *and all three together less than four right angles.* (XI. 21.)

Let the three given plane angles be ABC, DEF, GHK, any two of which are greater than the third, and all of them together less than four right angles.

It is required to make a solid angle contained by three plane angles equal to ABC, DEF, GHK, each to each.

From the straight lines which contain the angles cut off AB, BC, DE, EF, GH, HK, all equal to one another; and join AC, DF, GK:
then a triangle may be made of three straight lines equal to AC, DF, GK. (XI. 22.)



Let this be the triangle LMN, so that AC be equal to LM, DF to MN, and GK to LN; (I. 22.)
and about the triangle LMN describe a circle, (IV. 5.) and find its centre X, which will be either within the triangle, or in one of its sides, or without it. (III. 1.)

First, let the centre X be within the triangle, and join LX, MX, NX.
AB shall be greater than LX.
If not, AB must either be equal to, or less than LX.
First, let it be equal:
then, because AB is equal to LX, and that AB is also equal to BC, and LX to XM,
AB and BC are equal to LX and XM, each to each;
and the base AC is, by construction, equal to the base LM;
wherefore the angle ABC is equal to the angle LXM. (I. 8.)
For the same reason, the angle DEF is equal to the angle MXN, and the angle GHK to the angle NXL:
therefore the three angles ABC, DEF, GHK are equal to the three angles LXM, MXN, NXL:
but the three angles LXM, MXN, NXL are equal to four right angles; (I. 15. Cor. 2.)
therefore also the three angles ABC, DEF, GHK, are equal to four right angles:

but, by the hypothesis, they are less than four right angles; which is absurd:

therefore AB is not equal to LX.

But neither can AB be less than LX:

for, if possible, let it be less;

and upon the straight line LM, on the side of it on which is the centre X, describe the triangle LOM, the sides LO, OM of which are equal to AB, BC: (I. 22.)

and because the base LM is equal to the base AC,

the angle LOM is equal to the angle ABC. (I. 8.)

And AB, that is, LO is by the hypothesis less than LX:

wherefore LO, OM fall within the triangle LXM;

for if they fell upon its sides, or without it, they would be equal to, or greater than LX, XM: (I. 21.)

therefore the angle LOM, that is, the angle ABC, is greater than the angle LXM. (I. 21.)

In the same manner it may be proved that the angle DEF is greater than the angle MXN, and the angle GHK greater than the angle NXL:

therefore the three angles ABC, DEF, GHK are greater than the three angles LXM, MXN, NXL; that is, than four right angles: (I. 15. Cor. 2.)

but the same angles ABC, DEF, GHK are less than four right angles; which is absurd: (hyp.)

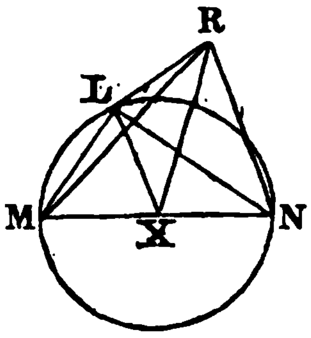
therefore AB is not less than LX:

and it has been proved that it is not equal to LX;

wherefore AB is greater than LX.

Next, let the centre X of the circle fall in one of the sides of the triangle, viz. in MN, and join XL.

In this case also, AB shall be greater than LX.



If not, AB is either equal to LX, or less than it.

First, let it be equal to LX:

therefore AB and BC, that is, DE and EF, are equal to MX and XL, that is, to MN:

but, by the construction, MN is equal to DF;

therefore DE, EF are equal to DF, which is impossible: (I. 20.)

wherefore AB is not equal to LX:

nor is it less; for then, much more, an absurdity would follow:

therefore AB is greater than LX.

But, let the centre X of the circle fall without the triangle LMN, and join LX, MX, NX.

In this case likewise AB shall be greater than LX.

If not, it is either equal to or less than LX.

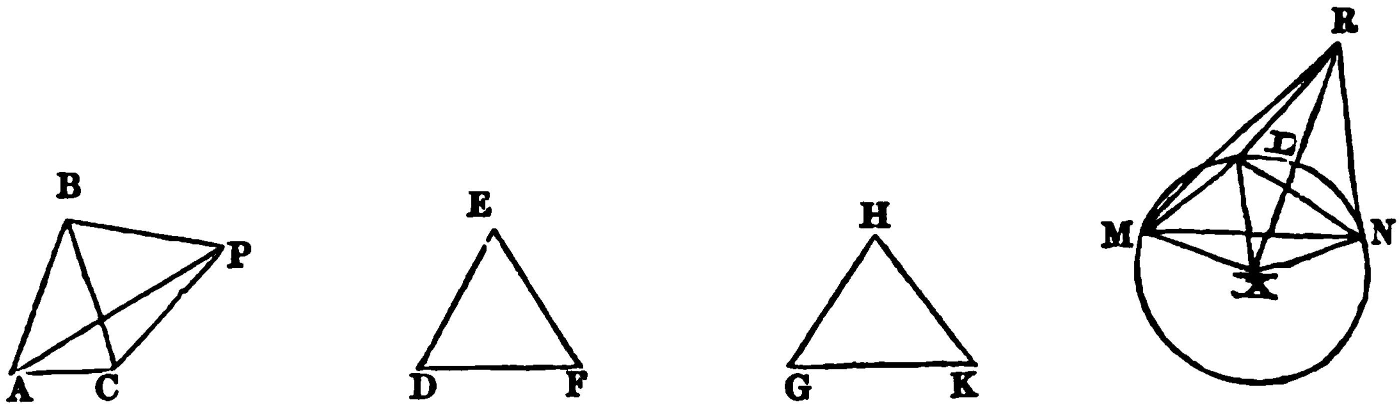
First, let it be equal:

it may be proved, in the same manner as in the first case, that the angle ABC is equal to the angle MXL, and GHK to LXN:

therefore the whole angle MXN is equal to the two angles ABC, GHK;

but ABC and GHK are together greater than the angle DEF; (hyp.)
therefore also the angle MXN is greater than DEF:
and because DE, EF are equal to MX, XN, each to each,
and the base DF to the base MN,
the angle MXN is equal to the angle DEF: (I. 8.)
but it has been proved, that it is greater than DEF, which is absurd.
Therefore AB is not equal to LX:
neither is it less;
for then, as has been proved in the first case,
the angle ABC is greater than the angle MXL, and the angle GHK greater than the angle LXN.

At the point B, in the straight line CB, make the angle CBP equal to the angle GHK,
and make BP equal to HK, and join CP, AP.



And because CB is equal to GH,
CB, BP are equal to GH, HK, each to each; and they contain equal angles;
wherefore the base CP is equal to the base GK, that is, to LN.
And in the isosceles triangles ABC, MXL,
because the angle ABC is greater than the angle MXL,
therefore the angle MLX at the base is greater than the angle ACB at the base. (I. 32.)

For the same reason, because the angle GHK or CBP is greater than the angle LXN,
the angle XLN is greater than the angle BCP:
therefore the whole angle MLN is greater than the whole angle ACP.
And because ML, LN are equal to AC, CP, each to each,
but the angle MLN is greater than the angle ACP,
the base MN is greater than the base AP: (I. 24.)
but MN is equal to DF;
therefore also DF is greater than AP.
Again, because DE, EF are equal to AB, BP, each to each,
but the base DF greater than the base AP,
the angle DEF is greater than the angle ABP: (I. 25.)
but ABP is equal to the two angles ABC, CBP, that is, to the two angles ABC, GHK;
therefore the angle DEF is greater than the two angles ABC, GHK:
but it is also less than these, which is impossible. (hyp.)
Therefore AB is not less than LX:
and it has been proved that it is not equal to it;
therefore AB is greater than LX.

From the point X erect XR at right angles to the plane of the circle LMN. (XI. 12.)

And because it has been proved in all the cases, that AB is greater than LX,

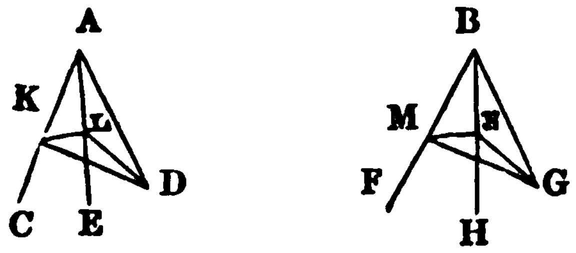
find a square equal to the excess of the square of *AB* above the square of *LX*,
and make *RX* equal to its side, and join *RL, RM, RN*.
The solid angle at *R* shall be the angle required.
Because *RX* is perpendicular to the plane of the circle *LMN*,
is perpendicular to each of the straight lines *LX, MX, NX*. (XI. def. 3.)
And because *LX* is equal to *MX*, and *XR* common, and at right
gles to each of them,
the base *RL* is equal to the base *RM*. (I. 4.)
For the same reason, *RN* is equal to each of the two *RL, RM*:
therefore the three straight lines *RL, RM, RN*, are all equal.
And because the square of *XR* is equal to the excess of the square
AB above the square of *LX*;
therefore the square of *AB* is equal to the squares of *LX, XR*:
but the square of *RL* is equal to the same squares, because *LXR* is a right angle; (I. 47.)
therefore the square of *AB* is equal to the square of *RL*,
and the straight line *AB* to *RL*.
But each of the straight lines *BC, DE, EF, GH, HK* is equal to *AB*,
and each of the two *RM, RN* is equal to *RL*;
therefore *AB, BC, DE, EF, GH, HK* are each of them equal to each of the straight lines *RL, RM, RN*.
And because *RL, RM* are equal to *AB, BC*, each to each,
and the base *LM* to the base *AC*;
the angle *LRM* is equal to the angle *ABC*. (I. 8.)
For the same reason, the angle *MRN* is equal to the angle *DEF*, and *NRL* to *GHK*.
Therefore there is made a solid angle at *R*, which is contained by
ıree plane angles *LRM, MRN, NRL*, which are equal to the three
iven plane angles *ABC, DEF, GHK*, each to each. Q.E.F.

PROPOSITION A. THEOREM.

If each of two solid angles be contained by three plane angles, which re equal to one another, each to each; the planes in which the equal ngles are, have the same inclination to one another.

Let there be two solid angles at the points *A, B*;
and let the angle at *A* be contained by the three plane angles *CAD, CAE, EAD*;
and the angle at *B* by the three plane angles *FBG, FBH, HBG*;
of which the angle *CAD* is equal to the angle *FBG*, and *CAE* to *FBH*, and *EAD* to *HBG*.
Then the planes in which the equal angles are, shall have the same inclination to one another.
In the straight line *AC* take any point *K*,
and from *K* draw in the plane *CAD* the straight line *KD* at right angles to *AC*, (I. 11.)
and in the plane *CAE* the straight line *KL* at right angles to the same *AC*:
therefore the angle *DKL* is the inclination of the plane *CAD* to the plane *CAE*. (XI. def. 6.)
In *BF* take *BM* equal to *AK*,

and from the point *M* draw in the planes *FBG, FBH,* the straight lines *MG, MN* at right angles to *BF*;
therefore the angle *GMN* is the inclination of the plane *FBG* to the plane *FBH.* (XI. def. 6.)



Join *LD, NG.*

And because in the triangles *KAD, MBG,* the angles *KAD, MBG* are equal, as also the right angles *AKD, BMG,* and that the sides *AK, BM,* adjacent to the equal angles, are equal to one another; (hyp.)
therefore *KD* is equal to *MG,* and *AD* to *BG*: (I. 26.)
for the same reason, in the triangles *KAL, MBN,*
KL is equal to *MN,* and *AL* to *BN*:
therefore in the triangles *LAD, NBG, LA, AD* are equal to *NB, BG,* each to each; and they contain equal angles:
therefore the base *LD* is equal to the base *NG.* (I. 4.)

Lastly, in the triangles *KLD, MNG,* the sides *DK, KL* are equal to *GM, MN,* each to each, and the base *LD* to the base *NG*;
therefore the angle *DKL* is equal to the angle *GMN*: (I. 8.)
but the angle *DKL* is the inclination of the plane *CAD* to the plane *CAE,*
and the angle *GMN* is the inclination of the plane *FBG* to the plane *FBH,*
which planes have therefore the same inclination to one another, (XI. def. 7.)

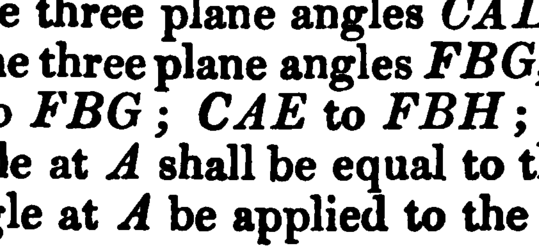
And in the same manner it may be demonstrated, that the other planes in which the equal angles are, have the same inclination to one another.

Therefore if each of two solid angles, &c. Q.E.D.

PROPOSITION B. THEOREM.

If two solid angles be contained, each by three plane angles which are equal to one another, each to each, and alike situated; these solid angles are equal to one another.

Let there be two solid angles at *A* and *B,* of which the solid angle at *A* is contained by the three plane angles *CAD, CAE, EAD*;
and that at *B,* by the three plane angles *FBG, FBH, HBG*; of which *CAD* is equal to *FBG*; *CAE* to *FBH*; and *EAD* to *HBG.*
Then the solid angle at *A* shall be equal to the solid angle at *B.*
Let the solid angle at *A* be applied to the solid angle at *B*:



and first, the plane angle *CAD* being applied to the plane angle *FBG,* so that the point *A* may coincide with the point *B,* and the straight line *AC* with *BF*;

then AD coincides with BG,
because the angle CAD is equal to the angle FBG:
and because the inclination of the plane CAE to the plane CAD is equal to the inclination of the plane FBH to the plane FBG, (XI. A.)
the plane CAE coincides with the plane FBH,
because the planes CAD, FBG coincide with one another:
and because the straight lines AC, BF coincide,
and that the angle CAE is equal to the angle FBH;
therefore AE coincides with BH:
and AD coincides with BG;
wherefore the plane EAD coincides with the plane HBG:
therefore the solid angle A coincides with the solid angle B, and consequently they are equal to one another. (I. ax. 8.) Q.E.D.

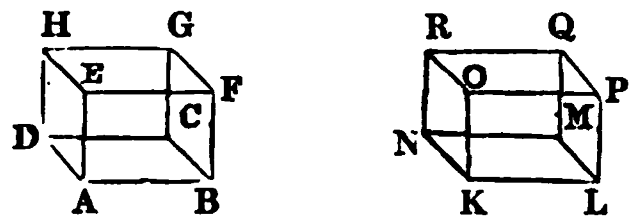
PROPOSITION C. THEOREM.

Solid figures which are contained by the same number of equal and imilar planes alike situated, and having none of their solid angles conzined by more than three plane angles, are equal and similar to one nother.

Let AG, KQ be two solid figures contained by the same number of imilar and equal planes, alike situated,

viz. let the plane AC be similar and equal to the plane KM; the lane AF to KP; BG to LQ; GD to QN; DE to NO; and, lastly, FH similar and equal to PR.

The solid figure AG shall be equal and similar to the solid figure KQ.



Because the solid angle at A is contained by the three plane angles BAD, BAE, EAD, which, by the hypothesis, are equal to the plane angles LKN, LKO, OKN, which contain the solid angle at K, each to each;

therefore the solid angle at A is equal to the solid angle at K. (XI. B.)

In the same manner, the other solid angles of the figures are equal to one another.

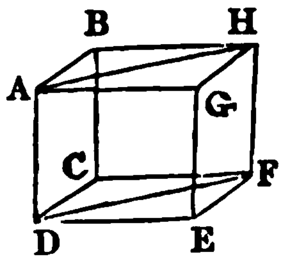
Let then the solid figure AG be applied to the solid figure KQ;
first, the plane figure AC being applied to the plane figure KM,
so that the straight line AB may coincide with KL,
the figure AC must coincide with the figure KM, because they are equal and similar:
therefore the straight lines AD, DC, CB coincide with KN, NM, ML, each with each;
and the points A, D, C, B with the points K, N, M, L:
and the solid angle at A coincides with the solid angle at K; (XI. B.)
wherefore the plane AF coincides with the plane KP, and the figure AF with the figure KP,
because they are equal and similar to one another:
therefore the straight lines AE, EF, FB coincide with KO, OP, PL;

and the points E, F with the points O, P.
In the same manner, the figure AH coincides with the figure KR, and the straight line DH with NR, and the point H with the point R.
And because the solid angle at B is equal to the solid angle at L,
it may be proved, in the same manner,
that the figure BG coincides with the figure LQ, and the straight line CG with MQ, and the point G with the point Q.
Therefore, since all the planes and sides of the solid figure AG coincide with the planes and sides of the solid figure KQ,
AG is equal and similar to KQ.
And in the same manner, any other solid figures whatever contained by the same number of equal and similar planes, alike situated, and having none of their solid angles contained by more than three plane angles, may be proved to be equal and similar to one another. Q.E.D.

PROPOSITION XXIV. THEOREM.

If a solid be contained by six planes, two and two of which are parallels, the opposite planes are similar and equal parallelograms.

Let the solid $CDGH$ be contained by the parallel planes AC, GF; BG, CE; FB, AE.
Its opposite planes shall be similar and equal parallelograms.



Because the two parallel planes BG, CE are cut by the plane AC,
their common sections AB, CD are parallel: (XI. 16.)
again, because the two parallel planes BF, AE are cut by the plane AC,
their common sections AD, BC are parallel: (XI. 16.)
and AB is parallel to CD;
therefore AC is a parallelogram.
In like manner it may be proved, that each of the figures CE, FG, GB, BF, AE is a parallelogram.
Join AH, DF;
And because AB is parallel to DC, and BH to CF;
the two straight lines AB, BH, which meet one another, are parallel to DC and CF, which meet one another, and are not in the same plane with the other two:
wherefore they contain equal angles; (XI. 10.)
therefore the angle ABH is equal to the angle DCF:
and because AB, BH are equal to DC, CF, each to each,
and the angle ABH equal to the angle DCF;
therefore the base AH is equal to the base DF, (I. 4.)
and the triangle ABH to the triangle DCF:
but the parallelogram BG is double of the triangle ABH, (I. 34.)
and the parallelogram CE double of the triangle DCF:
therefore the parallelogram BG is equal and similar to the parallelogram CE.

In the same manner it may be proved, that the parallelogram *AC* is qual and similar to the parallelogram *GF*, and the parallelogram *AE* ɔ *BF*.

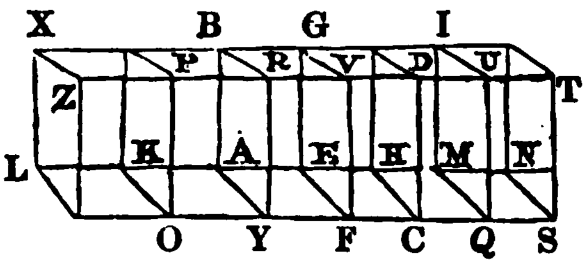
Therefore, if a solid, &c. Q.E.D.

PROPOSITION XXV. THEOREM.

If a solid parallelopiped be cut by a plane parallel to two of its opposite lanes; it divides the whole into two solids, the base of one of which shall e to the base of the other, as the one solid is to the other.

Let the solid parallelopiped *ABCD* be cut by the plane *EV*, which s parallel to the opposite planes *AR*, *HD*, and divides the whole into he two solids *ABFV*, *EGCD*.

As the base *AEFY* of the first is to the base *EHCF* of the other, so shall the solid *ABFV* be to the solid *EGCD*.



Produce *AH* both ways, and take any number of straight lines *HM*, *MN*, each equal to *EH*,

and any number *AK*, *KL*, each equal to *EA*,
and complete the parallelograms *LO*, *KY*, *HQ*, *MS*,
and the solids *LP*, *KR*, *HU*, *MT*.
Then, because the straight lines *LK*, *KA*, *AE* are all equal,
the parallelograms *LO*, *KY*, *AF* are equal; (I. 36.)
and likewise the parallelograms *KX*, *KB*, *AG*:
also the parallelograms *LZ*, *KP*, *AR* are equal, because they are opposite planes: (XI. 24.)
or the same reason, the parallelograms *EC*, *HQ*, *MS* are equal, (I. 36.)
and the parallelograms *HG*, *HI*, *IN*:
as also *HD*, *MU*, *NT*: (XI. 24.)
therefore three planes of the solid *LP* are equal and similar to three planes of the solid *KR*, as also to three planes of the solid *AV*:
but the three planes opposite to these three are equal and similar to them in the several solids, and none of their solid angles are contained by more than three plane angles; (XI. 24.)
therefore the three solids *LP*, *KR*, *AV* are equal to one another: (XI. C.)
for the same reason, the three solids *ED*, *HU*, *MT* are equal to one another:
therefore, what multiple soever the base *LF* is of the base *AF*,
the same multiple is the solid *LV* of the solid *AV*;
and whatever multiple the base *NF* is of the base *HF*,
the same multiple is the solid *NV* of the solid *ED*:
and if the base *LF* be equal to the base *NF*,
the solid *LV* is equal to the solid *NV*; (XI. C.)
and if the base *LF* be greater than the base *NF*,
the solid *LV* is greater than the solid *NV*; and if less, less.
Since then there are four magnitudes, viz. the two bases *AF*, *FH*, and the two solids *AV*, *ED*;

and that of the base AF and solid AV, the base LF and solid LV are any equimultiples whatever;
and of the base FH and solid ED, the base FN and solid NV are any equimultiples whatever:
and since it has been proved, that if the base LF is greater than the base FN, the solid LV is greater than the solid NV:
and if equal, equal; and if less, less;
therefore as the base AF is to the base FH, so is the solid AV to the solid ED. (v. def. 5.)
Wherefore, if a solid, &c. Q.E.D.

PROPOSITION XXVI. PROBLEM.

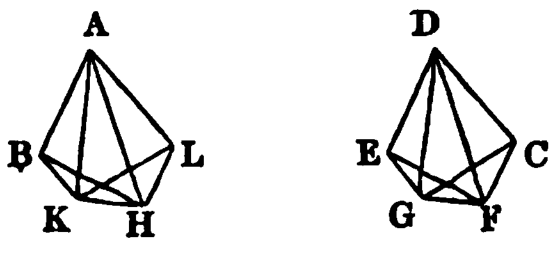
At a given point in a given straight line, to make a solid angle equal to a given solid angle contained by three plane angles.

Let AB be a given straight line, A a given point in it, and D a given solid angle contained by the three plane angles EDC, EDF, FDC.

It is required to make at the point A, in the straight line AB, a solid angle equal to the solid angle D.

In the straight line DF take any point F, from which draw FG perpendicular to the plane EDC, meeting that plane in G, (XI. 11.) and join DG:
at the point A, in the straight line AB, make the angle BAL equal to the angle EDC, (I. 23.)
and in the plane BAL make the angle BAK equal to the angle EDG;
then make AK equal to DG, and from the point K erect KH at right angles to the plane BAL, (XI. 12.) and make KH equal to GF, and join AH.

The solid angle at A which is contained by the three plane angles BAL, BAH, HAL shall be equal to the solid angle at D contained by the three plane angles EDC, EDF, FDC.



Take the equal straight lines AB, DE, and join HB, KB, FE, GE.
And because FG is perpendicular to the plane EDC, it makes right angles with every straight line meeting it in that plane: (XI. def. 3.)
therefore each of the angles FGD, FGE is a right angle.
For the same reason, HKA, HKB are right angles.
And because KA, AB are equal to GD, DE, each to each,
and that they contain equal angles,
therefore the base BK is equal to the base EG; (I. 4.)
and KH is equal to GF, and HKB, FGE are right angles, (constr.)
therefore HB is equal to FE. (I. 4.)
Again, because AK, KH are equal to DG, GF, each to each, and contain right angles,
the base AH is equal to the base DF;
and AB is equal to DE,

therefore, HA, AB are equal to FD, DE, each to each;
and the base HB is equal to the base FE;
therefore the angle BAH is equal to the angle EDF. (I. 8.)
For the same reason, the angle HAL is equal to the angle FDC:
because if AL and DC be made equal, and KL, HL, GC, FC be joined;
since the whole angle BAL is equal to the whole EDC, and the parts of them BAK, EDG are, by the construction, equal;
therefore the remaining angle KAL is equal to the remaining angle GDC:
and because KA, AL are equal to GD, DC, each to each, and contain equal angles,
the base KL is equal to the base GC; (I. 4.)
and KH is equal to GF;
so that LK, KH are equal to CG, GF, each to each;
and they contain right angles; (XI. def. 3.)
therefore the base HL is equal to the base FC: (I. 4.)
again, because HA, AL are equal to FD, DC, each to each, and the base HL to the base FC,
the angle HAL is equal to the angle FDC. (I. 8.)

Therefore, because the three plane angles BAL, BAH, HAL, which contain the solid angle at A, are equal to the three plane angles EDC, EDF, FDC, which contain the solid angle at D, each to each, and are situated in the same order,
the solid angle at A is equal to the solid angle at D. (XI. B.)

Therefore at a given point in a given straight line a solid angle has been made equal to a given solid angle contained by three plane angles. Q.E.F.

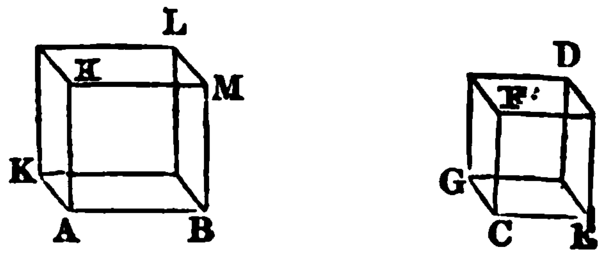
PROPOSITION XXVII. PROBLEM.

To describe, from a given straight line, a solid parallelopiped similar and similarly situated to one given.

Let AB be the given straight line, and CD the given solid parallelopiped.

It is required from AB to describe a solid parallelopiped similar and similarly situated to CD.

At the point A of the given straight line AB, make a solid angle equal to the solid angle at C, (XI. 26.)
and let BAK, KAH, HAB be the three plane angles which contain it,
so that BAK be equal to the angle ECG, and KAH to GCF, and HAB to FCE:
and as EC to CG, so make BA to AK; (VI. 12.)
and as GC to CF, so make KA to AH; (VI. 12.)
wherefore, ex æquali, as EC to CF, so is BA to AH: (V. 22.)
complete the parallelogram BH, and the solid AL.
Then AL shall be similar and similarly situated to CD.

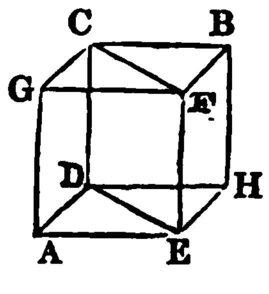


Because, as EC to GC, so is BA to AK,
the sides about the equal angles ECG, BAK, are proportionals;
therefore the parallelogram BK is similar to EG. (VI. def. 1.)
For the same reason, the parallelogram KH is similar to GF, and HB to FE;
wherefore three parallelograms of the solid AL are similar to three of the solid CD:
and the three opposite ones in each solid are equal and similar to these, each to each. (XI. 24.)
Also, because the plane angles which contain the solid angles of the figures are equal, each to each, and situated in the same order,
the solid angles are equal, each to each. (XI. B.)
Therefore the solid AL is similar to the solid CD. (XI. def. 11.)
Wherefore, from a given straight line AB, a solid parallelopiped AL has been described similar and similarly situated to the given one CD. Q.E.F.

PROPOSITION XXVIII. THEOREM.

If a solid parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes; it shall be cut into two equal parts.

Let AB be a solid parallelopiped, and DE, CF the diagonals of the opposite parallelograms AH, GB, viz. those which are drawn betwixt the equal angles in each.
And because CD, FE are each of them parallel to GA, and not in the same plane with it,
CD, FE are parallel; (XI. 9.)
wherefore the diagonal CF, DE, are in the plane in which the parallels are, and are themselves parallels: (XI. 16.)
and the plane $CDEF$ shall cut the solid AB into two equal parts.



Because the triangle CGF is equal to the triangle CBF, (I. 34.)
and the triangle DAE to DHE;
and that the parallelogram CA is equal and similar to the opposite one BE, (XI. 24.)
and the parallelogram GE to CH;
therefore the prism contained by the two triangles CGF, DAE, and the three parallelograms, CA, GE, EC, is equal to the prism contained by the two triangles CBF, DHE, and the three parallelograms BE, CH, EC; (XI. C.)
because they are contained by the same number of equal and similar planes, alike situated, and none of their solid angles are contained by more than three plane angles.
Therefore the solid AB is cut into two equal parts by the plane $CDEF$. Q.E.D.

PROPOSITION XXIX. THEOREM.

Solid parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are terminated in the same straight lines in the plane opposite to the base, are equal to one another.

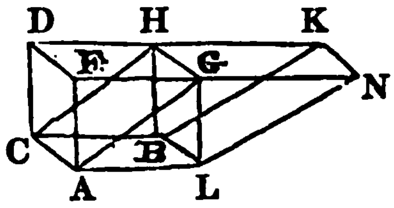
Let the solid parallelopipeds *AH, AK* be upon the same base *AB*, and of the same altitude,

and let their insisting straight lines *AF, AG, LM, LN* be terminated in the same straight line *FN*,

and *CD, CE, BH, BK* be terminated in the same straight line *DK*.

The solid *AH* shall be equal to the solid *AK*.

First, let the parallelograms *DG, HN*, which are opposite to the base *AB*, have a common side *HG*.



Then, because the solid *AH* is cut by the plane *AGHC* passing through the diagonals, *AG, CH*, of the opposite planes *ALGF, CBHD*, *AH* is cut into two equal parts by the plane *AGHC*; (XI. 28.)

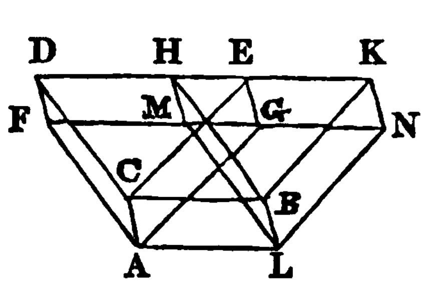
therefore the solid *AH* is double of the prism which is contained betwixt the triangles *ALG, CBH*:

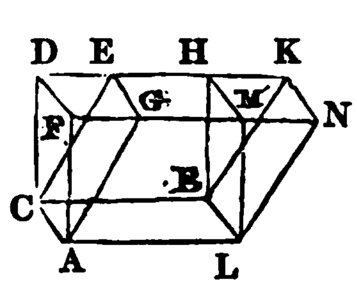
for the same reason, because the solid *AK* is cut by the plane *LGHB*, through the diagonals *LG, BH* of the opposite planes *ALNG, CBKH*,

the solid *AK* is double of the same prism which is contained betwixt the triangles *ALG, CBH*:

therefore the solid *AH* is equal to the solid *AK*. (I. ax. 6.)

Next let the parallelograms *DM, EN*, opposite to the base, have no common side.





Then, because *CH, CK*, are parallelograms,

CB is equal to each of the opposite sides *DH, EK*; (I. 34.)

wherefore *DH* is equal to *EK*:

add or take away the common part *HE*;

then *DE* is equal to *HK*: (I. ax. 2 or 3.)

wherefore also the triangle *CDE* is equal to the triangle *BHK*, (I. 38.)

and the parallelogram *DG* is equal to the parallelogram *HN*: (I. 36.)

for the same reason, the triangle *AFG* is equal to the triangle *LMN*:

and the parallelogram *CF* is equal to the parallelogram *BM*, and *CG* to *BN*; for they are opposite. (XI. 24.)

Therefore the prism which is contained by the two triangles *AFG, CDE*, and the three parallelograms *AD, DG, GC* is equal to the prism contained by the two triangles *LMN, BHK*, and the three parallelograms *BM, MK, KL*. (XI. C.)

If therefore the prism *LMN, BHK* be taken from the solid of

which the base is the parallelogram *AB*, and in which *FDKN* is the one opposite to it; and if from this same solid there be taken the prism *AFG*, *CDE*;

the remaining solid, viz. the parallelopiped *AH*, is equal to the remaining parallelopiped *AK*. (I. ax. 3.)

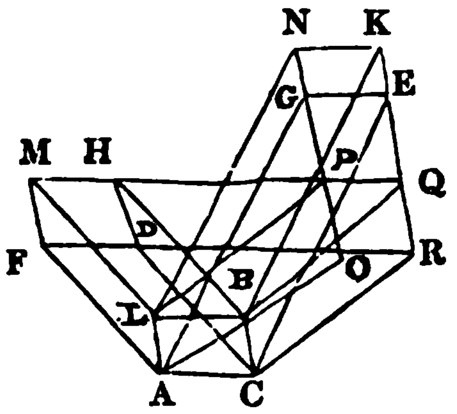
Therefore solid parallelopipeds, &c. Q.E.D.

PROPOSITION XXX. THEOREM.

Solid parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are not terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the parallelopipeds *CM*, *CN* be upon the same base *AB*, and of the same altitude, but their insisting straight lines *AF*, *AG*, *LM*, *LN*, *CD*, *CE*, *BH*, *BK* not terminated in the same straight lines.

The solids *CM*, *CN* shall be equal to one another.



Produce *FD*, *MH*, and *NG*, *KE*, and let them meet one another in the points *O*, *P*, *Q*, *R*; and join *AO*, *LP*, *BQ*, *CR*.

And because the plane *LBHM* is parallel to the opposite plane *ACDF*,

and that the plane *LBHM* is that in which are the parallels *LB*, *MHPQ*, in which also is the figure *BLPQ*;

and the plane *ACDF* is that in which are the parallels *AC*, *FDOR*, in which also is the figure *CAOR*;

therefore the figures *BLPQ*, *CAOR* are in parallel planes:

in like manner, because the plane *ALNG* is parallel to the opposite plane *CBKE*,

and that the plane *ALNG* is that in which are the parallels *AL*, *OPGN*, in which also is the figure *ALPO*;

and the plane *CBKE* is that in which are the parallels *CB*, *RQEK*, in which also is the figure *CBQR*;

therefore the figures *ALPO*, *CBQR* are in parallel planes:

and the planes *ACBL*, *ORQP* are parallel; (hyp.)

therefore the solid *CP* is a parallelopiped:

but the solid *CM* is equal to the solid *CP*, (XI. 29.)

because they are upon the same base *ACBL*, and their insisting straight lines *AF*, *AO*; *CD*, *CR*; *LM*, *LP*; *BH*, *BQ* are in the same straight lines *FR*, *MQ*:

and the solid *CP* is equal to the solid *CN*, (XI. 29.)

for they are upon the same base *ACBL*, and their insisting straight lines *AO*, *AG*; *LP*, *LN*; *CR*, *CE*; *BQ*, *BK* are in the same straight lines *ON*, *RK*:

therefore the solid *CM* is equal to the solid *CN*.

Wherefore solid parallelopipeds, &c. Q.E.D.

PROPOSITION XXXI. THEOREM.

Solid parallelopipeds, which are upon equal bases, and of the same altitude, are equal to one another.

Let the solid parallelopipeds AE, CF be upon equal bases AB, CD, and be of the same altitude.

The solid AE shall be equal to the solid CF.

First, let the insisting straight lines be at right angles to the bases AB, CD,

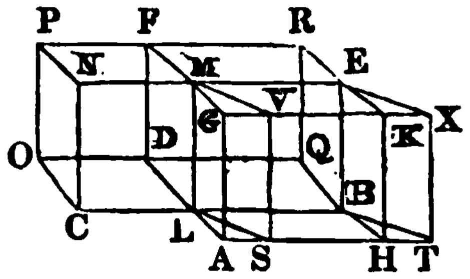
and let the bases be placed in the same plane, and so that the sides CL, LB may be in a straight line;

therefore the straight line LM, which is at right angles to the plane in which the bases are, in the point L, is common to the two solids AE, CF: (XI. 13.)

let the other insisting lines of the solids be AG, HK, BE; DF, OP, CN:

and first, let the angle ALB be equal to the angle CLD:

then AL, LD are in a straight line. (I. 14.)



Produce OD, HB, and let them meet in Q,

and complete the solid parallelopiped LR, the base of which is the parallelogram LQ, and of which LM is one of its insisting straight lines.

Therefore, because the parallelogram AB is equal to CD,

as the base AB is to the base LQ, so is the base CD to the base LQ. (V. 7.)

And because the solid parallelopiped AR is cut by the plane $LMEB$, which is parallel to the opposite planes AK, DR;

as the base AB is to the base LQ, so is the solid AE to the solid LR: (XI. 25.)

for the same reason, because the solid parallelopiped CR is cut by the plane $LMFD$, which is parallel to the opposite planes CP, BR;

as the base CD to the base LQ, so is the solid CF to the solid LR:

but as the base AB to the base LQ, so the base CD to the base LQ, as before was proved;

therefore, as the solid AE to the solid LR, so is the solid CF to the solid LR: (V. 11.)

and therefore the solid AE is equal to the solid CF. (V. 9.)

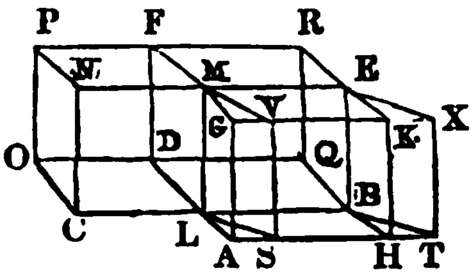
But let the solid parallelopipeds SE, CF be upon equal bases SB, CD, and be of the same altitude, and let their insisting straight lines be at right angles to the bases;

and place the bases SB, CD in the same plane, so that CL, LB may be in a straight line;

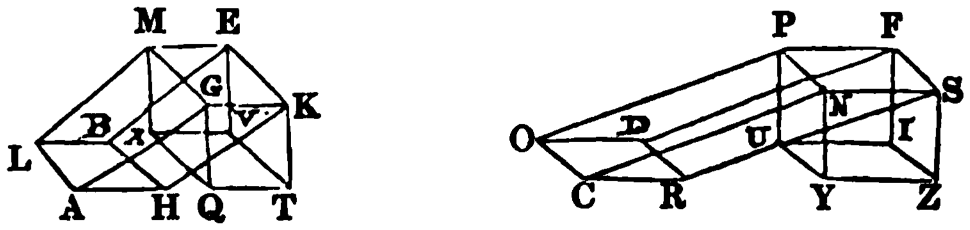
and let the angles SLB, CLD be unequal:

the solid SE shall be equal to the solid CF.

Produce DL, TS until they meet in A, and from B draw BH parallel to DA;



and let HB, OD produced meet in Q, and complete the solids AE, LR;
therefore the solid AE is equal to the solid SE; (XI. 29.)
because they are upon the same base LE, and of the same altitude, and their insisting straight lines, viz. LA, LS, BH, BT; MG, MV, EK, EX, are in the same straight lines AT, GX:
and because the parallelogram AB is equal to SB, (I. 35.)
for they are upon the same base LB, and between the same parallels LB, AT;
and that the base SB is equal to the base CD;
therefore the base AB is equal to the base CD;
and the angle ALB is equal to the angle CLD;
therefore, by the first case, the solid AE is equal to the solid CF:
but the solid AE is equal to the solid SE, as was demonstrated;
therefore the solid SE is equal to the solid CF.
But, if the insisting straight lines AG, HK, BE, LM; CN, RS, DF, OP be not at right angles to the bases AB, CD;
in this case likewise the solid AE shall be equal to the solid CF.

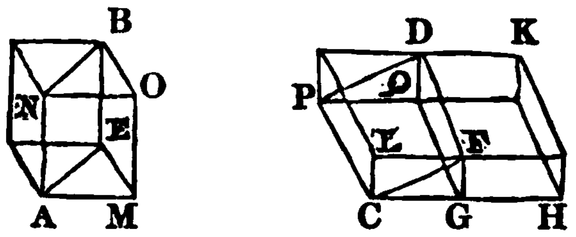


From the points G, K, E, M; N, S, F, P draw the straight lines GQ, KT, EV, MX; NY, SZ, FI, PU, perpendicular to the plane in which are the bases AB, CD; (XI. 11.)
and let them meet it in the points Q, T, V, X; Y, Z, I, U;
and join QT, TV, VX, XQ; YZ, ZI, IU, UY.
Then, because GQ, KT are at right angles to the same plane,
they are parallel to one another: (XI. 6.)
and MG, EK are parallels;
therefore the planes MQ, ET, of which one passes through MG, GQ, and the other through EK, KT, which are parallel to MG, GQ, and not in the same plane with them, are parallel to one another; (XI. 15.)
for the same reason, the planes MV, GT are parallel to one another;
therefore the solid QE is a parallelopiped.
In like manner it may be proved, that the solid YF is a parallelopiped.
But, from what has been demonstrated, the solid EQ is equal to the solid FY,
because they are upon equal bases MK, PS, and of the same altitude,
and have their insisting straight lines at right angles to the bases:
and the solid EQ is equal to the solid AE, (XI. 29. or 30.)
and the solid FY to the solid CF,
because they are upon the same bases and of the same altitude;
therefore the solid AE is equal to the solid CF.
Wherefore, solid parallelopipeds, &c. Q.E.D.

PROPOSITION XXXII. THEOREM.

Solid parallelopipeds which have the same altitude, are to one another
; their bases.

Let AB, CD be solid parallelopipeds of the same altitude.
They shall be to one another as their bases;
that is, as the base AE to the base CF, so shall the solid AB be to the solid CD.



To the straight line FG apply the parallelogram FH equal to AE,
. 45. Cor.)
so that the angle FGH may be equal to the angle LCG;
and upon the base FH complete the solid parallelopiped GK, one of
hose insisting lines is FD, whereby the solids CD, GK must be of the
me altitude.

Therefore the solid AB is equal to the solid GK, (XI. 31.)
ecause they are upon equal bases AE, FH, and are of the same altitude:
and because the solid parallelopiped CK is cut by the plane DG,
which is parallel to its opposite planes,
ıe base HF is to the base FC, as the solid HD to the solid DC: (XI. 25.)
but the base HF is equal to the base AE, and the solid GK to the solid AB;
therefore, as the base AE to the base CF, so is the solid AB to the solid CD.

Wherefore, solid parallelopipeds, &c. Q.E.D.

COR. From this it is manifest, that prisms upon triangular bases, of
he same altitude, are to one another as their bases.

Let the prisms, the bases of which are the triangles AEM, CFG, and
NBO, PDQ the triangles opposite to them, have the same altitude:
they shall be to one another as their bases.

Complete the parallelograms AE, CF, and the solid parallelopipeds
AB, CD, in the first of which let MO, and in the other let GQ be one
of the insisting lines.

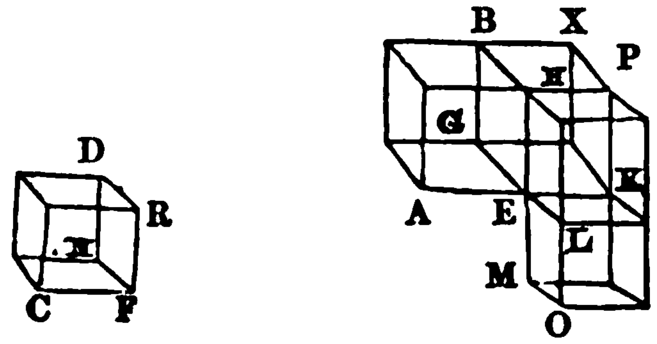
And because the solid parallelopipeds AB, CD have the same altitude,
they are to one another as the base AE is to the base CF:
wherefore the prisms, which are their halves, are to one another,
as the base AE to the base CF; (XI. 28.)
that is, as the triangle AEM to the triangle CFG.

PROPOSITION XXXIII. THEOREM.

Similar solid parallelopipeds are one to another in the triplicate ratio
f their homologous sides.

Let AB, CD be similar solid parallelopipeds, and the side AE homo-
'gous to the side CF.

The solid AB shall have to the solid CD the triplicate ratio of that which AE has to CF.



Produce AE, GE, HE, and in these produced take EK equal to CF, EL equal to FN, and EM equal to FR;

and complete the parallelogram KL, and the solid KO.

Because KE, EL are equal to CF, FN, each to each,

and the angle KEL equal to the angle CFN,

because it is equal to the angle AEG, which is equal to CFN,

by reason that the solids AB, CD are similar;

therefore the parallelogram KL is similar and equal to the parallelogram CN.

For the same reason the parallelogram MK is similar and equal to CR, and also OE to FD.

Therefore three parallelograms of the solid KO are equal and similar to three parallelograms of the solid CD:

and the three opposite ones in each solid are equal and similar to these: (xi. 24.)

therefore the solid KO is equal and similar to the solid CD. (xi. c.)

Complete the parallelogram GK; and upon the bases GK, KL, complete the solids EX, LP, so that EH be an insisting straight line in each of them, whereby they must be of the same altitude with the solid AB.

And because the solids AB, CD are similar,

and, by permutation, as AE is to CF, so is EG to FN, and so is EH to FR:

but FC is equal to EK, and FN to EL, and FR to EM;

therefore, as AE to EK, so is EG to EL, and so is HE to EM:

but as AE to EK, so is the parallelogram AG to the parallelogram GK; (vi. 1.)

and as GE to EL, so is GK to KL; (vi. 1.)

and as HE to EM, so is PE to KM: (vi. 1.)

therefore, as the parallelogram AG to the parallelogram GK, so is GK to KL, and PE to KM:

but as AG to GK, so is the solid AB to the solid EX; (xi. 25.)

and as GK to KL, so is the solid EX to the solid PL; (xi. 25.)

and as PE to KM, so is the solid PL to the solid KO; (xi. 25.)

and therefore as the solid AB to the solid EX, so is EX to PL, and PL to KO:

but if four magnitudes be continual proportionals, the first is said to have to the fourth, the triplicate ratio of that which it has to the second; (v. def. 11.)

therefore the solid AB has to the solid KO, the triplicate ratio of that which AB has to EX:

but as AB is to EX, so is the parallelogram AG to the parallelogram GK, and the straight line AE to the straight line EK;

wherefore the solid AB has to the solid KO, the triplicate ratio of that which AE has to EK:

but the solid KO is equal to the solid CD,

and the straight line EK is equal to the straight line CF;

therefore the solid AB has to the solid CD, the triplicate ratio of that which the side AE has to the homologous side CF.

Therefore, similar solid parallelopipeds, &c. Q.E.D.

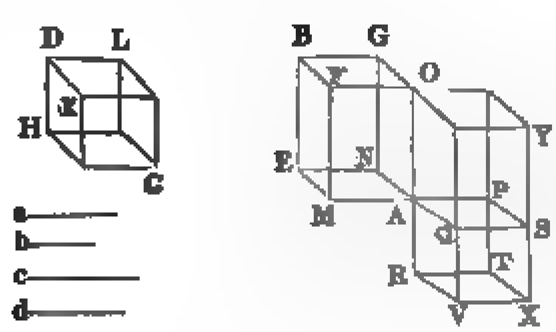
COR.—From this it is manifest, that if four straight lines be continual proportionals, as the first is to the fourth, so is the solid parallelopiped described from the first to the similar solid similarly described from the second; because the first straight line has to the fourth the triplicate ratio of that which it has to the second.

PROPOSITION D. THEOREM.

Solid parallelopipeds which are contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each, have to one another the ratio which is the same with the ratio compounded of the ratios of their sides.

Let AB, CD be solid parallelopipeds, of which AB is contained by the parallelograms AE, AF, AG, which are equiangular, each to each, to the parallelograms CH, CK, CL, which contain the solid CD.

The ratio which the solid AB has to the solid CD, shall be the same with that which is compounded of the ratios of the sides AM to DL, AN to DK, and AO to DH.



Produce MA, NA, OA to P, Q, R, so that AP be equal to DL, AQ to DK, and AR to DH;

and complete the solid parallelopiped AX contained by the parallelograms AS, AT, AV similar and equal to CH, CK, CL, each to each.

Therefore the solid AX is equal to the solid CD. (XI. C.)

Complete likewise the solid AY, the base of which is AS, and AO one of its insisting straight lines.

Take any straight line a,

and as MA to AP, so make a to b; (VI. 12.)

and as NA to AQ, so make b to c;

and as AO to AR, so c to d.

Then, because the parallelogram AE is equiangular to AS,

AE is to AS, as the straight line a to c, as is demonstrated in Prop. 23, Book VI.:

and the solids AB, AY, being betwixt the parallel planes BOY, EAS, are of the same altitude;

therefore the solid AB is to the solid AY, as the base AE to the base AS; that is, as the straight line a is to c. (XI. 32.)

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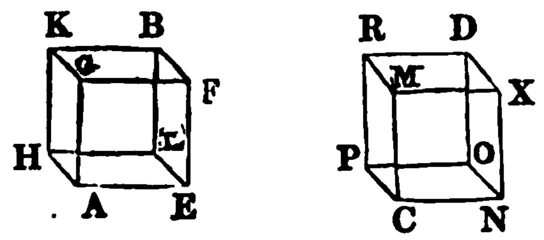
And the solid AY is to the solid AX, as the base OQ is to the base QR; (XI. 25.)
that is, as the straight line OA to AR;
that is, as the straight line c to the straight line d.
And because the solid AB is to the solid AY, as a is to c,
and the solid AY to the solid AX, as c is to d;
ex æquali, the solid AB is to the solid AX, or CD which is equal to it, as the straight line a is to d.
But the ratio of a to d is said to be compounded of the ratios of a to b, b to c, and c to d, (V. def. A.)
which are the same with the ratios of the sides MA to AB, NA to AQ, and OA to AR, each to each:
and the sides, AP, AE, AR are equal to the sides DL, DK, DH, each to each:
therefore the solid AB has to the solid CD the ratio which is the same with that which is compounded of the ratios of the sides AM to DL, AN to DK, and AO to DH. Q.E.D.

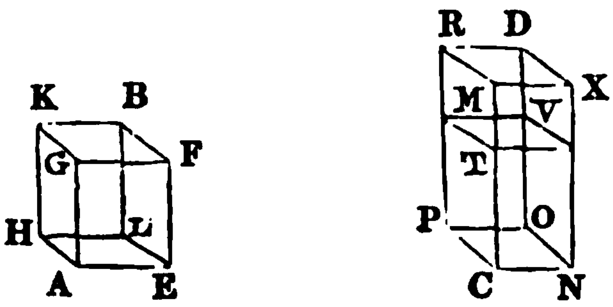
PROPOSITION XXXIV. THEOREM.

The bases and altitudes of equal solid parallelopipeds, are reciprocally proportional: and conversely, if the bases and altitudes be reciprocally proportional, the solid parallelopipeds are equal.

Let AB, CD be two solid parallelopipeds:
and first, let the insisting straight lines AG, EF, LB, HK; CM, NX, OD, PR be at right angles to the bases.
If the solid AB be equal to the solid CD, their bases shall be reciprocally proportional to their altitudes;
that is, as the base EH is to the base NP, so shall CM be to AG.
If the base EH be equal to the base NP,



then because the solid AB is likewise equal to the solid CD,
CM shall be equal to AG:
because if the bases EH, NP be equal, but the altitudes AG, CM be not equal, neither shall the solid AB be equal to the solid CD:
but the solids are equal, by the hypothesis;
therefore the altitude CM is not unequal to the altitude AG; that is, they are equal.
Wherefore, as the base EH to the base NP, so is CM to AG.
Next, let the bases EH, NP not be equal, but EH greater than the other:



then since the solid AB is equal to the solid CD,
CM is therefore greater than AG:
for, if it be not, neither also in this case would the solids AB, CD be equal, which, by the hypothesis, are equal.

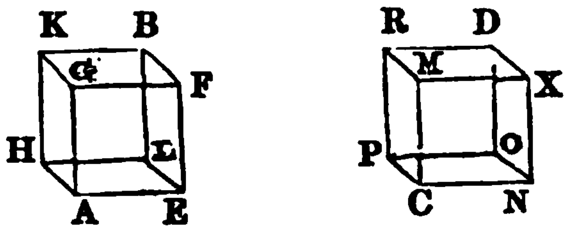
Make then CT equal to AG, and complete the solid parallelopiped CV, of which the base is NP, and altitude CT.

Because the solid AB is equal to the solid CD,
therefore the solid AB is to the solid CV, as the solid CD to the solid CV: (v. 7.)
but as the solid AB to the solid CV, so is the base EH to the base NP; (xi. 32.)
for the solids AB, CV are of the same altitude:
and as the solid CD to CV so is the base MP to the base PT, (xi. 25.) and so is the straight line MC to CT; (vi. 1.)
and CT is equal to AG:
therefore, as the base EH to the base NP, so is MC to AG.

Wherefore the bases of the solid parallelopipeds AB, CD are reciprocally proportional to their altitudes.

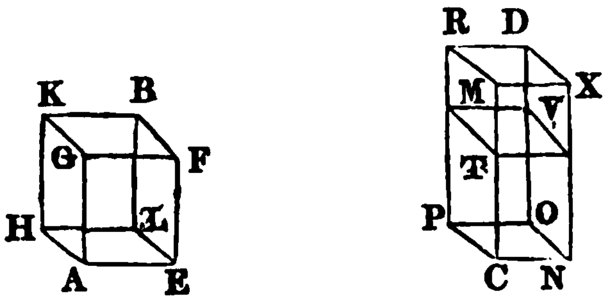
Let now the bases of the solid parallelopipeds AB, CD be reciprocally proportional to their altitudes, viz. as the base EH is to the base NP, so let CM be to AG.

The solid AB shall be equal to the solid CD.



If the base EH be equal to the base NP,
then since EH is to NP, as the altitude of the solid CD is to the altitude of the solid AB,
therefore the altitude of CD is equal to the altitude of AB: (v. A.)
but solid parallelopipeds upon equal bases, and of the same altitude, are equal to one another; (xi. 31.)
therefore the solid AB is equal to the solid CD.

But let the bases EH, NP be unequal, and let EH be the greater of the two.



Therefore, since, as the base EH to the base NP, so is CM the altitude of the solid CD to AG the altitude of AB,
CM is greater than AG. (v. A.)
Therefore, as before, take CT equal to AG, and complete the solid CV.
And because the base EH is to the base NP, as CM to AG,
and that AG is equal to CT,
therefore the base EH is to the base NP, as MC to CT.
But as the base EH is to NP, so is the solid AB to the solid CV; (xi. 32.)
for the solids AB, CV are of the same altitude:
and as MC to CT, so is the base MP to the base PT, (vi. 1.) and the solid CD to the solid CV: (xi. 25.)

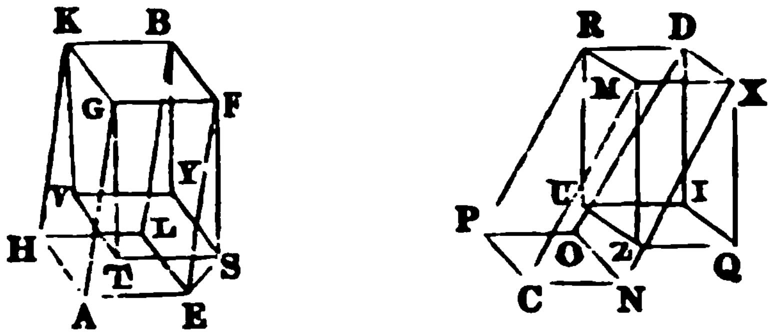
therefore as the solid AB to the solid CV, so is the solid CD to the solid CV;

that is, each of the solids AB, CD has the same ratio to the solid CV;

and therefore the solid AB is equal to the solid CD. (v. 9.)

Second general case. Let the insisting straight lines FE, BL, GA, KH; XN, DO, MC, RP not be at right angles to the bases of the solids.

In this case, likewise, if the solids AB, CD be equal, their bases shall be reciprocally proportional to their altitudes, viz. the base EH shall be to the base NP, as the altitude of the solid CD to the altitude of the solid AB.



From the points F, B, K, G; X, D, R, M, draw perpendiculars to the planes in which are the bases EH, NP, meeting those planes in the points S, Y, V, T; Q, I, U, Z; and complete the solids FV, XU, which are parallelopipeds, as was proved in the last part of Prop. 31, of this book.

Because the solid AB is equal to the solid CD, and that the solid AB is equal to the solid BT, for they are upon the same base FK, and of the same altitude; (xi. 30. or 29.)

and that the solid CD is equal to the solid DZ, being upon the same base XR, and of the same altitude; (xi. 30. or 29.)

therefore the solid BT is equal to the solid DZ:

but the bases are reciprocally proportional to the altitudes of equal solid parallelopipeds of which the insisting straight lines are at right angles to their bases, as before was proved;

therefore as the base FK to the base XR, so is the altitude of the solid DZ to the altitude of the solid BT:

and the base FK is equal to the base EH,

and the base XR to the base NP;

wherefore, as the base EH to the base NP, so is the altitude of the solid DZ to the altitude of the solid BT:

but the altitudes of the solids DZ, DC, as also of the solids BT, BA, are the same;

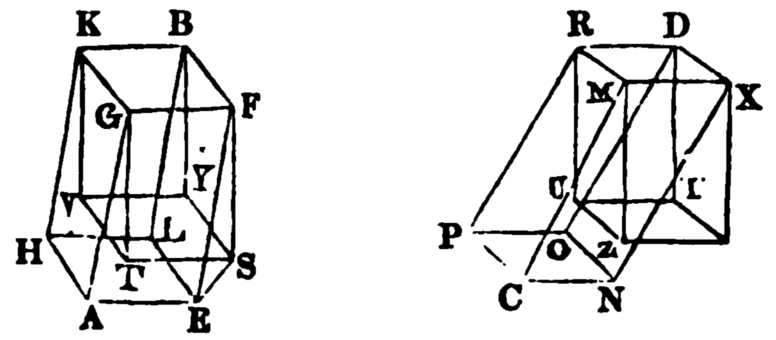
therefore as the base EH to the base NP, so is the altitude of the solid CD to the altitude of the solid AB;

that is, the bases of the solid parallelopipeds AB, CD are reciprocally proportional to their altitudes.

Next, let the bases of the solids AB, CD be reciprocally proportional to their altitudes,

viz. the base EH to the base NP, as the altitude of the solid CD to the altitude of the solid AB.

The solid AB shall be equal to the solid CD.

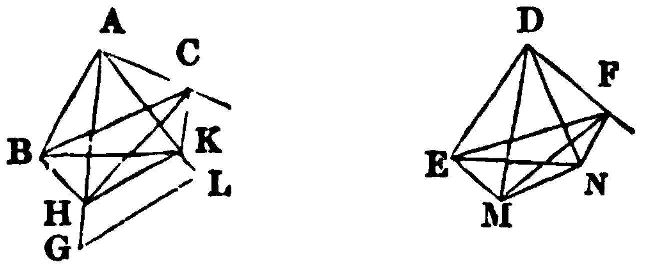


The same construction being made;
because, as the base EH to the base NP, so is the altitude of the solid CD to the altitude of the solid AB;
and that the base EH is equal to the base FK, and NP to XR;
therefore the base FK is to the base XR, as the altitude of the solid CD to the altitude of AB:
but the altitudes of the solids AB, BT are the same, as also of CD and DZ;
therefore as the base FK to the base XR, so is the altitude of the solid DZ to the altitude of the solid BT:
wherefore the bases of the solids BT, DZ are reciprocally proportional to their altitudes:
and their insisting straight lines are at right angles to the bases;
vherefore, as was before proved, the solid BT is equal to the solid DZ:
ut BT is equal to the solid BA, and DZ to the solid DC, (XI. 30. or 29.)
because they are upon the same bases, and of the same altitude;
therefore the solid AB is equal to the solid CD.
Therefore, the bases, &c. Q. E. D.

PROPOSITION XXXV. THEOREM.

If, from the vertices of two equal plane angles, there be drawn two traight lines elevated above the planes in which the angles are, and conaining equal angles with the sides of those angles, each to each; and if n the lines above the planes there be taken any points, and from them perpendiculars be drawn to the planes in which the first-named angles are; and from the points in which they meet the planes, straight lines be drawn o the vertices of the angles first named: these straight lines shall contain equal angles with the straight lines which are above the planes of the angles.

Let BAC, EDF be two equal plane angles:
and from the points A, D let the straight lines AG, DM be elevated above the planes of the angles, making equal angles with their sides, each to each,
viz. the angle GAB equal to the angle MDE, and GAC to MDF;
and in AG, DM let any points G, M be taken,
and from them let perpendiculars GL, MN be drawn to the planes BAC, EDF, (XI. 11.) meeting these planes in the points L, N; and join LA, ND.
The angle GAL shall be equal to the angle MDN.



Make AH equal to DM, and through H draw HK parallel to GL:
but GL is perpendicular to the plane BAC;
wherefore HK is perpendicular to the same plane. (XI. 8.)
From the points K, N, to the straight lines AB, AC, DE, DF, draw perpendiculars KB, KC, NE, NF; and join HB, BC, ME, EF.

Because HK is perpendicular to the plane BAC,
the plane HBK which passes through HK is at right angles to the plane BAC; (xi. 18.)
and AB is drawn in the plane BAC at right angles to the common section BK of the two planes;
therefore AB is perpendicular to the plane HBK, (xi. def. 4.)
and makes right angles with every straight line meeting it in that plane: (xi. def. 3.)
but BH meets it in that plane;
therefore ABH is a right angle:
for the same reason DEM is a right angle, and is therefore equal to the angle ABH:
and the angle HAB is equal to the angle MDE: (hyp.)
therefore in the two triangles HAB, MDE there are two angles in one, equal to two angles in the other, each to each, and one side equal to one side, opposite to one of the equal angles in each, viz. HA equal to DM;
therefore the remaining sides are equal, each to each: (i. 26.)
wherefore AB is equal to DE.
In the same manner, if HC and MF be joined,
it may be demonstrated that AC is equal to DF:
therefore, since AB is equal to DE, BA and AC are equal to ED and DF, each to each;
and the angle BAC is equal to the angle EDF; (hyp.)
wherefore the base BC is equal to the base EF, (i. 4.)
and the remaining angles to the remaining angles:
therefore the angle ABC is equal to the angle DEF:
and the right angle ABK is equal to the right angle DEN;
whence the remaining angle CBK is equal to the remaining angle FEN:
for the same reason, the angle BCK is equal to the angle EFN:
therefore, in the two triangles BCK, EFN, there are two angles in one equal to two angles in the other, each to each, and one side equal to one side adjacent to the equal angles in each, viz. BC equal to EF;
therefore the other sides are equal to the other sides;
BK then is equal to EN, and AB is equal to DE;
wherefore AB, BK are equal to DE, EN, each to each;
and they contain right angles;
wherefore the base AK is equal to the base DN.
And since AH is equal to DM,
the square of AH is equal to the square of DM:
but the squares of AK, KH are equal to the square of AH, because AKH is a right angle; (i. 47.)
and the squares of DN, NM are equal to the square of DM, for DNM is a right angle:
wherefore the squares of AK, KH are equal to the squares of DN, NM:
and of these the square of AK is equal to the square of DN;
therefore the remaining square of KH is equal to the remaining square of NM;
and the straight line KH to the straight line NM:
and because HA, AK are equal to MD, DN, each to each,
and the base HK to the base MN, as has been proved;
therefore the angle HAK is equal to the angle MDN. (i. 8.)
Therefore, if from the vertices, &c. Q.E.D.

COR.—From this it is manifest, that if from the vertices of two equal plane angles, there be elevated two equal straight lines containing equal angles with the sides of the angles each to each; the perpendiculars drawn from the extremities of the equal straight lines to the planes of the first angles are equal to one another.

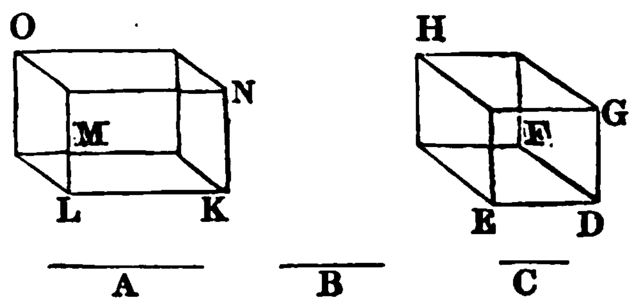
Another demonstration of the Corollary.

Let the plane angles *BAC, EDF* be equal to one another, and let *AH, DM* be two equal straight lines above the planes of the angles, containing equal angles with *BA, AC; ED, DF,* each to each, viz. the angle *HAB* equal to *MDE,* and *HAC* equal to the angle *MDF;*
and from *H, M,* let *HK, MN* be perpendiculars to the planes *BAC, EDF:*
HK shall be equal to *MN.*
Because the solid angle at *A* is contained by the three plane angles *BAC, BAH, HAC,* which are, each to each, equal to the three plane angles *EDF, EDM, MDF,* containing the solid angle at *D;*
the solid angles at *A* and *D* are equal, and therefore coincide with one another;
to wit, if the plane angle *BAC* be applied to the plane angle *EDF,*
the straight line *AH* coincides with *DM,* as was shewn in Prop. B. of this book:
and because *AH* is equal to *DM,*
the point *H* coincides with the point *M:*
wherefore *HK,* which is perpendicular to the plane *BAC,* coincides with *MN* which is perpendicular to the plane *EDF,* (XI. 13.)
because these planes coincide with one another.
Therefore, *HK* is equal to *MN.* Q.E.D.

PROPOSITION XXXVI. THEOREM.

If three straight lines be proportionals, the solid parallelopiped described from all three, as its sides, is equal to the equilateral parallelopiped described from the mean proportional, one of the solid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the solid angles of the other figure.

Let *A, B, C* be three proportionals, viz. *A* to *B,* as *B* to *C.*
The solid described from *A, B, C* shall be equal to the equilateral solid described from *B,* equiangular to the other.



Take a solid angle *D* contained by three plane angles *EDF, FDG, GDE;*
and make each of the straight lines *ED, DF, DG* equal to *B,*
and complete the solid parallelopiped *DH;*

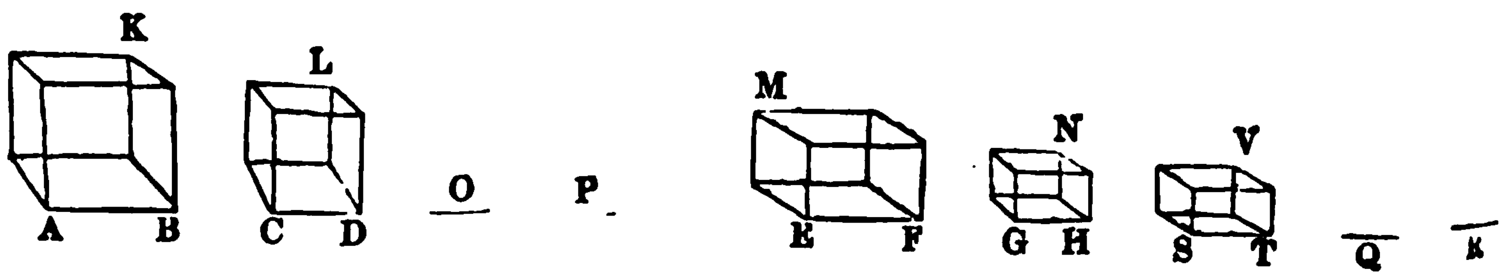
make LK equal to A, and at the point K in the straight line LK, make a solid angle contained by the three plane angles LKM, MKN, NKL, equal to the angles EDF, FDG, GDE, each to each; (XI. 26.)
and make KN equal to B, and KM equal to C; and complete the solid parallelopiped KO.

And because, as A is to B, so is B to C,
and that A is equal to LK, and B to each of the straight lines DE, DF, and C to KM;
therefore LK is to ED, as DF to KM;
that is, the sides about the equal angles are reciprocally proportional;
therefore the parallelogram LM is equal to EF: (VI. 14.)
and because EDF, LKM are two equal plane angles, and the two equal straight lines DG, KN are drawn from their vertices above their planes, and contain equal angles with their sides;
therefore the perpendiculars from the points G, N to the planes EDF, LKM are equal to one another: (XI. 35. Cor.)
therefore the solids KO, DH are of the same altitude:
and they are upon equal bases LM, EF;
and therefore they are equal to one another: (XI. 31.)
but the solid KO is described from the three straight lines A, B, C, and the solid DH from the straight line B.
Therefore if three straight lines, &c. Q.E.D.

PROPOSITION XXXVII. THEOREM.

If four straight lines be proportionals, the similar solid parallelopipeds similarly described from them shall also be proportionals. And if the similar parallelopipeds similarly described from four straight lines be proportionals, the straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, GH be proportionals,
viz. as AB to CD, so EF to GH;
and let the similar parallelopipeds AK, CL, EM, GN be similarly described from them.
AK shall be to CL, as EM to GN.



Make AB, CD, O, P continual proportionals,
as also EF, GH, Q, R: (VI. 11.)
and because as AB is to CD, so EF to GH;
and that CD is to O, as GH to Q, (V. 11.)
and O to P, as Q to R;
therefore, ex æquali, AB is to P, as EF to R: (V. 22.)
but as AB to P, so is the solid AK to the solid CL; and as EF to R, so is the solid EM to the solid GN; (XI. 33. Cor.)
therefore as the solid AK to the solid CL, so is the solid EM to the solid GN. (V. 11.)

Next, let the solid AK be to the solid CL, as the solid EM to the id GN.

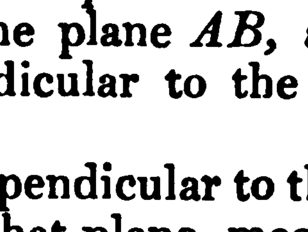
The straight line AB shall be to CD, as EF to GH.

Take as AB to CD, so EF to ST,
and from ST describe a solid parallelopiped SV similar and similarly situated to either of the solids EM, GN. (XI. 27.)
And because AB is to CD, as EF to ST,
and that from AB, CD the solid parallelopipeds AK, CL are similarly described;
d in like manner the solids EM, SV from the straight lines EF, ST;
therefore AK is to CL, as EM to SV;
but, by the hypothesis, AK is to CL, as EM to GN;
therefore GN is equal to SV: (V. 9.)
but it is likewise similar and similarly situated to SV;
therefore the planes which contain the solids GN, SV are similar and equal, and their homologous sides GH, ST equal to one another:
and because as AB to CD, so EF to ST,
and that ST is equal to GH,
therefore AB is to CD, as EF to GH.
Therefore, if four straight lines, &c. Q.E.D.

PROPOSITION XXXVIII. THEOREM.

"*If a plane be perpendicular to another plane, and a straight line be ·awn from a point in one of the planes perpendicular to the other plane, :is straight line shall fall on the common section of the planes.*

"Let the plane CD be perpendicular to the plane AB, and let AD e their common section:
if any point E be taken in the plane CD, the perpendicular drawn from E to the plane AB shall fall on AD.



"For, if it does not, let it, if possible, fall elsewhere, as EF;
and let it meet the plane AB in the point F;
and from F draw, in the plane AB, a perpendicular FG to DA, I. 12.) which is also perpendicular to the plane CD; (XI. def. 4.) and oin EG.
Then, because FG is perpendicular to the plane CD, and the straight line EG, which is in that plane, meets it;
therefore FGE is a right angle: (XI. def. 3.)
but EF is also at right angles to the plane AB;
and therefore EFG is a right angle;
wherefore two of the angles of the triangle EFG are equal together to two right angles; which is absurd: (I. 17.)
therefore the perpendicular from the point E to the plane AB does not fall elsewhere than upon the straight line AD;
it therefore falls upon it.
If therefore a plane, &c. Q.E.D."

PROPOSITION XXXIX. THEOREM.

In a solid parallelopiped, if the sides of two of the opposite planes be divided, each into two equal parts, the common section of the planes passing through the points of division, and the diameter of the solid parallelopiped, cut each other into two equal parts.

Let the sides of the opposite planes *CF*, *AH*, of the solid parallelopiped *AF*, be divided each into two equal parts in the points *K*, *L*, *M*, *N*; *X*, *O*, *P*, *R*; and join *KL*, *MN*, *XO*, *PR*:

and because *DK*, *CL* are equal and parallel,

KL is parallel to *DC*: (I. 33.)

for the same reason, *MN* is parallel to *BA*:

and *BA* is parallel to *DC*;

therefore, because *KL*, *BA* are each of them parallel to *DC*, and not in the same plane with it, *KL* is parallel to *BA*: (XI. 9.)

and because *KL*, *MN* are each of them parallel to *BA*, and not in the same plane with it,

KL is parallel to *MN*: (XI. 9.)

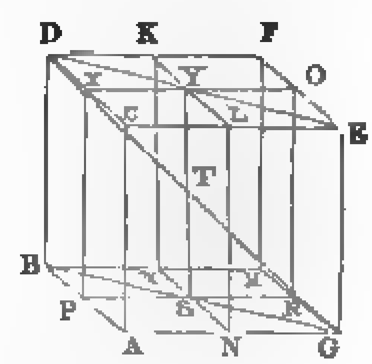
wherefore *KL*, *MN* are in one plane.

In like manner, it may be proved, that *XO*, *PR* are in one plane.

Let *YS* be the common section of the planes *KN*, *XR*;

and *DG* the diameter of the solid parallelopiped *AF*:

YS and *DG* shall meet, and cut one another into two equal parts.



Join *DY*, *YE*, *BS*, *SG*.

Because *DX* is parallel to *OE*, the alternate angles *DXY*, *YOE* are equal to one another: (I. 29.)

and because *DX* is equal to *OE*, and *XY* to *YO*, and that they contain equal angles,

the base *DY* is equal to the base *YE*, and the other angles are equal; (I. 4.)

therefore the angle *XYD* is equal to the angle *OYE*,

and *DYE* is a straight line: (I. 14.)

for the same reason *BSG* is a straight line, and *BS* equal to *SG*.

And because *CA* is equal and parallel to *DB*, and also equal and parallel to *EG*;

therefore *DB* is equal and parallel to *EG*: (XI. 9.)

and *DE*, *BG* join their extremities;

therefore *DE* is equal and parallel to *BG*: (I. 33.)

and *DG*, *YS* are drawn from points in the one, to points in the other; and are therefore in one plane:

whence it is manifest, that *DG*, *YS* must meet one another:

let them meet in *T*.

And because DE is parallel to BG,
the alternate angles EDT, BGT are equal; (XI. 29.)
and the angle DTY is equal to the angle GTS: (I. 15.)
herefore in the triangles DTY, GTS there are two angles in the equal to two angles in the other, and one side equal to one side, site to two of the equal angles, viz. DY to GS;
for they are the halves of DE, BG:
therefore the remaining sides are equal, each to each: (I. 26.)
wherefore DT is equal to TG, and YT equal to TS.
Wherefore, if in a solid, &c. Q.E.D.

PROPOSITION XL. THEOREM.

If there be two triangular prisms of the same altitude, the base of one hich is a parallelogram, and the base of the other a triangle; if the llelogram be double of the triangle, the prisms shall be equal to one her.

Let the prisms $ABCDEF$, $GHKLMN$ be of the same altitude, the whereof is contained by the two triangles ABE, CDF, and the e parallelograms AD, DE, EC; and the other by the two triangles K, LMN, and the three parallelograms LH, HN, NG; and let one hem have a parallelogram AF, and the other a triangle GHK, for base.

If the parallelogram AF be double of the triangle GHK, the prism $CDEF$ shall be equal to the prism $GHKLMN$.

Complete the solids AX, GO:
nd because the parallelogram AF is double of the triangle GHK;
and the parallelogram HK double of the same triangle; (I. 34.)
therefore the parallelogram AF is equal to HK:
but solid parallelopipeds upon equal bases, and of the same altitude, are equal to one another; (XI. 31.)
therefore the solid AX is equal to the solid GO:
and the prism $ABCDEF$ is half of the solid AX; (XI. 28.)
and the prism $GHKLMN$ half of the solid GO: (XI. 28.)
therefore the prism $ABCDEF$ is equal to the prism $GHKLMN$.
Wherefore, if there be two, &c. Q.E.D.

NOTES TO BOOK XI.

The Eleventh Book of the Elements commences with the definitions of the Geometry of Planes and Solids, and then proceeds to demonstrate the most elementary properties of straight lines and planes, solid angles and parallelopipeds.

The solids considered in the eleventh and twelfth books are Geometrical solids, portions of space bounded by surfaces which are supposed capable of penetrating and intersecting one another.

In the first six books, all the diagrams employed in the demonstrations are supposed to be in the same plane, which may lie in any position whatever, and be extended in every direction, and there is no difficulty in representing them roughly on any plane surface; this, however, is not the case with the diagrams employed in the demonstrations in the eleventh and twelfth books, which cannot be so intelligibly represented on a plane surface on account of the perspective. A more exact conception may be attained, by adjusting pieces of paper to represent the different planes, and drawing lines upon them as the constructions may require, and by fixing pins to represent the lines which are perpendicular to, or inclined to any planes.

Any plane may be conceived to move round any fixed point in that plane, either in its own plane, or in any direction whatever; and if there be two fixed points in the plane, the plane cannot move in its own plane, but may move round the straight line which passes through the two fixed points in the plane, and may assume every possible position of the planes which pass through that line, and every different position of the plane will represent a different plane; thus, an indefinite number of planes may be conceived to pass through a straight line which will be the common intersection of all the planes. Hence, it is manifest, that though two points fix the position of a straight line in a plane, neither do two points nor a straight line fix the position of a plane in space. If however, three points, not in the same straight line, be conceived to be fixed in the plane, it will be manifest, that the plane cannot be moved round, either in its own plane or in any other direction, and therefore is fixed.

Also any conditions which involve the consideration of three fixed points not in the same straight line, will fix the position of a plane in space; as two straight lines which meet or intersect one another or two parallel straight lines in the plane.

Def. v. When a straight line meets a plane, it is inclined at different angles to the different lines in that plane which may meet it; and it is manifest that the inclination of the line to the plane is not determined by its meeting *any line* in that plane. The inclination of the line to the plane can only be determined by its inclination to some fixed line in the plane. If a point be taken in the line different from that point where the line meets the plane, and a perpendicular be drawn to meet the plane in another point; then these two points in the plane will fix the position of the line which passes through them in that plane, and the angle contained by this line and the given line, will measure the inclination of the line to the plane; and it will be found to be the least angle which can be formed with the given line and any other straight line in the plane.

If two perpendiculars be drawn upon a plane from the extremities of a straight line which is inclined to that plane, the straight line in the plane intercepted between the perpendiculars is called the *projection* of the line on that plane; and it is obvious that the inclination of a straight line to a plane is equal to the inclination of the straight line to its *projection* on the plane. If however, the line be parallel to the

plane, the projection of the line is of the same length as the line itself; in all other cases the projection of the line is less than the line, being the base of a right-angled triangle, the hypothenuse of which is the line itself.

The inclination of two lines to each other, which do not meet, is measured by the angle contained by two lines drawn through the same point and parallel to the two given lines.

Def. VI. Planes are distinguished from one another by their inclinations, and the inclinations of two planes to one another will be found to be measured by the acute angle formed by two straight lines drawn in the planes, and perpendicular to the straight line which is the common intersection of the two planes.

It is also obvious that the inclination of one plane to another will be measured by the angle contained between two straight lines drawn from the same point, and perpendicular, one on each of the two planes.

The intersection of two planes suggests a new conception of the straight line.

Def. IX. When a solid angle is contained by three plane angles, each plane which contains one plane angle, is fixed by the position of the other two, and consequently, only one solid angle can be formed by three plane angles. But when a solid angle is formed by more than three plane angles, if one of the planes be considered fixed in position, there are no conditions which fix the position of the rest of the planes which contain the solid angle, and hence, an indefinite number of solid angles, unequal to one another, may be formed by the same plane angles, when the number of plane angles is more than three.

Def. X is restored, as it is found in the editions of Euclid, by Dr Barrow and others. It appears to be universally true, supposing the planes to be similarly situated, in which are contained the corresponding equal plane angles of each figure.

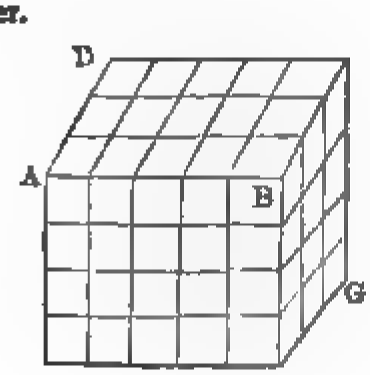
Def. A. Parallelopipeds are solid figures in some respects analogous to parallelograms, and remarks might be made on parallelopipeds similar to those which were made on parallelograms in the notes to Book II, p. 67; and every right-angled parallelopiped *may be said to be contained* by any three of the straight lines which contain the three right angles by which any one of the solid angles of the figure is formed; or more briefly, by the three adjacent edges of the parallelopiped.

As all lines are measured by lines, and all surfaces by surfaces, so all solids are measured by solids. The cube is the figure assumed as the measure of solids or volumes, and the unit of volume is that cube, the edge of which is one unit in length.

If the edges of a rectangular parallelopiped can be divided into units of the same length, a numerical expression for the number of cubic units in the parallelopiped may be found, by a process similar to that by which a numerical expression for the area of a rectangle was found.

Let AB, AC, AD be the adjacent edges of a rectangular parallelopiped AG, and let AB contain 5 units, AC, 4 units, and AD, 3 units in length.

Then if through the points of division of AB, AC, AD, planes be drawn parallel to the faces BG, BD, AE respectively, the parallelopiped will be divided into cubic units, all equal to one another.



And since the rectangle $ABEC$ contains 5×4 square untis, (note, p. 68.) and that for every linear unit in AD there is a layer of 5×4 cubic units corresponding to it;

consequently, there are $5 \times 4 \times 3$ cubic units in the whole parallelopiped AG.

That is, the product of the three numbers which express the number of linear units in the three edges, will give the number of cubic units in the parallelopiped, and therefore will be the arithmetical representation of its volume.

And generally, if AB, AC, AD; instead of 5, 4 and 3, consisted of a, b and c linear units, it may be shewn, in a similar manner, that the volume of the parallelopiped would contain abc cubic units, and the product abc would be a proper representation of the volume of the parallelopiped.

If the three sides of the figure were equal to one another, or b and c each equal to a, the figure would become a cube, and its volume would be represented by aaa, or a^3.

Prop. VI. From the diagram, the following important construction may be made.

If from B a perpendicular BF be drawn to the opposite side DE of the triangle DBE, and AF be joined; then AF shall be perpendicular to DE, and the angle AFB measures the inclination of the planes AED and BED.

Prop. XIX. It is also obvious, that if three planes intersect one another; and if the first be perpendicular to the second, and the second be perpendicular to the third; the first shall be perpendicular to the third; also the intersections of every two shall be perpendicular to one another.

Prop. XXXIII. Algebraically.

Let the adjacent edges of the solid AB contain a, b, c units,

and those of the solid CD contain a', b', c' units respectively.

Also, let V, V' denote their volumes.

Then $V = abc$, and $V' = a'b'c'$.

But since the parallelopipeds are similar, $\therefore \dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$;

$$\text{Hence } \frac{V}{V'} = \frac{abc}{a'b'c'} = \frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'} = \frac{a}{a'} \cdot \frac{a}{a'} \cdot \frac{a}{a'} = \frac{a^3}{a'^3} = \frac{b^3}{b'^3} = \frac{c^3}{c'^3}.$$

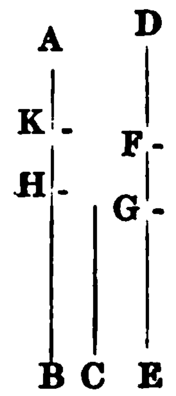
In a similar manner, it may be shewn that the volumes of all similar solid figures bounded by planes, are proportional to the cubes of their homologous edges.

BOOK XII.

LEMMA I.

If from the greater of two unequal magnitudes, there be taken more an its half, and from the remainder more than its half; and so on: ere shall at length remain a magnitude less than the least of the pro- osed magnitudes. (Book x. Prop. 1.)

et AB and C be two unequal magnitudes, of which AB is the greater.
If from AB there be taken more than its half,
and from the remainder more than its half, and so on;
there shall at length remain a magnitude less than C.



For C may be multiplied so as at length to become greater than AB.
Let it be so multiplied, and let DE its multiple be greater than AB,
and let DE be divided into DF, FG, GE, each equal to C.
From AB take BH greater than its half,
and from the remainder AH take HK greater than its half, and so on,
until there be as many divisions in AB as there are in DE:
and let the divisions in AB be AK, KH, HB;
and the divisions in DE be DF, FG, GE.
And because DE is greater than AB,
and that EG taken from DE is not greater than its half, but BH taken from AB is greater than its half;
therefore the remainder GD is greater than the remainder HA.
Again, because GD is greater than HA, and that GF is not greater han the half of GD, but HK is greater than the half of HA;
therefore the remainder FD is greater than the remainder AK:
and FD is equal to C,
therefore C is greater than AK;
that is, AK is less than C. Q.E.D.

And if only the halves be taken away, the same thing may in the ıme way be demonstrated.

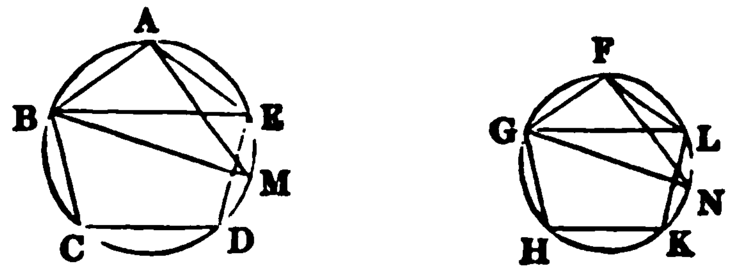
PROPOSITION I. THEOREM.

Similar polygons inscribed in circles, are to one another as the squares f their diameters.

Let $ABCDE$, $FGHKL$ be two circles, and in them the similar polygons $ABCDE$, $FGHKL$;

and let BM, GN be the diameters of the circles:

as the polygon $ABCDE$ is to the polygon $FGHKL$, so shall the square of BM be to the square of GN.



Join BE, AM, GL, FN.

And because the polygon $ABCDE$ is similar to the polygon $FGHKL$,

the angle BAE is equal to the angle GFL,

and as BA to AE, so is GF to FL:

therefore the two triangles BAE, GFL having one angle in one equal to one angle in the other, and the sides about the equal angles proportionals, are equiangular;

and therefore the angle AEB is equal to the angle FLG:

but AEB is equal to AMB, because they stand upon the same circumference: (III. 21.)

and the angle FLG is, for the same reason, equal to the angle FNG:

therefore also the angle AMB is equal to FNG:

and the right angle BAM is equal to the right angle GFN; (III. 31.)

wherefore the remaining angles in the triangles ABM, FGN are equal, and they are equiangular to one another:

therefore as BM to GN, so is BA to GF; (VI. 4.)

and therefore the duplicate ratio of BM to GN, is the same with the duplicate ratio of BA to GF: (V. def. 10. and V. 22.)

but the ratio of the square of BM to the square of GN, is the duplicate ratio of that which BM has to GN; (VI. 20.)

and the ratio of the polygon $ABCDE$ to the polygon $FGHKL$ is the duplicate of that which BA has to GF: (VI. 20.)

therefore as the polygon $ABCDE$ is to the polygon $FGHKL$, so is the square of BM to the square of GN.

Wherefore, similar polygons, &c. Q.E.D.

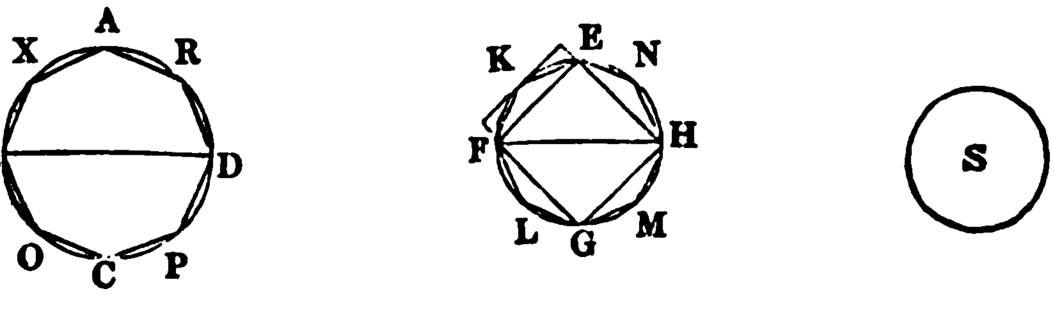
PROPOSITION II. THEOREM.

Circles are to one another as the squares of their diameters.

Let $ABCD$, $EFGH$ be two circles, and BD, FH their diameters.

As the square of BD to the square of FH, so shall the circle $ABCD$ be to the circle $EFGH$.

For, if it be not so, the square of BD must be to the square of FH, as the circle $ABCD$ is to some space either less than the circle $EFGH$, or greater than it.



First, if possible, let it be to a space *S* less than a circle *EFGH*;
and in the circle *EFGH* inscribe the square *EFGH*. (IV. 6.)
This square is greater than half of the circle *EFGH*;
because, if through the points *E, F, G, H*, there be drawn tangents to the circle,
the square *EFGH* is half of the square described about the circle: (I. 47.)
and the circle is less than the square described about it;
therefore the square *EFGH* is greater than half of the circle.

Divide the circumferences *EF, FG, GH, HE*, each into two equal parts in the points *K, L, M, N*, and join *EK, KF, FL, LG, GM, HM, HN, NE*;
therefore each of the triangles *EKF, FLG, GMH, HNE*, is greater than half of the segment of the circle in which it stands;
because, if straight lines touching the circle be drawn through the points *K, L, M, N*, and the parallelograms upon the straight lines *EF, FG, GH, HE* be completed,
each of the triangles *EKF, FLG, GMH, HNE* is the half of the parallelogram in which it is: (I. 41.)
but every segment is less than the parallelogram in which it is;
wherefore each of the triangles *EKF, FLG, GMH, HNE* is greater than half the segment of the circle which contains it.

Again, if the remaining circumferences be divided each into two equal parts, and their extremities be joined by straight lines, by continuing to do this, there will at length remain segments of the circle, which together are less than the excess of the circle *EFGH* above the space *S*;
because, by the preceding Lemma, if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than the least of the proposed magnitudes.

Let then the segments *EK, KF, FL, LG, GM, MH, HN, NE* be those that remain, and are together less than the excess of the circle *EFGH* above *S*:
therefore the rest of the circle, viz. the polygon *EKFLGMHN* is greater than the space *S*.

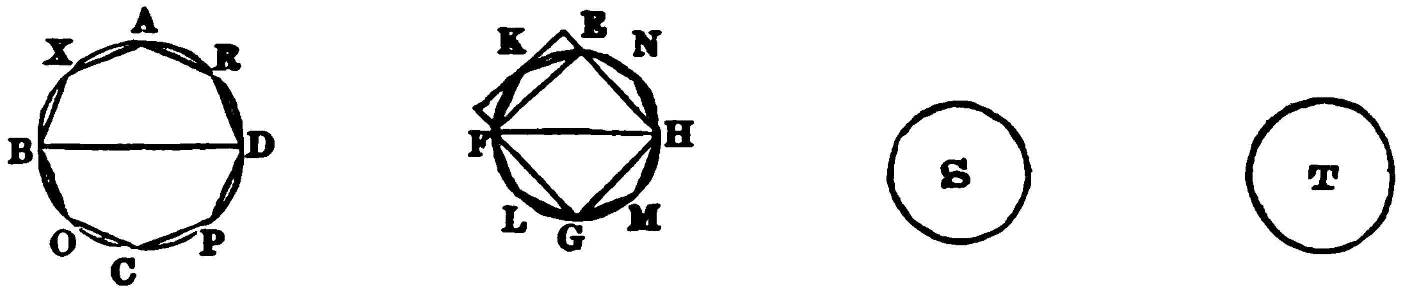
Describe likewise in the circle *ABCD* the polygon *AXBOCPDR* similar to the polygon *EKFLGMHN*:
as therefore the square of *BD* is to the square of *FH*, so is the polygon *AXBOCPDR* to the polygon *EKFLGMHN*: (XII. 1.)
but the square of *BD* is also to the square of *FH*, as the circle *ABCD* is to the space *S*; (hyp.)
therefore as the circle *ABCD* is to the space *S*, so is the polygon *AXBOCPDR* to the polygon *EKFLGMHN*: (V. 11.)
but the circle *ABCD* is greater than the polygon contained in it;
wherefore the space *S* is greater than the polygon *EKFLGMHN*: (V. 14.)
but it is likewise less, as has been demonstrated; which is impossible.

Therefore the square of *BD* is not to the square of *FH*, as the circle *ABCD* is to any space less than the circle *EFGH*.

In the same manner, it may be demonstrated, that neither is the square of *FH* to the square of *BD*, as the circle *EFGH* is to any space less than the circle *ABCD*.

Nor is the square of *BD* to the square of *FH*, as the circle *ABCD* is to any space greater than the circle *EFGH*.

For, if possible, let it be so to T, a space greater than the circle $EFGH$



therefore, inversely, as the square of FH to the square of BD, so is the space T to the circle $ABCD$;

but as the space T is to the circle $ABCD$, so is the circle $EFGH$ to some space, which must be less than the circle $ABCD$, (v. 14.)

because the space T is greater, by hypothesis, than the circle $EFGH$;

therefore as the square of FH is to the square of BD, so is the circle $EFGH$ to a space less than the circle $ABCD$, which has been demonstrated to be impossible;

therefore the square of BD is not to the square of FH as the circle $ABCD$ is to any space greater than the circle $EFGH$:

and it has been demonstrated, that neither is the square of BD to the square of FH, as the circle $ABCD$ to any space less than the circle $EFGH$:

wherefore, as the square of BD to the square of FH, so is the circle $ABCD$ to the circle $EFGH$.

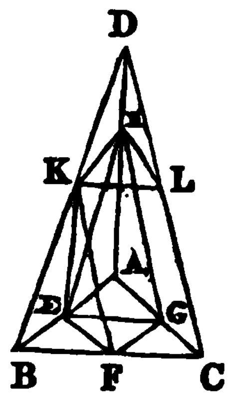
Circles, therefore, are, &c. Q. E. D.

PROPOSITION III. THEOREM.

Every pyramid having a triangular base, may be divided into two equal and similar pyramids having triangular bases, and which are similar to the whole pyramid; and into two equal prisms which together are greater than half of the whole pyramid.

Let there be a pyramid of which the base is the triangle ABC, and its vertex the point D.

The pyramid $ABCD$ may be divided into two equal and similar pyramids having triangular bases, and similar to the whole; and into two equal prisms which together shall be greater than half of the whole pyramid.



Divide AB, BC, CA, AD, DB, DC, each into two equal parts in the points E, F, G, H, K, L,

and join EH, EG, GH, HK, KL, LH, EK, KF, FG.

Because AE is equal to EB, and AH to HD,

HE is parallel to DB: (vi. 2.)

for the same reason, HK is parallel to AB:

herefore *HEBK* is a parallelogram, and *HK* equal to *EB*: (I. 34.)
but *EB* is equal to *AE*;
therefore also *AE* is equal to *HK*:
and *AH* is equal to *HD*;
wherefore *EA*, *AH* are equal to *KH*, *HD*, each to each;
and the angle *EAH* is equal to the angle *KHD*; (I. 29.)
therefore the base *EH* is equal to the base *KD*,
nd the triangle *AEH* equal and similar to the triangle *HKD*. (I. 4.)

For the same reason, the triangle *AGH* is equal and similar to the iangle *HLD*.

Again, because the two straight lines *EH*, *HG*, which meet one iother, are parallel to *KD*, *DL*, that meet one another, and are not in ie same plane with them,
they contain equal angles; (XI. 10.)
therefore the angle *EHG* is equal to the angle *KDL*;
and because *EH*, *HG* are equal to *KD*, *DL*, each to each,
and the angle *EHG* equal to the angle *KDL*;
therefore the base *EG* is equal to the base *KL*,
nd the triangle *EHG* equal and similar to the triangle *KDL*. (I. 4.)

For the same reason, the triangle *AEG* is also equal and similar to the triangle *HKL*.

Therefore the pyramid of which the base is the triangle *AEG*, and of hich the vertex is the point *H*, is equal and similar to the pyramid, he base of which is the triangle *KHL*, and vertex the point *D*. (XI. C.)

And because *HK* is parallel to *AB*, a side of the triangle *ADB*,
the triangle *ADB* is equiangular to the triangle *HDK*, and their sides are proportionals: (VI. 4.)
therefore the triangle *ADB* is similar to the triangle *HDK*:
nd for the same reason, the triangle *DBC* is similar to the triangle *DKL*;
and the triangle *ADC* to the triangle *HDL*;
and also the triangle *ABC* to the triangle *AEG*;
but the triangle *AEG* is similar to the triangle *HKL*, as before was proved;
therefore the triangle *ABC* is similar to the triangle *HKL*: (VI. 21.)
and therefore the pyramid of which the base is the triangle *ABC*, nd vertex the point *D*, is similar to the pyramid of which the base is ie triangle *HKL*, and vertex the same point *D*: (XI. B. & XI. def. 11.)
but the pyramid of which the base is the triangle *HKL*, and vertex ie point *D*, is similar, as has been proved, to the pyramid the base of hich is the triangle *AEG*, and vertex the point *H*;
wherefore the pyramid, the base of which is the triangle *ABC*, and ertex the point *D*, is similar to the pyramid of which the base is the riangle *AEG*, and vertex *H*:
therefore each of the pyramids *AEGH*, *HKLD* is similar to the whole pyramid *ABCD*.

And because *BF* is equal to *FC*,
the parallelogram *EBFG* is double of the triangle *GFC*: (I. 41.)
but when there are two prisms of the same altitude, of which one ias a parallelogram for its base, and the other a triangle that is half of he parallelogram,
these prisms are equal to one another; (XI. 40.)
therefore the prism having the parallelogram *EBFG* for its base, nd the straight line *KH* opposite to it, is equal to the prism having he triangle *GFC* for its base, and the triangle *HKL* opposite to it;

for they are of the same altitude, because they are between the parallel planes ABC, HKL: (XI. 15.)

and it is manifest that each of these prisms is greater than either of the pyramids of which the triangles AEG, HKL are the bases, and the vertices the points H, D;

because, if EF be joined, the prism having the parallelogram $EBFG$ for its base, and KH the straight line opposite to it, is greater than the pyramid of which the base is the triangle EBF, and vertex the point K:

but this pyramid is equal to the pyramid, the base of which is the triangle AEG, and vertex the point H; (XI. C.)

because they are contained by equal and similar planes:

wherefore the prism having the parallelogram $EBFG$ for its base, and opposite side KH, is greater than the pyramid of which the base is the triangle AEG, and vertex the point H:

and the prism of which the base is the parallelogram $EBFG$, and opposite side KH, is equal to the prism having the triangle GFC for its base, and HKL the triangle opposite to it;

and the pyramid of which the base is the triangle AEG, and vertex H, is equal to the pyramid of which the base is the triangle HKL, and vertex D:

therefore the two prisms before mentioned are greater than the two pyramids of which the bases are the triangles AEG, HKL, and vertices the points H, D.

Therefore the whole pyramid of which the base is the triangle ABC, and vertex the point D, is divided into two equal pyramids similar to one another, and to the whole pyramid; and into two equal prisms; and the two prisms are together greater than half of the whole pyramid.

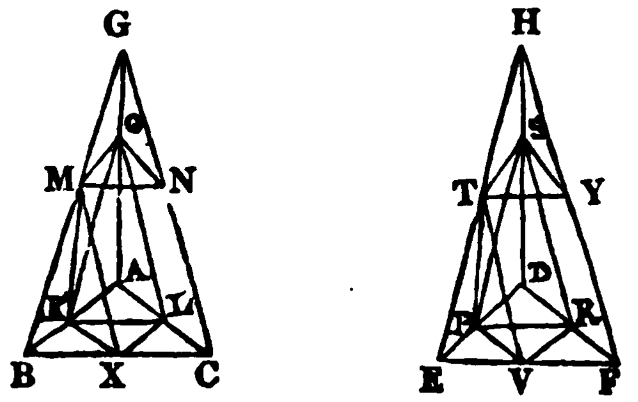
Q.E.D.

PROPOSITION IV. THEOREM.

If there be two pyramids of the same altitude, upon triangular bases, and each of them be divided into two equal pyramids similar to the whole pyramid, and also into two equal prisms; and if each of these pyramids be divided in the same manner as the first two, and so on: as the base of one of the first two pyramids is to the base of the other, so shall all the prisms in one of them be to all the prisms in the other, that are produced by the same number of divisions.

Let there be two pyramids of the same altitude upon the triangular bases ABC, DEF, and having their vertices in the points G, H; and let each of them be divided into two equal pyramids similar to the whole, and into two equal prisms; and let each of the pyramids thus made be conceived to be divided in the like manner, and so on.

As the base ABC is to the base DEF, so shall all the prisms in the pyramid $ABCG$ be to all the prisms in the pyramid $DEFH$ made by the same number of divisions.



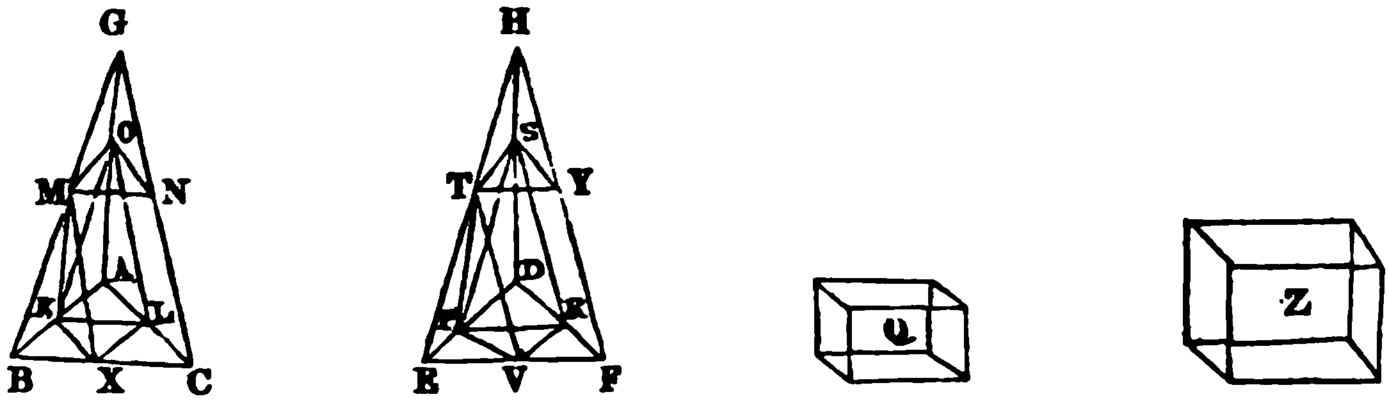
Make the same construction as in the foregoing proposition:
and because BX is equal to XC, and AL to LC,
therefore XL is parallel to AB, (VI. 2.)
and the triangle ABC similar to the triangle LXC.
For the same reason, the triangle DEF is similar to RVF.
And because BC is double of CX, and EF double of FV,
therefore BC is to CX, as EF to FV: (V. C.)
and upon BC, CX are described the similar and similarly situated ctilineal figures ABC, LXC;
and upon EF, FV, in like manner, are described the similar figures DEF, RFV:
therefore, as the triangle ABC is to the triangle LXC, so is the triangle DEF to the triangle RVF, (VI. 22.)
and, by permutation, as the triangle ABC to the triangle DEF, so is the triangle LXC to the triangle RVF.
And because the planes ABC, OMN, as also the planes DEF, STY, are parallel, (XI. 15.)
the perpendiculars drawn from the points G, H to the bases ABC, EF, which, by the hypothesis, are equal to one another, shall be cut ch into two equal parts by the planes OMN, STY, (XI. 17.)
because the straight lines GC, HF are cut into two equal parts in the points N, Y by the same planes:
herefore the prisms $LXCOMN$, $RVFSTY$ are of the same altitude;
and therefore, as the base LXC to the base RVF, that is, as the iangle ABC to the triangle DEF, so is the prism having the triangle XC for its base, and OMN the triangle opposite to it, to the prism of hich the base is the triangle RVF, and the opposite triangle STY: I. 32. Cor.)
and because the two prisms in the pyramid $ABCG$ are equal to one another,
nd also the two prisms in the pyramid $DEFH$ equal to one another;
as the prism of which the base is the parallelogram $KBXL$ and posite side MO, to the prism having the triangle LXC for its base, d OMN the triangle opposite to it; so is the prism of which the base the parallelogram $PEVR$, and opposite side TS, to the prism of iich the base is the triangle RVF, and opposite triangle STY: (V. 7.)
therefore, componendo, as the prisms $KBXLMO$, $LXCOMN$, tother, are to the prism $LXCOMN$, so are the prisms $PEVRTS$, $VFSTY$ to the prism $RVFSTY$;
and permutando, as the prisms $KBXLMO$, $LXCOMN$ are to the isms $PEVRTS$, $RVFSTY$, so is the prism $LXCOMN$ to the prism $VFSTY$:
but as the prism $LXCOMN$ to the prism $RVFSTY$, so is, as has been proved, the base ABC to the base DEF;
therefore, as the base ABC to the base DEF, so are the two prisms in the pyramid $ABCG$ to the two prisms in the pyramid $DEFH$:
and likewise if the pyramids now made, for example, the two $OMNG$, $STYH$, be divided in the same manner;
as the base OMN is to the base STY, so are the two prisms in the pyramid $OMNG$ to the two prisms in the pyramid $STYH$;
but the base OMN is to the base STY, as the base ABC to the base DEF;

therefore, as the base *ABC* to the base *DEF*, so are the two prisms in the pyramid *ABCG* to the two prisms in the pyramid *DEFH*;
and so are the two prisms in the pyramid *OMNG* to the two prisms in the pyramid *STYH*;
and so are all four to all four:
and the same thing may be shewn of the prisms made by dividing the pyramids *AKLO* and *DPRS*, and of all made by the same number of divisions. Q. E. D.

PROPOSITION V. THEOREM.

Pyramids of the same altitude which have triangular bases, are to one another as their bases.

Let the pyramids of which the triangles *ABC*, *DEF* are the bases, and of which the vertices are the points *G*, *H*, be of the same altitude.
As the base *ABC* to the base *DEF*, so shall the pyramid *ABCG* be to the pyramid *DEFH*.



For, if it be not so, the base *ABC* must be to the base *DEF*, as the pyramid *ABCG* to a solid either less than the pyramid *DEFH*, or greater than it.
First, if possible, let it be to a solid less than it, viz. to the solid *Q*:
and divide the pyramid *DEFH* into two equal pyramids, similar to the whole, and into two equal prisms;
therefore these two prisms are greater than the half of the whole pyramid. (XII. 3.)
And again, let the pyramids made by this division be in like manner divided, and so on until the pyramids which remain undivided in the pyramid *DEFH* be, all of them together, less than the excess of the pyramid *DEFH* above the solid *Q*: (XII. Lem. 1.)
let these, for example, be the pyramids *DPRS*, *STYH*:
therefore the prisms, which make the rest of the pyramid *DEFH*, are greater than the solid *Q*.
Divide likewise the pyramid *ABCG* in the same manner, and into as many parts, as the pyramid *DEFH*.
Therefore, as the base *ABC* to the base *DEF*, so are the prisms in the pyramid *ABCG* to the prisms in the pyramid *DEFH*: (XII. 4.)
but as the base *ABC* to the base *DEF*, so, by hypothesis, is the pyramid *ABCG* to the solid *Q*;
and therefore, as the pyramid *ABCG* to the solid *Q*, so are the prisms in the pyramid *ABCG* to the prisms in the pyramid *DEFH*;
but the pyramid *ABCG* is greater than the prisms contained in it;
wherefore also the solid *Q* is greater than the prisms in the pyramid *DEFH*; (V. 14.)

but it is also less, which is impossible.

Therefore the base ABC is not to the base DEF, as the pyramid $ABCG$ to any solid which is less than the pyramid $DEFH$.

In the same manner it may be demonstrated, that the base DEF is not to the base ABC, as the pyramid $DEFH$ to any solid which is less than the pyramid $ABCG$.

Nor can the base ABC be to the base DEF, as the pyramid $ABCG$ to any solid which is greater than the pyramid $DEFH$.

For, if it be possible, let it be so to a greater, viz. the solid Z.

And because the base ABC is to the base DEF as the pyramid $ABCG$ to the solid Z;

by inversion, as the base DEF to the base ABC, so is the solid Z to the pyramid $ABCG$:

but as the solid Z is to the pyramid $ABCG$, so is the pyramid $DEFH$ to some solid, which must be less than the pyramid $ABCG$, (v. 14.)

because the solid Z is greater than the pyramid $DEFH$;

and therefore, as the base DEF to the base ABC, so is the pyramid $DEFH$ to a solid less than the pyramid $ABCG$;

the contrary to which has been proved:

therefore the base ABC is not to the base DEF, as the pyramid $ABCG$ to any solid which is greater than the pyramid $DEFH$.

And it has been proved, that neither is the base ABC to the base DEF, as the pyramid $ABCG$ to any solid which is less than the pyramid $DEFH$.

Therefore, as the base ABC is to the base DEF, so is the pyramid $ABCG$ to the pyramid $DEFH$.

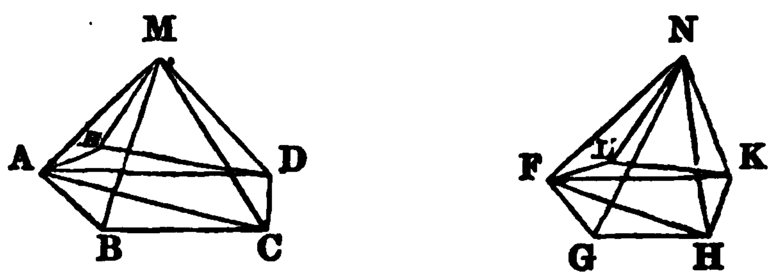
Wherefore, pyramids, &c. Q.E.D.

PROPOSITION VI. THEOREM.

Pyramids of the same altitude which have polygons for their bases, are to one another as their bases.

Let the pyramids which have the polygons $ABCDE$, $FGHKL$ for their bases, and their vertices in the points M, N, be of the same altitude.

As the base $ABCDE$ is to the base $FGHKL$, so shall the pyramid $ABCDEM$ be to the pyramid $FGHKLN$.



Divide the base $ABCDE$ into the triangles ABC, ACD, ADE,

and the base $FGHKL$ into the triangles FGH, FHK, FKL;

and upon the bases ABC, ACD, ADE let there be as many pyramids of which the common vertex is the point M,

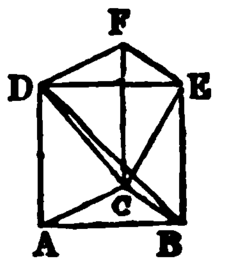
and upon the remaining bases as many pyramids having their common vertex in the point N.

Therefore, since the triangle *ABC* is to the triangle *FGH*, as the pyramid *ABCM* to the pyramid *FGHN*; (XII. 5.)
and the triangle *ACD* to the triangle *FGH*, as the pyramid *ACDM* to the pyramid *FGHN*;
and also the triangle *ADE* to the triangle *FGH*, as the pyramid *ADEM* to the pyramid *FGHN*;
as all the first antecedents to their common consequent, so are all the other antecedents to their common consequent; (V. 24. Cor. 2.)
that is, as the base *ABCDE* to the base *FGH*, so is the pyramid *ABCDEM* to the pyramid *FGHN*:
and for the same reason, as the base *FGHKL* to the base *FGH*, so is the pyramid *FGHKLN* to the pyramid *FGHN*:
and, by inversion, as the base *FGH* to the base *FGHKL*, so is the pyramid *FGHN* to the pyramid *FGHKLN*:
then, because, as the base *ABCDE* to the base *FGH*, so is the pyramid *ABCDEM* to the pyramid *FGHN*;
and as the base *FGH* to the base *FGHKL*, so is the pyramid *FGHN* to the pyramid *FGHKLN*;
therefore, ex æquali, as the base *ABCDE* to the base *FGHKL*, so the pyramid *ABCDEM* to the pyramid *FGHKLN*. (V. 22.)
Therefore, pyramids, &c. Q.E.D.

PROPOSITION VII. THEOREM.

Every prism having a triangular base may be divided into three pyramids that have triangular bases, and are equal to one another.

Let there be a prism of which the base is the triangle *ABC*, and *DEF* the triangle opposite to it.
The prism *ABCDEF* may be divided into three equal pyramids having triangular bases.



Join *BD*, *EC*, *CD*.
And because *ABED* is a parallelogram of which *BD* is the diameter,
the triangle *ABD* is equal to the triangle *EBD*; (I. 34.)
therefore the pyramid of which the base is the triangle *ABD*, and vertex the point *C*, is equal to the pyramid of which the base is the triangle *EBD*, and vertex the point *C*: (XII. 5.)
but this pyramid is the same with the pyramid the base of which is the triangle *EBC*, and vertex the point *D*;
for they are contained by the same planes:
therefore the pyramid of which the base is the triangle *ABD*, and vertex the point *C*, is equal to the pyramid, the base of which is the triangle *EBC*, and vertex the point *D*.
Again, because *FCBE* is a parallelogram of which the diameter is *CE*,
the triangle *ECF* is equal to the triangle *ECB*; (I. 34.)
therefore the pyramid of which the base is the triangle *ECB*, and

vertex the point D, is equal to the pyramid, the base of which is the triangle ECF, and vertex the point D:

but the pyramid of which the base is the triangle ECB, and vertex the point D, has been proved equal to the pyramid of which the base is the triangle ABD, and vertex the point C:

therefore the prism $ABCDEF$ is divided into three equal pyramids having triangular bases, viz. into the pyramids $ABDC$, $EBDC$, $ECFD$.

And because the pyramid of which the base is the triangle ABD, and vertex the point C, is the same with the pyramid of which the base is the triangle ABC, and vertex the point D,

for they are contained by the same planes;

and that the pyramid of which the base is the triangle ABD, and vertex the point C, has been demonstrated to be a third part of the prism, the base of which is the triangle ABC, and DEF the opposite triangle;

therefore the pyramid of which the base is the triangle ABC, and vertex the point D, is the third part of the prism which has the same base, viz. the triangle ABC, and DEF its opposite triangle. Q.E.D.

Cor. 1. From this it is manifest, that every pyramid is the third part of a prism which has the same base, and is of an equal altitude with it: for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

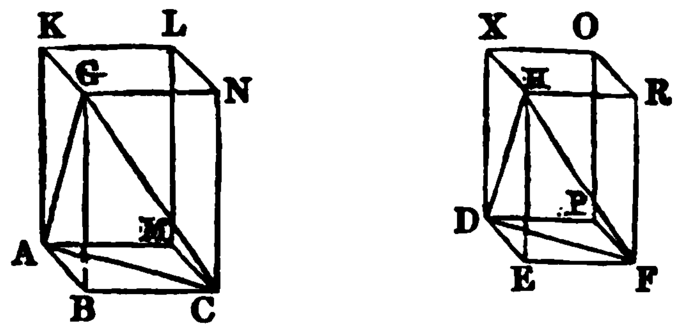
Cor. 2. Prisms of equal altitudes are to one another as their bases; because the pyramids upon the same bases, and of the same altitude, are to one another as their bases. (XII. 6.)

PROPOSITION VIII. THEOREM.

Similar pyramids, having triangular bases, are one to another in the triplicate ratio of that of their homologous sides.

Let the pyramids having the triangles ABC, DEF for their bases, and the points G, H for their vertices, be similar and similarly situated.

The pyramid $ABCG$ shall have to the pyramid $DEFH$, the triplicate ratio of that which the side BC has to the homologous side EF.



Complete the parallelograms $ABCM$, $GBCN$, $ABGK$, and the solid parallelopiped $BGML$, contained by these planes and those opposite to them:

and, in like manner, complete the solid parallelopiped $EHPO$ contained by the three parallelograms $DEFP$, $HEFR$, $DEHX$, and those opposite to them.

And because the pyramid $ABCG$ is similar to the pyramid $DEFH$,

the angle ABC is equal to the angle DEF, (XI. def. 11.)

and the angle GBC to the angle HEF, and ABG to DEH:

and AB is to BC, as DE to EF; (VI. def. 1.)

that is, the sides about the equal angles are proportionals:
wherefore the parallelogram *BM* is similar to *EP*:
for the same reason, the parallelogram *BN* is similar to *ER*, and *BK* to *EX*:
therefore the three parallelograms *BM, BN, BK* are similar to the three *EP, ER, EX*:
but the three *BM, BN, BK* are equal and similar to the three which are opposite to them, (XI. 24.)
and the three *EP, ER, EX* equal and similar to the three opposite to them:
wherefore the solids *BGML, EHPO* are contained by the same number of similar planes:
and their solid angles are equal; (XI. B.)
and therefore the solid *BGML* is similar to the solid *EHPO*: (XI. def. 11.)
but similar solid parallelopipeds have the triplicate ratio of that which their homologous sides have; (XI. 33.)
therefore the solid *BGML* has to the solid *EHPO* the triplicate ratio of that which the side *BC* has to the homologous side *EF*:
but as the solid *BGML* is to the solid *EHPO*, so is the pyramid *ABCG* to the pyramid *DEFH*; (V. 15.)
because the pyramids are the sixth part of the solids, since the prism, which is the half of the solid parallelopiped, (XI. 28.) is triple of the pyramid: (XII. 7.)
wherefore, likewise the pyramid *ABCG* has to the pyramid *DEFH*, the triplicate ratio of that which *BC* has to the homologous side *EF*. Q.E.D.

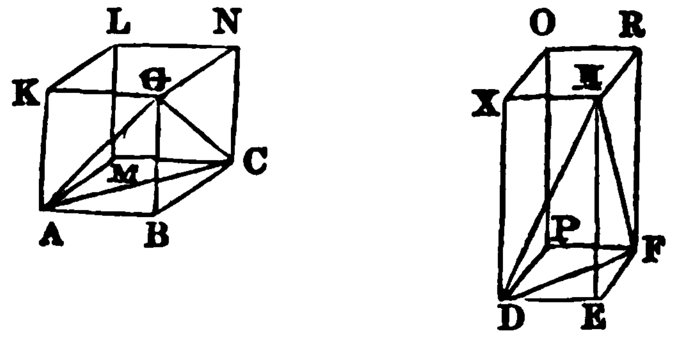
COR. From this it is evident, that similar pyramids which have multangular bases, are likewise to one another in the triplicate ratio of their homologous sides. For they may be divided into similar pyramids having triangular bases, because the similar polygons, which are their bases, may be divided into the same number of similar triangles homologous to the whole polygons: therefore, as one of the triangular pyramids in the first multangular pyramid is to one of the triangular pyramids in the other, (V. 12.) so are all the triangular pyramids in the first to all the triangular pyramids in the other; that is, so is the first multangular pyramid to the other: but one triangular pyramid is to its similar triangular pyramid, in the triplicate ratio of their homologous sides; and therefore the first multangular pyramid has to the other, the triplicate ratio of that which one of the sides of the first has to the homologous side of the other.

PROPOSITION IX. THEOREM.

The bases and altitudes of equal pyramids having triangular bases are reciprocally proportional: and conversely, triangular pyramids, of which the bases and altitudes are reciprocally proportional, are equal to one another.

Let the pyramids of which the triangles *ABC, DEF* are the bases, and which have their vertices in the points *G, H*, be equal to one another.

The bases and altitudes of the pyramids *ABCG, DEFH* shall be reciprocally proportional,
viz. the base *ABC* shall be to the base *DEF*, as the altitude of the pyramid *DEFH* to the altitude of the pyramid *ABCG*.



Complete the parallelograms *AC, AG, GC, DF, DH, HF*;
and the solid parallelopipeds *BGML, EHPO* contained by these planes and those which are opposite to them.
And because the pyramid *ABCG* is equal to the pyramid *DEFH*,
and that the solid *BGML* is sextuple of the pyramid *ABCG*, (XI. 28. and XII. 7.)
and the solid *EHPO* sextuple of the pyramid *DEFH*;
therefore the solid *BGML* is equal to the solid *EHPO*: (V. ax. 1.)
but the bases and altitudes of equal solid parallelopipeds are reciprocally proportional; (XI. 34.)
therefore as the base *BM* to the base *EP*, so is the altitude of the solid *EHPO* to the altitude of the solid *BGML*:
but as the base *BM* to the base *EP*, so is the triangle *ABC* to the triangle *DEF*; (V. 15.)
therefore as the triangle *ABC* to the triangle *DEF*, so is the altitude of the solid *EHPO* to the altitude of the solid *BGML*:
but the altitude of the solid *EHPO* is the same with the altitude of the pyramid *DEFH*;
and the altitude of the solid *BGML* is the same with the altitude of the pyramid *ABCG*;
therefore, as the base *ABC* to the base *DEF*, so is the altitude of the pyramid *DEFH* to the altitude of the pyramid *ABCG*:
wherefore, the bases and altitudes of the pyramids *ABCG, DEFH* are reciprocally proportional.
Again, let the bases and altitudes of the pyramids *ABCG, DEFH* be reciprocally proportional,
viz. the base *ABC* to the base *DEF*, as the altitude of the pyramid *DEFH* to the altitude of the pyramid *ABCG*.
The pyramid *ABCG* shall be equal to the pyramid *DEFH*.
The same construction being made,
because as the base *ABC* to the base *DEF*, so is the altitude of the pyramid *DEFH* to the altitude of the pyramid *ABCG*;
and as the base *ABC* to the base *DEF*, so is the parallelogram *BM* to the parallelogram *EP*:
therefore the parallelogram *BM* is to *EP*, as the altitude of the pyramid *DEFH* to the altitude of the pyramid *ABCG*:
but the altitude of the pyramid *DEFH* is the same with the altitude of the solid parallelopiped *EHPO*;
and the altitude of the pyramid *ABCG* is the same with the altitude of the solid parallelopiped *BGML*:
therefore, as the base *BM* to the base *EP*, so is the altitude of the

solid parallelopiped *EHPO* to the altitude of the solid parallelopiped *BGML*.

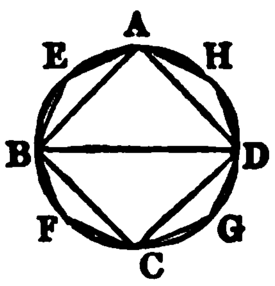
But solid parallelopipeds having their bases and altitudes reciprocally proportional, are equal to one another; (XI. 34.)
therefore the solid parallelopiped *BGML* is equal to the solid parallelopiped *EHPO*:
and the pyramid *ABCG* is the sixth part of the solid *BGML*,
and the pyramid *DEFH* is the sixth part of the solid *EHPO*;
therefore the pyramid *ABCG* is equal to the pyramid *DEFH*. (V. ax. 2.)
Therefore the bases, &c. Q.E.D.

PROPOSITION X. THEOREM.

Every cone is the third part of a cylinder which has the same base and is of an equal altitude with it.

Let a cone have the same base with a cylinder, viz. the circle *ABCD*, and the same altitude.

The cone shall be the third part of the cylinder; that is, the cylinder shall be triple of the cone.
If the cylinder be not triple of the cone,
it must either be greater than the triple, or less than it.
First, if possible, let it be greater than the triple;



and inscribe the square *ABCD* in the circle:
this square is greater than the half of the circle *ABCD*.
Upon the square *ABCD* erect a prism of the same altitude with the cylinder;
this prism shall be greater than half of the cylinder:
for let a square be described about the circle, and let a prism be erected upon the square, of the same altitude with the cylinder:
then the inscribed square is half of that circumscribed;
and upon these square bases are erected solid parallelopipeds, viz. the prisms of the same altitude;
therefore the prism upon the square *ABCD* is the half of the prism upon the square described about the circle;
because they are to one another as their bases: (XI. 32.)
and the cylinder is less than the prism upon the square described about the circle *ABCD*:
therefore the prism upon the square *ABCD* of the same altitude with the cylinder, is greater than half of the cylinder.
Bisect the circumferences *AB*, *BC*, *CD*, *DA*, in the points *E*, *F*, *G*, *H*;
and join *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, *HA*:
then, each of the triangles *AEB*, *BFC*, *CGD*, *DHA* is greater than the half of the segment of the circle in which it stands, as was shewn in Prop. II. of this book.

Erect prisms upon each of these triangles, of the same altitude with the cylinder;

each of these prisms shall be greater than half of the segment of the cylinder in which it is;

because if through the points *E, F, G, H,* parallels be drawn to *AB, BC, CD, DA,*

and parallelograms be completed upon the same *AB, BC, CD, DA,*

and solid parallelopipeds be erected upon the parallelograms;

the prisms upon the triangles *AEB, BFC, CGD, DHA,* are the halves of the solid parallelopipeds; (XII. 7. Cor. 2.)

and the segments of the cylinder which are upon the segments of the circle cut off by *AB, BC, CD, DA,* are less than the solid parallelopipeds which contain them;

therefore the prisms upon the triangles *AEB, BFC, CGD, DHA,* are greater than half of the segments of the cylinder in which they are:

therefore, if each of the circumferences be divided into two equal parts, and straight lines be drawn from the points of division to the extremities of the circumferences, and upon the triangles thus made prisms be erected of the same altitude with the cylinder, and so on, there must at length remain some segments of the cylinder which together are less than the excess of the cylinder above the triple of the cone: (XII. Lem. 1.)

let them be those upon the segments of the circle *AE, EB, BF, FC, CG, GD, DH, HA*;

therefore the rest of the cylinder, that is, the prism of which the base is the polygon *AEBFCGDH,* and of which the altitude is the same with that of the cylinder, is greater than the triple of the cone:

but this prism is triple of the pyramid upon the same base, of which the vertex is the same with the vertex of the cone; (XII. 7. Cor. 1.)

therefore the pyramid upon the base *AEBFCGDH,* having the same vertex with the cone, is greater than the cone, of which the base is the circle *ABCD*:

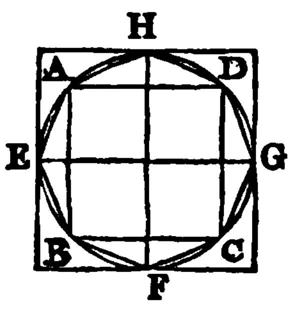
but it is also less, for the pyramid is contained within the cone; which is impossible:

therefore the cylinder is not greater than the triple of the cone.

Nor can the cylinder be less than the triple of the cone.

Let it be less, if possible;

therefore, inversely, the cone is greater than the third part of the cylinder.



In the circle *ABCD* inscribe a square:

this square is greater than the half of the circle:

and upon the square *ABCD* erect a pyramid, having the same vertex with the cone;

this pyramid is greater than the half of the cone;

because, as was before demonstrated, if a square be described about the circle, the square *ABCD* is the half of it:

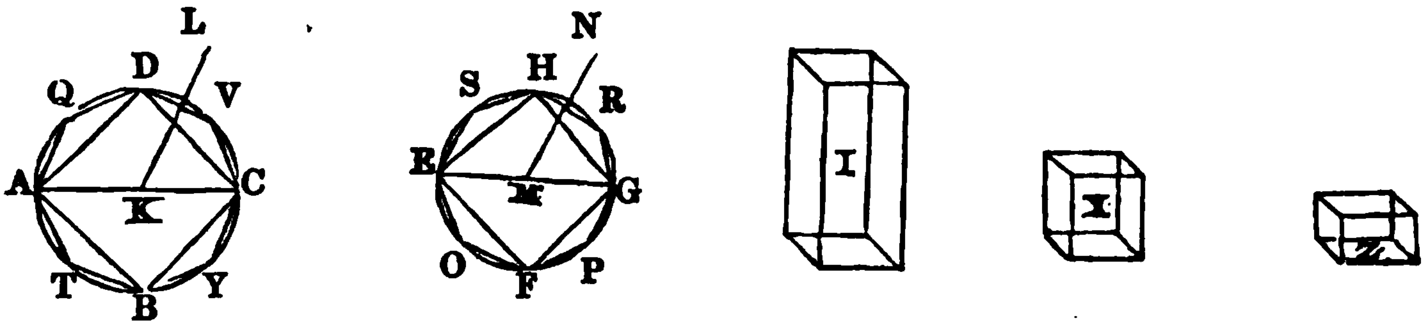
and if upon these squares there be erected solid parallelopipeds of the same altitude with the cone, which are also prisms,
the prism upon the square *ABCD* is the half of that which is upon the square described about the circle;
for they are to one another as their bases; as are also the third parts of them: (XI. 32.)
therefore the pyramid, the base of which is the square *ABCD*, is half of the pyramid upon the square described about the circle:
but this last pyramid is greater than the cone which it contains;
therefore the pyramid upon the square *ABCD*, having the same vertex with the cone, is greater than the half of the cone.
Bisect the circumferences *AB, BC, CD, DA* in the points *E, F, G, H*, and join *AE, EB, BF, FC, CG, GD, DH, HA*:
therefore each of the triangles *AEB, BFC, CGD, DHA* is greater than half of the segment of the circle in which it is:
upon each of these triangles erect pyramids having the same vertex with the cone:
therefore each of these pyramids is greater than the half of the segment of the cone in which it is, as before was demonstrated of the prisms and segments of the cylinder:
and thus dividing each of the circumferences into two equal parts, and joining the points of division and their extremities by straight lines, and upon the triangles erecting pyramids having their vertices the same with that of the cone, and so on, there must at length remain some segments of the cone, which together are less than the excess of the cone above the third part of the cylinder: (XII. Lem. 1.)
let these be the segments upon *AE, EB, BF, FC, CG, GD, DH, HA*:
therefore the rest of the cone, that is, the pyramid of which the base is the polygon *AEBFCGDH*, and of which the vertex is the same with that of the cone, is greater than the third part of the cylinder:
but this pyramid is the third part of the prism upon the same base *AEBFCGDH*, and of the same altitude with the cylinder;
therefore this prism is greater than the cylinder of which the base is the circle *ABCD*:
but it is also less, for it is contained within the cylinder; which is impossible:
therefore the cylinder is not less than the triple of the cone.
And it has been demonstrated that neither is it greater than the triple:
therefore the cylinder is triple of the cone, or, the cone is the third part of the cylinder.
Wherefore, every cone, &c. Q.E.D.

PROPOSITION XI. THEOREM.

Cones and cylinders of the same altitude, are to one another as their bases.

Let the cones and cylinders, of which the bases are the circles *ABCD, EFGH*, and the axes *KL, MN*, and *AC, EG* the diameters of their bases, be of the same altitude.

As the circle *ABCD* to the circle *EFGH*, so shall the cone *AL* be to the cone *EN*.



If it be not so, the circle $ABCD$ must be to the circle $EFGH$, as the cone AL to some solid either less than the cone EN, or greater than it.

First, let it be to a solid less than EN, viz. to the solid X;
and let Z be the solid which is equal to the excess of the cone EN above the solid X;
therefore the cone EN is equal to the solids X, Z together.
In the circle $EFGH$ inscribe the square $EFGH$;
therefore this square is greater than the half of the circle:
upon the square $EFGH$ erect a pyramid of the same altitude with the cone;
this pyramid shall be greater than half of the cone:
for, if a square be described about the circle, and a pyramid be erected upon it, having the same vertex with the cone, the pyramid inscribed in the cone is half of the pyramid circumscribed about it,
because they are to one another as their bases: (XII. 6.)
but the cone is less than the circumscribed pyramid;
therefore the pyramid of which the base is the square $EFGH$, and its vertex the same with that of the cone, is greater than half of the cone.

Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points O, P, R, S,
and join EO, OF, FP, PG, GR, RH, HS, SE:
therefore each of the triangles EOF, FPG, GRH, HSE is greater than half of the segment of the circle in which it is:
upon each of these triangles erect a pyramid having the same vertex with the cone;
each of these pyramids is greater than the half of the segment of the cone in which it is:
and thus dividing each of these circumferences into two equal parts, and from the points of division drawing straight lines to the extremities of the circumferences, and upon each of the triangles thus made erecting pyramids having the same vertex with the cone, and so on, there must at length remain some segments of the cone which are together less than the solid Z; (Lemma.)
let these be the segments upon EO, OF, FP, PG, GR, RH, HS, SE:
therefore the remainder of the cone, viz. the pyramid of which the base is the polygon $EOFPGRHS$, and its vertex the same with that of the cone, is greater than the solid X.

In the circle $ABCD$ inscribe the polygon $ATBYCVDQ$ similar to the polygon $EOFPGRHS$, and upon it erect a pyramid having the same vertex with the cone AL:
and because as the square of AC is to the square of EG, so is the polygon $ATBYCVDQ$ to the polygon $EOFPGRHS$; (XII. 1.)
and as the square of AC to the square of EG, so is the circle $ABCD$ to the circle $EFGH$; (XII. 2.)
therefore the circle $ABCD$ is to the circle $EFGH$, as the polygon $ATBYCVDQ$ to the polygon $EOFPGRHS$: (V. 11.)

but as the circle *ABCD* to the circle *EFGH*, so is the cone *AL* to the solid *X*;

and as the polygon *ATBYCVDQ* to the polygon *EOFPGRHS*, so is the pyramid of which the base is the first of these polygons, and vertex *L*, to the pyramid of which the base is the other polygon, and its vertex *N*: (XII. 6.)

therefore, as the cone *AL* to the solid *X*, so is the pyramid of which the base is the polygon *ATBYCVDQ*, and vertex *L*, to the pyramid the base of which is the polygon *EOFPGRHS*, and vertex *N*:

but the cone *AL* is greater than the pyramid contained in it;

therefore the solid *X* is greater than the pyramid in the cone *EN*: (V. 14.)

but it is less, as was shewn; which is absurd:

therefore the circle *ABCD* is not to the circle *EFGH*, as the cone *AL* to any solid which is less than the cone *EN*.

In the same manner it may be demonstrated, that the circle *EFGH* is not to the circle *ABCD*, as the cone *EN* to any solid less than the cone *AL*.

Nor can the circle *ABCD* be to the circle *EFGH*, as the cone *AL* to any solid greater than the cone *EN*.

For, if it be possible, let it be so to the solid *I*, which is greater than the cone *EN*:

therefore, by inversion, as the circle *EFGH* to the circle *ABCD*, so is the solid *I* to the cone *AL*:

but as the solid *I* to the cone *AL*, so is the cone *EN* to some solid, which must be less than the cone *AL*; (V. 14.)

because the solid *I* is greater than the cone *EN*;

therefore, as the circle *EFGH* is to the circle *ABCD*, so is the cone *EN* to a solid less than the cone *AL*, which was shewn to be impossible:

therefore the circle *ABCD* is not to the circle *EFGH*, as the cone *AL* is to any solid greater than the cone *EN*.

And it has been demonstrated, that neither is the circle *ABCD* to the circle *EFGH*, as the cone *AL* to any solid less than the cone *EN*:

therefore the circle *ABCD* is to the circle *EFGH*, as the cone *AL* to the cone *EN*:

but as the cone is to the cone, so is the cylinder to the cylinder, (V. 15.)

because the cylinders are triple of the cones, each of each, (XII. 10.)

therefore as the circle *ABCD* to the circle *EFGH*, so are the cylinders upon them of the same altitude.

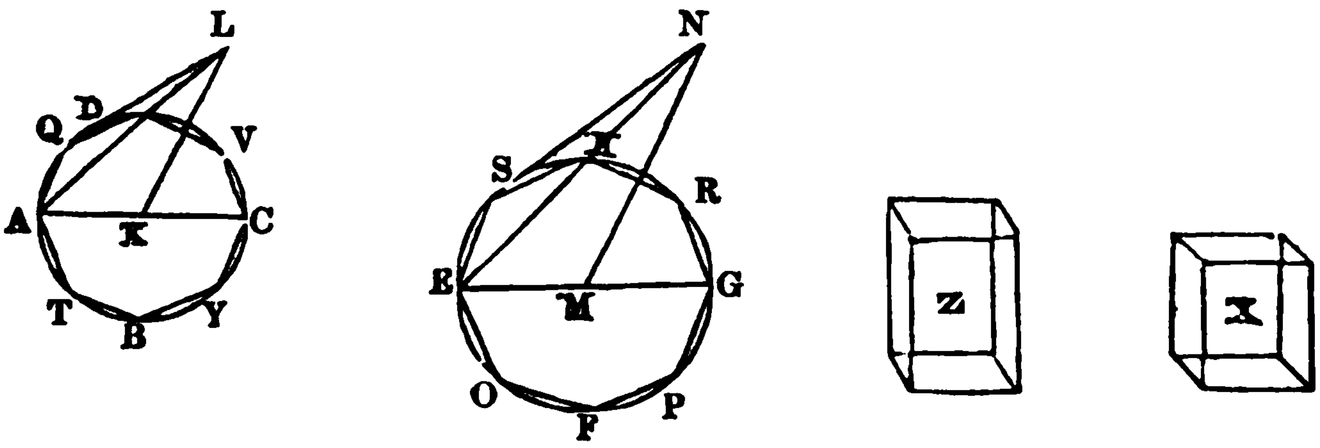
Wherefore, cones and cylinders of the same altitude are to one another as their bases. Q.E.D.

PROPOSITION XII. THEOREM.

Similar cones and cylinders have to one another the triplicate ratio of that which the diameters of their bases have.

Let the cones and cylinders of which the bases are the circles *ABCD*, *EFGH*, and the diameters of the bases *AC*, *EG*, and *KL*, *MN* the axes of the cones or cylinders, be similar.

The cone of which the base is the circle *ABCD* and vertex the point *L*, shall have to the cone of which the base is the circle *EFGH* and vertex *N*, the triplicate ratio of that which *AC* has to *EG*.



For, if the cone $ABCDL$ has not to the cone $EFGHN$ the triplicate ratio of that which AC has to EG,

the cone $ABCDL$ must have the triplicate of that ratio to some solid which is less or greater than the cone $EFGHN$.

First, if possible, let it have it to a less, viz. to the solid X.

Make the same construction as in the preceding proposition, and it may be demonstrated the very same way as in that proposition, that the pyramid of which the base is the polygon $EOFPGRHS$, and vertex N, is greater than the solid X.

Inscribe also in the circle $ABCD$ the polygon $ATBYCVDQ$ similar to the polygon $EOFPGRHS$, upon which erect a pyramid having the same vertex with the cone;

and let LAQ be one of the triangles containing the pyramid upon the polygon $ATBYCVDQ$, the vertex of which is L;

and let NES be one of the triangles containing the pyramid upon the polygon $EOFPGRHS$ of which the vertex is N, and join KQ, MS.

Then, because the cone $ABCDL$ is similar to the cone $EFGHN$, AC is to EG as the axis KL to the axis MN; (XI. def. 24.)

and as AC to EG, so is AK to EM; (V. 15.)

therefore as AK to EM, so is KL to MN;

and alternately, AK to KL, as EM to MN:

and the right angles AKL, EMN are equal:

therefore, the sides about these equal angles being proportionals, the triangle AKL is similar to the triangle EMN. (VI. 6.)

Again, because AK is to KQ, as EM to MS, and that these sides are about equal angles AKQ, EMS, because these angles are, each of them, the same part of four right angles at the centres K, M;

therefore the triangle AKQ is similar to the triangle EMS. (VI. 6.)

And because it has been shewn that as AK to KL, so is EM to MN, and that AK is equal to KQ, and EM to MS:

therefore as QK to KL, so is SM to MN:

and therefore, the sides about the right angles QKL, SMN being proportionals,

the triangle LKQ is similar to the triangle NMS.

And because of the similarity of the triangles AKL, EMN, as LA is to AK, so is NE to EM;

and by the similarity of the triangles AKQ, EMS, as KA to AQ, so ME to ES:

therefore, ex æquali, LA is to AQ, as NE to ES. (V. 22.)

Again, because of the similarity of the triangles LQK, NSM, as LQ to QK, so NS to SM;

and from the similarity of the triangles KAQ, MES,

as KQ to QA, so MS to SE:
therefore, ex æquali, LQ is to QA, as NS to SE: (v. 22.)
and it was proved that QA is to AL, as SE to EN:
therefore, again, ex æquali, as QL to LA, so is SN to NE:
wherefore the triangles LQA, NSE, having the sides about all their angles proportionals, are equiangular and similar to one another: (vi. 5.)
and therefore the pyramid of which the base is the triangle AKQ, and vertex L, is similar to the pyramid the base of which is the triangle EMS, and vertex N,
because their solid angles are equal to one another, and they are contained by the same number of similar planes: (xi. B.)
but similar pyramids which have triangular bases have to one another the triplicate ratio of that which their homologous sides have; (xii. 8.)
therefore the pyramid $AKQL$ has to the pyramid $EMSN$ the triplicate ratio of that which AK has to EM.

In the same manner, if straight lines be drawn from the points D, V, C, Y, B, T to K,
and from the points H, R, G, P, F, O to M,
and pyramids be erected upon the triangles having the same vertices with the cones,
it may be demonstrated that each pyramid in the first cone has to each in the other, taking them in the same order, the triplicate ratio of that which the side AK has to the side EN;
that is, which AC has to EG:
but as one antecedent to its consequent, so are all the antecedents to all the consequents; (v. 12.)
therefore as the pyramid $AKQL$ to the pyramid $EMSN$, so is the whole pyramid the base of which is the polygon $DQATBYCV$, and vertex L, to the whole pyramid of which the base is the polygon $HSEOFPGR$, and vertex N:
wherefore also the first of these two last-named pyramids has to the other the triplicate ratio of that which AC has to EG:
but, by the hypothesis, the cone of which the base is the circle $ABCD$, and vertex L, has to the solid X, the triplicate ratio of that which AC has to EG;
therefore, as the cone of which the base is the circle $ABCD$, and vertex L, is to the solid X, so is the pyramid the base of which is the polygon $DQATBYCV$, and vertex L, to the pyramid the base of which is the polygon $HSEOFPGR$, and vertex N:
but the said cone is greater than the pyramid contained in it;
therefore the solid X is greater than the pyramid, the base of which is the polygon $HSEOFPGR$, and vertex N: (v. 14.)
but it is also less; which is impossible:
therefore the cone, of which the base is the circle $ABCD$ and vertex L, has not to any solid which is less than the cone of which the base is the circle $EFGH$, and vertex N, the triplicate ratio of that which AC has to EG.

In the same manner it may be demonstrated, that neither has the cone $EFGHN$ to any solid which is less than the cone $ABCDL$, the triplicate ratio of that which EG has to AC.

Nor can the cone $ABCDL$ have to any solid which is greater than the cone $EFGHN$, the triplicate ratio of that which AC has to EG.

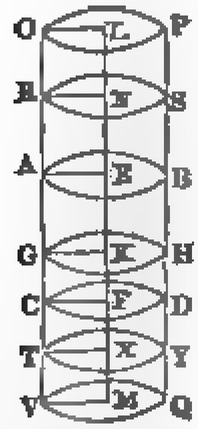
For, if it be possible, let it have it to a greater, viz. to the solid Z:
therefore, inversely, the solid Z has to the cone $ABCDL$, the triplicate ratio of that which EG has to AC:
but as the solid Z is to the cone $ABCDL$, so is the cone $EFGHN$ to some solid, which must be less than the cone $ABCDL$, (v. 14.) because the solid Z is greater than the cone $EFGHN$;
therefore the cone $EFGHN$ has to a solid which is less than the cone $ABCDL$, the triplicate ratio of that which EG has to AC, which was demonstrated to be impossible:
therefore the cone $ABCDL$ has not to any solid greater than the cone $EFGHN$, the triplicate ratio of that which AC has to EG:
and it was demonstrated, that it could not have that ratio to any solid less than the cone $EFGHN$;
therefore the cone $ABCDL$ has to the cone $EFGHN$, the triplicate ratio of that which AC has to EG:
it as the cone is to the cone, so the cylinder to the cylinder; (v. 15.)
for every cone is the third part of the cylinder upon the same base, and of the same altitude: (XII. 10.)
therefore also the cylinder has to the cylinder, the triplicate ratio of that which AC has to EG.

Wherefore, similar cones, &c. Q. E. D.

PROPOSITION XIII. THEOREM.

If a cylinder be cut by a plane parallel to its opposite planes, or bases; divides the cylinder into two cylinders, one of which is to the other as e axis of the first to the axis of the other.

Let the cylinder AD be cut by the plane GH parallel to the oppote planes AB, CD, meeting the axis EF in the point K,
and let the line GH be the common section of the plane GH and the surface of the cylinder AD.
Let $AEFC$ be the parallelogram in any position of it, by the revoition of which about the straight line EF the cylinder AD is described;
and let GK be the common section of the plane GH, and the plane $AEFC$.
And because the parallel planes AB, GH are cut by the plane $lEKG$,
their common sections AE, KG, with it, are parallel: (XI. 16.)
wherefore AK is a parallelogram,
ınd GK equal to EA the straight line from the centre of the circle AB:
for the same reason, each of the straight lines drawn from the point K to the line GH may be proved to be equal to those which are drawn rom the centre of the circle AB to its circumference, and are therefore ll equal to one another;
therefore the line GH is the circumference of a circle of which the centre is the point K: (I. def. 15.)
therefore the plane GH divides the cylinder AD into the cylinders AH, GD;
for they are the same which would be described by the revolution of the parallelograms AK, GF about the straight lines EK, KF:
and it is to be shewn, that the cylinder AH is to the cylinder HC, as the axis EK to the axis KF.



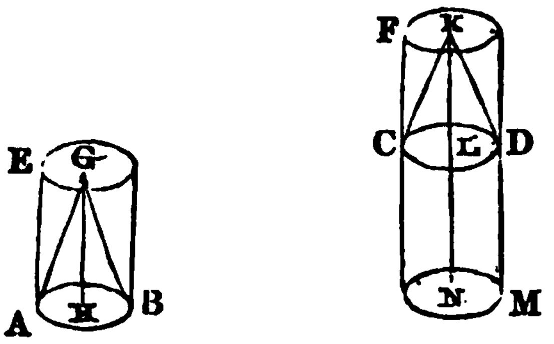
Produce the axis EF both ways:
and take any number of straight lines EN, NL, each equal to EK;
and any number FX, XM, each equal to FK;
and let planes parallel to AB, CD, pass through the points L, N, X, M:
therefore the common sections of these planes with the cylinder produced are circles, the centres of which are the points L, N, X, M, as was proved of the plane GH;
and these planes cut off the cylinders PR, RB, DT, TQ.
And because the axes LN, NE, EK, are all equal,
therefore the cylinders PR, RB, BG, are to one another as their bases: (XII. 11.)
but their bases are equal, and therefore the cylinders PR, RB, BG, are equal:
and because the axes LN, NE, EK, are equal to one another,
as also the cylinders PR, RB, BG,
and that there are as many axes as cylinders:
therefore, whatever multiple the axis KL is of the axis KE, the same multiple is the cylinder PG of the cylinder GB:
for the same reason, whatever multiple the axis MK is of the axis KF, the same multiple is the cylinder QG of the cylinder GD:
and if the axis KL be equal to the axis KM, the cylinder PG is equal to the cylinder GQ;
and if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if less, less:
therefore, since there are four magnitudes, viz. the axes EK, KF, and the cylinders BG, GD;
and that of the axis EK and cylinder BG there have been taken any equimultiples whatever, viz. the axis KL and cylinder PG,
and of the axis KF and cylinder GD, any equimultiples whatever, viz. the axis KM and cylinder GQ;
and since it has been demonstrated, that if the axis KL be greater than the axis KM,
the cylinder PG is greater than the cylinder GQ;
and if equal, equal; and if less, less:
therefore as the axis EK is to the axis KF, so is the cylinder BG to the cylinder GD.
Wherefore, if a cylinder, &c. Q.E.D.

PROPOSITION XIV. THEOREM.

Cones and cylinders upon equal bases are to one another as their ltitudes.

Let the cylinders EB, FD, be upon the equal bases AB, CD.
As the cylinder EB to the cylinder FD, so shall the axis GH be to the axis KL.

Produce the axis KL to the point N, and make LN equal to the xis GH,
and let CM be a cylinder of which the base is CD, and axis LN.
Then because the cylinders EB, CM, have the same altitude, they are to one another as their bases: (XII. 11.)
but their bases are equal, therefore also the cylinders EB, CM, are equal:

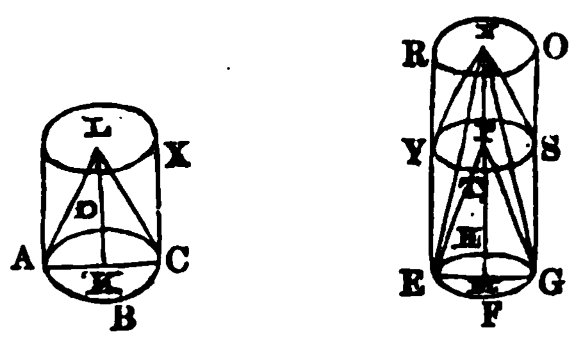


and because the cylinder FM is cut by the plane CD parallel to its opposite planes,
as the cylinder CM to the cylinder FD, so is the axis LN to the axis KL: (XII. 13.)
but the cylinder CM is equal to the cylinder EB, and the axis LN to the axis GH;
therefore as the cylinder EB to the cylinder FD, so is the axis GH to the axis KL:
and as the cylinder EB to the cylinder FD, so is the cone ABG to the cone CDK, (V. 15.)
because the cylinders are triple of the cones: (XII. 10.)
therefore also the axis GH is to the axis KL, as the cone ABG to the cone CDK, and as the cylinder EB to the cylinder FD.
Wherefore cones, &c. Q.E.D.

PROPOSITION XV. THEOREM.

The bases and altitudes of equal cones and cylinders are reciprocally proportional; and conversely, if the bases and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

Let the circles $ABCD$, $EFGH$, the diameters of which are AC, EG, be the bases, and KL, MN, the axes, as also the altitudes, of equal cones and cylinders;
and let ALC, ENG be the cones, and AX, EO the cylinders:
the bases and altitudes of the cylinders AX, EO shall be reciprocally proportional;
that is, as the base $ABCD$ to the base $EFGH$, so shall the altitude MN be to the altitude KL.



Either the altitude MN is equal to the altitude KL, or these altitudes are not equal.

First let them be equal;
and the cylinders AX, EO being also equal,
and cones and cylinders of the same altitude being to one another as their bases, (XII. 11.)
therefore the base $ABCD$ is equal to the base $EFGH$: (V. A.)
and as the base $ABCD$ is to the base $EFGH$, so is the altitude MN to the altitude KL.

But let the altitudes KL, MN, be unequal, and MN the greater of the two,
and from MN take MP equal to KL,
and through the point P cut the cylinder EO by the plane TYS, parallel to the opposite planes of the circles $EFGH$, RO;
therefore the common section of the plane TYS and the cylinder EO is a circle, and consequently ES is a cylinder, the base of which is the circle $EFGH$, and altitude MP:
and because the cylinder AX is equal to the cylinder EO,
as AX is to the cylinder ES, so is the cylinder EO to the same ES: (V. 7.)
but as the cylinder AX to the cylinder ES, so is the base $ABCD$ to the base $EFGH$; (XII. 11.)
for the cylinders AX, ES are of the same altitude;
and as the cylinder EO to the cylinder ES, so is the altitude MN to the altitude MP, (XII. 13.)
because the cylinder EO is cut by the plane TYS parallel to its opposite planes;
therefore as the base $ABCD$ to the base $EFGH$, so is the altitude MN to the altitude MP:
but MP is equal to the altitude KL;
wherefore as the base $ABCD$ to the base $EFGH$, so is the altitude MN to the altitude KL:
that is, the bases and altitudes of the equal cylinders AX, EO, are reciprocally proportional.

But let the bases and altitudes of the cylinders AX, EO be reciprocally proportional,
viz. the base $ABCD$ to the base $EFGH$, as the altitude MN to the altitude KL:
the cylinder AX shall be equal to the cylinder EO.

First, let the base $ABCD$ be equal to the base $EFGH$:
then because as the base $ABCD$ is to the base $EFGH$, so is the altitude MN to the altitude KL;
MN is equal to KL; (V. A.)
and therefore the cylinder AX is equal to the cylinder EO. (XII. 11.)

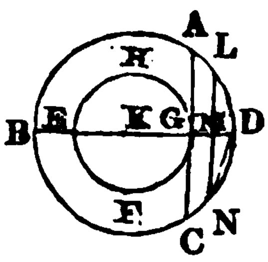
But let the bases $ABCD$, $EFGH$ be unequal,
and let $ABCD$ be the greater:
and because as $ABCD$ is to the base $EFGH$, so is the altitude MN to the altitude KL;

therefore MN is greater than KL. (v. A.)
Then, the same construction being made as before,
because as the base $ABCD$ to the base $EFGH$, so is the altitude MN to the altitude KL;
and because the altitude KL is equal to the altitude MP;
therefore the base $ABCD$ is to the base $EFGH$, as the cylinder AX to the cylinder ES; (XII. 12.)
and as the altitude MN to the altitude MP or KL, so is the cylinder EO to the cylinder ES:
therefore the cylinder AX is to the cylinder ES, as the cylinder EO is to the same ES:
whence the cylinder AX is equal to the cylinder EO:
and the same reasoning holds in cones. Q.E.D.

PROPOSITION XVI. PROBLEM.

In the greater of two circles that have the same centre, to inscribe a polygon of an even number of equal sides, that shall not meet the lesser circle.

Let $ABCD$, $EFGH$ be two given circles having the same centre K.
It is required to inscribe in the greater circle $ABCD$, a polygon of an even number of equal sides, that shall not meet the lesser circle.

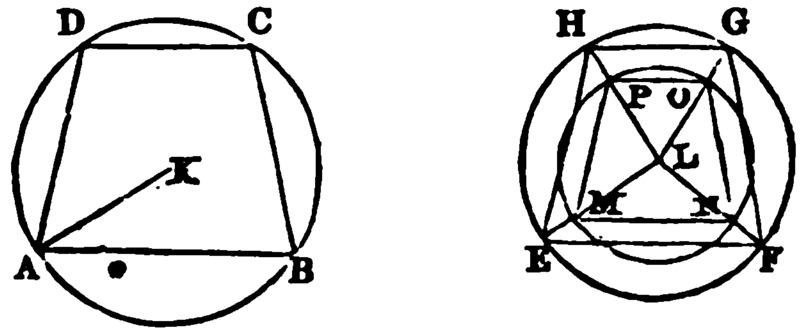


Through the centre K draw the straight line BD,
and from the point G, where it meets the circumference of the lesser circle,
draw GA at right angles to BD, and produce it to C;
therefore AC touches the circle $EFGH$: (III. 16. Cor.)
then, if the circumference BAD be bisected, and the half of it be again bisected, and so on, there must at length remain a circumference less than AD: (XII. Lem. 1.)
let this be LD;
and from the point L draw LM perpendicular to BD, and produce it to N;
and join LD, DN:
therefore LD is equal to DN:
and because LN is parallel to AC, and that AC touches the circle $EFGH$;
therefore LN does not meet the circle $EFGH$;
and much less shall the straight lines LD, DN, meet the circle $EFGH$:
so that if straight lines equal to LD be applied in the circle $ABCD$ from the point L around to N,
there shall be inscribed in the circle a polygon of an even number of equal sides not meeting the lesser circle. Q.E.F.

LEMMA II.

If two trapeziums ABCD, EFGH *be inscribed in the circles, the centres of which are the points* K, L; *and if the sides* AB, DC *be parallel, as also* EF, HG; *and the other four sides* AD, BC, EH, FG, *be all equal to one another; but the side* AB *greater than* EF, *and* DC *greater than* HG: *the straight line* KA *from the centre of the circle in which the greater sides are, is greater than the straight line* LE *drawn from the centre to the circumference of the other circle.*

If it be possible, let KA be not greater than LE;
then KA must be either equal to it, or less than it.



First, let KA be equal to LE:
therefore, because in two equal circles AD, BC, in the one, are equal to EH, FG in the other, the circumferences AD, BC, are equal to the circumferences EH, FG; (III. 28.)
but because the straight lines AB, DC are respectively greater than EF, GH;
the circumferences AB, DC are greater than EF, HG;
therefore the whole circumference $ABCD$ is greater than the whole $EFGH$:
but it is also equal to it, which is impossible:
therefore the straight line KA is not equal to LE.
But let KA be less than LE,
and make LM equal to KA,
and from the centre L, and distance LM, describe the circle $MNOP$,
meeting the straight lines LE, LF, LG, LH, in M, N, O, P;
and join MN, NO, OP, PM, which are respectively parallel to, and less than EF, FG, GH, HE: (VI. 2.)
then because EH is greater than MP,
AD is greater than MP;
and the circles $ABCD$, $MNOP$ are equal;
therefore the circumference AD is greater than MP:
for the same reason, the circumference BC is greater than NO:
and because the straight line AB is greater than EF, which is greater than MN,
much more is AB greater than MN:
therefore the circumference AB is greater than MN;
and for the same reason, the circumference DC is greater than PO:
therefore the whole circumference $ABCD$ is greater than the whole $MNOP$:
but it is likewise equal to it, which is impossible:
therefore KA is not less than LE:
nor is it equal to it;
therefore the straight line KA must be greater than LE. Q.E.D.

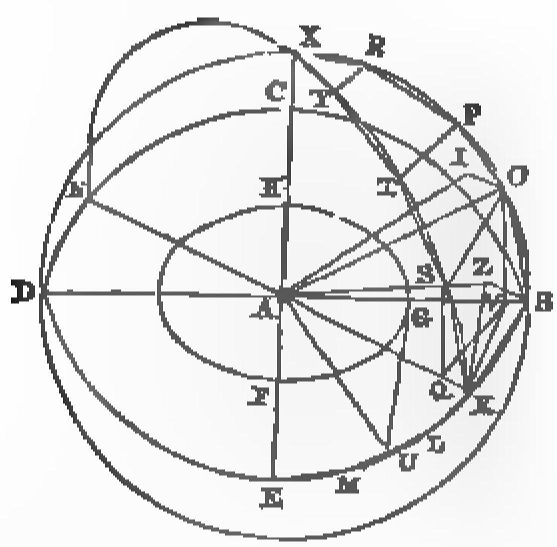
COR. And if there be an isosceles triangle, the sides of which are equal to AD, BC, but its base less than AB, the greater of the two sides AB, DC; the straight line KA may, in the same manner, be demonstrated to be greater than the straight line drawn from the centre to the circumference of the circle described about the triangle.

PROPOSITION XVII. PROBLEM.

In the greater of two spheres which have the same centre, to inscribe a solid polyhedron, the superficies of which shall not meet the lesser sphere.

Let there be two spheres about the same centre A.

It is required to inscribe in the greater a solid polyhedron, the superficies of which shall not meet the lesser sphere.



Let the spheres be cut by a plane passing through the centre;
the common sections of it with the spheres shall be circles;
because the sphere is described by the revolution of a semicircle about the diameter remaining unmoveable;
so that in whatever position the semicircle be conceived, the common section of the plane in which it is with the superficies of the sphere is the circumference of a circle;
and this is a great circle of the sphere, because the diameter of the sphere, which is likewise the diameter of the circle, is greater than any straight line in the circle or sphere. (III. 15.)
Let then the circle made by the section of the plane with the greater sphere be $BCDE$, and with the lesser sphere be FGH;
and draw the two diameters BD, CE at right angles to one another;
and in $BCDE$, the greater of the two circles, inscribe a polygon of an even number of equal sides not meeting the lesser circle FGH; (XII. 16.)
and let its sides in BE, the fourth part of the circle, be BK, KL, LM, ME;
join KA, and produce it to N;
and from A draw AX at right angles to the plane of the circle $BCDE$, meeting the superficies of the sphere in the point X: (XI. 12.)
and let planes pass through AX, and each of the straight lines BD, KN, which from what has been said, shall produce great circles on the

superficies of the sphere, and let BXD, KXN be the semicircles thus made upon the diameters, BD, KN:

therefore, because XA is at right angles to the plane of the circle $BCDE$,

every plane which passes through XA is at right angles to the plane of the circle $BCDE$; (XI. 18.)

wherefore the semicircles BXD, KXN are at right angles to that plane:

and because the semicircles BED, BXD, KXN upon the equal diameters BD, KN, are equal to one another;

their halves BE, BX, KX, are equal to one another,

therefore as many sides of the polygon as are in BE, so many are there in BX, KX, equal to the sides BK, KL, LM, ME:

let these polygons be described, and their sides be BO, OP, PR, RX; KS, ST, TY, YX;

and join OS, PT, RY;

and from the points O, S, draw OV, SQ perpendiculars to AB, AK:

and because the plane $BOXD$ is at right angles to the plane $BCDE$, and in one of them $BOXD$, OV is drawn perpendicular to AB the common section of the planes,

therefore OV is perpendicular to the plane $BCDE$: (XI. def. 4.)

for the same reason SQ is perpendicular to the same plane,

because the plane KSX is at right angles to the plane $BCDE$.

Join VQ:

and because in the equal semicircles BXD, KXN,

the circumferences BO, KS are equal,

and OV, SQ are perpendicular to their diameters,

therefore OV is equal to SQ, and BV equal to KQ: (I. 26.)

but the whole BA is equal to the whole KA,

therefore the remainder VA is equal to the remainder QA;

therefore as BV is to VA, so is KQ to QA;

wherefore VQ is parallel to BK: (VI. 2.)

and because OV, SQ are each of them at right angles to the plane of the circle $BCDE$,

OV is parallel to SQ: (XI. 6.)

and it has been proved, that it is also equal to it;

therefore QV, SO are equal and parallel: (I. 33.)

and because QV is parallel to SO, and also to KB;

OS is parallel to BK; (XI. 9.)

and therefore BO, KS, which join them are in the same plane in which these parallels are,

and the quadrilateral figure $KBOS$ is in one plane:

and if PB, TK be joined, and perpendiculars be drawn from the points P, T, to the straight lines AB, AK,

it may be demonstrated, that TP is parallel to KB in the very same way that SO was shewn to be parallel to the same KB;

wherefore TP is parallel to SO, (XI. 9.)

and the quadrilateral figure $SOPT$ is in one plane:

for the same reason the quadrilateral $TPRY$ is in one plane:

and the figure YRX is also in one plane: (XI. 2.)

therefore, if from the points O, S, P, T, R, Y, there be drawn straight lines to the point A, there will be formed a solid polyhedron between the circumferences BX, KX, composed of pyramids, the bases of which are the quadrilaterals $KBOS$, $SOPT$, $TPRY$, and the triangle YRX, and of which the common vertex is the point A:

and if the same construction be made upon each of the sides KL, LM, ME, as has been done upon BK,

and the like be done also in the other three quadrants, and in the other hemisphere;

there will be formed a solid polyhedron inscribed in the sphere, composed of pyramids, the bases of which are the aforesaid quadrilateral figures, and the triangle YRX, and those formed in the like manner in the rest of the sphere, the common vertex of them all being the point A.

Also the superficies of this solid polyhedron shall not meet the lesser sphere in which is the circle FGH.

For, from the point A draw AZ perpendicular to the plane of the quadrilateral $KBOS$, meeting it in Z, and join BZ, ZK: (XI. 11.)

and because AZ is perpendicular to the plane $KBOS$,

it makes right angles with every straight line meeting it in that plane;

therefore AZ is perpendicular to BZ and ZK:

and because AB is equal to AK, and that the squares of AZ, ZB are equal to the square of AB, and the squares of AZ, ZK to the square of AK; (I. 47.)

therefore the squares of AZ, ZB are equal to the squares of AZ, ZK:

take from these equals the square of AZ,

and the remaining square of BZ is equal to the remaining square of ZK,

and therefore the straight line BZ is equal to ZK:

in the like manner it may be demonstrated, that the straight lines drawn from the point Z to the points O, S, are equal to BZ or ZK;

therefore the circle described from the centre Z, and distance ZB, will pass through the points K, O, S,

and $KBOS$ will be a quadrilateral figure in the circle:

and because KB is greater than QV, and QV equal to SO,

therefore KB is greater than SO:

but KB is equal to each of the straight lines BO, KS;

wherefore each of the circumferences cut off by KB, BO, KS, is greater than that cut off by OS;

and these three circumferences, together with a fourth equal to one of them, are greater than the same three together with that cut off by OS;

that is, than the whole circumference of the circle;

therefore the circumference subtended by KB is greater than the fourth part of the whole circumference of the circle $KBOS$,

and consequently the angle BZK at the centre is greater than a right angle:

and because the angle BZK is obtuse,

the square of BK is greater than the squares of BZ, ZK; (II. 12.)

that is, greater than twice the square of BZ.

Join KV:

and because in the triangle KBV, OBV,

KB, BV are equal to OB, BV, and that they contain equal angles;

the angle KBV is equal to the angle OVB: (I. 4.)

and OVB is a right angle;

therefore also KVB is a right angle:

and because BD is less than twice DV;

the rectangle contained by BD, BV is less than twice the rectangle DV, VB;

that is, the square of KB is less than twice the square of KV: (VI. 8.)

but the square of KB is greater than twice the square of BZ;
therefore the square of KV is greater than the square of BZ:
and because BA is equal to AK,
and that the squares of BZ, ZA are equal together to the square of BA,
and the squares of KV, VA to the square of AK;
therefore the squares of BZ, ZA are equal to the squares of KV, VA;
and of these the square of KV is greater than the square of BZ;
therefore the square of VA is less than the square of ZA,
and the straight line AZ greater than VA:
much more then is AZ greater than AG;
because, in the preceding proposition,
it was shewn that KV falls without the circle FGH:
and AZ is perpendicular to the plane $KBOS$,
and is therefore the shortest of all the straight lines that can be drawn from A, the centre of the sphere, to that plane.
Therefore the plane $KBOS$ does not meet the lesser sphere.

And that the other planes between the quadrants BX, KX, fall without the lesser sphere, is thus demonstrated.

From the point A draw AI perpendicular to the plane of the quadrilateral $SOPT$, and join IO;

and, as was demonstrated of the plane $KBOS$ and the point Z, in the same way it may be shewn that the point I is the centre of a circle described about $SOPT$;
and that OS is greater than PT;
and PT was shewn to be parallel to OS:
therefore, because the two trapeziums $KBOS$, $SOPT$ inscribed in circles, have their sides BK, OS parallel, as also OS, PT;
and their other sides BO, KS, OP, ST all equal to one another,
and that BK is greater than OS, and OS greater than PT,
therefore the straight line ZB is greater than IO. (XII. Lem. 2.)
Join AO, which will be equal to AB;
and because AIO, AZB are right angles,
the squares of AI, IO are equal to the square of AO or of AB;
that is, to the squares of AZ, ZB;
and the square of ZB is greater than the square of IO,
therefore the square of AZ is less than the square of AI;
and the straight line AZ less than the straight line AI:
and it was proved, that AZ is greater than AG;
much more then is AI greater than AG:
therefore the plane $SOPT$ falls wholly without the lesser sphere.
In the same manner it may be demonstrated,
that the plane $TPRY$ falls without the same sphere, as also the triangle YRX, viz. by XII. Lem. 2. Cor.

And after the same way it may be demonstrated, that all the planes which contain the solid polyhedron, fall without the lesser sphere.

Therefore in the greater of two spheres, which have the same centre, a solid polyhedron is inscribed, the superficies of which does not meet the lesser sphere. Q.E.F.

But the straight line AZ may be demonstrated to be greater than AG otherwise, and in a shorter manner, without the help of Prop. XVI., as follows.

From the point G draw GU at right angles to AG, and join AU.

If then the circumference BE be bisected, and its half again

bisected, and so on, there will at length be left a circumference less than the circumference which is subtended by a straight line equal to *GU*, inscribed in the circle *BCDE*:

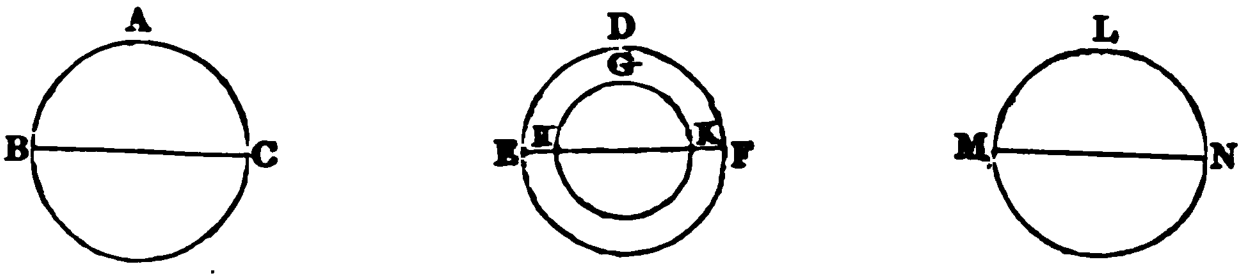
let this be the circumference *KB*:
therefore the straight line *KB* is less than *GU*:
and because the angle *BZK* is obtuse, as was proved in the preceding,
therefore *BK* is greater than *BZ*:
but *GU* is greater than *BK*;
much more then is *GU* greater than *BZ*,
and the square of *GU* than the square of *BZ*:
and *AU* is equal to *AB*;
therefore the square of *AU*, that is, the squares of *AG*, *GU* are equal to the square of *AB*, that is, to the squares of *AZ*, *ZB*:
but the square of *BZ* is less than the square of *GU*;
therefore the square of *AZ* is greater than the square of *AG*,
and the straight line *AZ* consequently greater than the straight line *AG*.

COR.—And if in the lesser sphere there be inscribed a solid polyhedron, by drawing straight lines betwixt the points in which the straight lines from the centre of the sphere drawn to all the angles of the solid polyhedron in the greater sphere meet the superficies of the lesser; in the same order in which are joined the points in which the same lines from the centre meet the superficies of the greater sphere: the solid polyhedron in the sphere *BCDE* shall have to this other solid polyhedron the triplicate ratio of that which the diameter of the sphere *BCDE* has to the diameter of the other sphere. For if these two solids be divided into the same number of pyramids, and in the same order, the pyramids shall be similar to one another, each to each: because they have the solid angles at their common vertex, the centre of the sphere, the same in each pyramid, and their other solid angles at the bases, equal to one another, each to each, (XI. B.) because they are contained by three plane angles, each equal to each; and the pyramids are contained by the same number of similar planes; and are therefore similar to one another, each to each: (XI. def. 11.) but similar pyramids have to one another the triplicate ratio of their homologous sides: (XII. 8. Cor.) therefore the pyramid of which the base is the quadrilateral *KBOS*, and vertex *A*, has to the pyramid in the other sphere of the same order, the triplicate ratio of their homologous sides, that is, of that ratio, which *AB* from the centre of the greater sphere has to the straight line from the same centre to the superficies of the lesser sphere. And in like manner, each pyramid in the greater sphere has to each of the same order in the less, the triplicate ratio of that which *AB* has to the semi-diameter of the less sphere. And as one antecedent is to its consequent, so are all the antecedents to all the consequents. Wherefore the whole solid polyhedron in the greater sphere has to the whole solid polyhedron in the other, the triplicate ratio of that which *AB* the semi-diameter of the first has to the semi-diameter of the other; that is, which the diameter *BD* of the greater has to the diameter of the other sphere.

PROPOSITION XVIII. THEOREM.

Spheres have to one another the triplicate ratio of that which their diameters have.

Let *ABC, DEF* be two spheres, of which the diameters are *BC, EF*.
The sphere *ABC* shall have to the sphere *DEF* the triplicate ratio of that which *BC* has to *EF*.



For, if it has not, the sphere *ABC* must have to a sphere either less or greater than *DEF*, the triplicate ratio of that which *BC* has to *EF*.
First, if possible, let it have that ratio to a less, viz. to the sphere *GHK*;
and let the sphere *DEF* have the same centre with *GHK*:
and in the greater sphere *DEF* inscribe a solid polyhedron, the superficies of which does not meet the lesser sphere *GHK*; (XII. 17.)
and in the sphere *ABC* inscribe another similar to that in the sphere *DEF*:
therefore the solid polyhedron in the sphere *ABC* has to the solid polyhedron in the sphere *DEF*, the triplicate ratio of that which *BC* has to *EF*. (XII. 17. Cor.)
But the sphere *ABC* has to the sphere *GHK*, the triplicate ratio of that which *BC* has to *EF*;
therefore, as the sphere *ABC* to the sphere *GHK*, so is the solid polyhedron in the sphere *ABC* to the solid polyhedron in the sphere *DEF*:
but the sphere *ABC* is greater than the solid polyhedron in it;
therefore also the sphere *GHK* is greater than the solid polyhedron in the sphere *DEF*: (V. 14.)
but it is also less, because it is contained within it, which is impossible:
therefore the sphere *ABC* has not to any sphere less than *DEF*, the triplicate ratio of that which *BC* has to *EF*.
In the same manner, it may be demonstrated, that the sphere *DEF* has not to any sphere less than *ABC*, the triplicate ratio of that which *EF* has to *BC*.
Nor can the sphere *ABC* have to any sphere greater than *DEF*, the triplicate ratio of that which *BC* has to *EF*:
for, if it can, let it have that ratio to a greater sphere *LMN*:
therefore, by inversion, the sphere *LMN* has to the sphere *ABC*, the triplicate ratio of that which the diameter *EF* has to the diameter *BC*.
But as the sphere *LMN* to *ABC*, so is the sphere *DEF* to some sphere, which must be less than the sphere *ABC*, (V. 14.)
because the sphere *LMN* is greater than the sphere *DEF*;
therefore the sphere *DEF* has to a sphere less than *ABC* the triplicate ratio of that which *EF* has to *BC*;
which was shewn to be impossible:
therefore the sphere *ABC* has not to any sphere greater than *DEF*, the triplicate ratio of that which *BC* has to *EF*:
and it was demonstrated, that neither has it that ratio to any sphere less than *DEF*.
Therefore the sphere *ABC* has to the sphere *DEF*, the triplicate ratio of that which *BC* has to *EF*. Q. E. D.

NOTES TO BOOK XII.

THIS book treats of the properties of prisms and cylinders, pyramids and cones. A new principle is introduced called "the method of Exhaustions," which may be applied for the purpose of finding the areas and ratios of circles, and the relations of the surfaces and of the volumes of cones, spheres and cylinders.

The first comparison of rectilinear areas is made in the first book of the Elements by the principle of superposition, where two triangles are coincident in all respects; next, comparison is made between triangles and other rectilinear figures when they are not coincident.

In the sixth book, similar triangles are compared by shewing that they are in the duplicate ratio of their homologous sides, and then by dividing similar polygons into the same number of similar triangles, and shewing that the polygons are also in the duplicate ratio of any of their homologous sides. In the eleventh book, similar rectilinear solids are compared by shewing that their volumes are to one another in the triplicate ratio of their homologous sides.

"The method of Exhaustions" is founded on the principle of *exhausting* a magnitude by continually taking away a certain part of it, as is explained in the tenth book of the Elements, where Euclid states, that two quantities are equal, whose difference is less than any assignable quantity. If A and A' be two magnitudes of the same kind, and if d and d' be any other magnitudes of the same kind, such that $A' + d$ is always greater than, and $A' - d'$ always less than A, however small d and d' may be made, then A' is equal to A.

The method of exhaustions may be applied to find the circumference and area of a circle. A rectilinear figure may be inscribed in the circle and a similar one circumscribed about it, and then by continually doubling the number of sides of the inscribed and circumscribed polygons, by this principle, it may be demonstrated, that the area of the circle is less than the area of the circumscribed polygon, but greater than the area of the inscribed polygon; and that as the number of sides of the polygon is increased, and consequently the magnitude of each diminished, the differences between the circle and the inscribed and circumscribed polygons are continually exhausted, and at length become less than any assignable difference; and the area of the circle is the limit to which the inscribed and circumscribed polygons continually approach, as the number of sides is increased.

Also in comparing two unequal circles, two similar polygons are inscribed in the circles, and then by doubling the number of sides continually, it is shewn that the limit to which the ratio of the areas of the rectilineal figures continually approach is the same as the ratio of the circles.

In a similar way it will be seen, that the principle is applied to the surfaces and volumes of cones, cylinders and spheres.

Prop. II. (1.) For there is some square equal to the circle $ABCD$; let P be the side of it, and to three straight lines BD, FH, and P, there can be a fourth proportional; let this be Q: therefore (VI. 22.) the squares of these four straight lines are proportionals; that is, to the squares of BD, FH, and the circle $ABCD$ it is possible there may be a fourth proportional. Let this be S. And in like manner are to be understood some things in some of the following propositions.

(2.) For as, in the foregoing note, it was explained how it was possible there could be a fourth proportional to the squares of *BD*, *FH*, and the circle *ABCD*, which was named *S*; so, in like manner, there can be a fourth proportional to this other space, named *T*, and the circles *ABCD*, *EFGH*. And the like is to be understood in some of the following propositions.

(3.) Because, as a fourth proportional to the squares of *BD*, *FH*, and the circle *ABCD*, is possible, and that it can neither be less nor greater than the circle *EFGH*, it must be equal to it. SIMSON.

Prop. IV. Because *GO* is equal to *OA*, and *GM* to *MB*, therefore (VI. 2.) *OM* is parallel to *AB*; in the same manner *ON* is parallel to *AC*; therefore (XI. 15.) the plane *MON* is parallel to the plane *BAC*. SIMSON.

Prop. XI. Vertex is put in place of altitude, which is in the Greek, because the pyramid, in what follows, is supposed to be circumscribed about the cone, and so must have the same vertex. And the same change is made in some places following. SIMSON.

The thirteenth book of the Elements relates to equilateral and equiangular figures inscribed in circles, and to the five regular solids.

Two books which treat of the inscriptions of the five regular solids in one another and in spheres, are frequently found annexed to the Elements as the fifteenth and sixteenth books; these however, were composed by Hypsicles of Alexandria.

There is a continuation of the same subject by Flussas, which has been appended to the Elements, and called the sixteenth book of the Elements.

ON THE

ANCIENT GEOMETRICAL ANALYSIS.

SYNTHESIS, or the method of composition, is a mode of reasoning which begins with something given, and ends with something required, either to be done or to be proved. This may be termed a *direct process*, as it leads from principles to consequences.

Analysis, or the method of resolution, is the reverse of synthesis, and thus may be considered an *indirect process*, a method of reasoning from consequences to principles.

The synthetic method is pursued by Euclid in his elements of Geometry. He commences with certain assumed principles, and proceeds to the solution of problems and the demonstration of theorems by undeniable and successive inferences from them.

The Geometrical Analysis was a process employed by the ancient Geometers, both for the discovery of the solution of problems and for the investigation of the truth of theorems. In the analysis of a *problem*, the quæsita, or what is required to be done, is supposed to have been effected, and the consequences are traced by a series of geometrical constructions and reasonings, till at length they terminate in the data of the problem, or in some previously demonstrated or admitted truth, whence the direct solution of the problem is deduced.

In the Synthesis of a *problem*, however, the last consequence of the analysis is assumed as the first step of the process, and by proceeding in a contrary order through the several steps of the analysis until the process terminate in the quæsita, the solution of the problem is effected.

But if, in the analysis, we arrive at a consequence which contradicts any truth demonstrated in the Elements, or which is inconsistent with the data of the problem, the problem must be impossible: and further, if in certain relations of the given magnitudes the construction be possible, while in other relations it is impossible, the discovery of these relations will become a necessary part of the solution of the problem.

In the analysis of a *theorem*, the question to be determined, is, whether by the application of the geometrical truths proved in the Elements, the predicate is consistent with the hypothesis. This point is ascertained by assuming the predicate to be true, and by deducing the successive consequences of this assumption combined with proved geometrical truths, till they terminate in the hypothesis of the theorem or some demonstrated truth. The theorem will be proved synthetically by retracing, in order, the steps of the investigation pursued in the analysis, till they terminate in the predicate, which was assumed in the analysis. This process will constitute the demonstration of the theorem.

If the assumption of the truth of the predicate in the analysis lead to some consequence which is inconsistent with any demonstrated truth,

the false conclusion thus arrived at indicates the falsehood of the predicate; and by reversing the process of the analysis, it may be demonstrated, that the theorem cannot be true.

It may be here remarked, that the geometrical analysis is more extensively useful in discovering the solution of problems than for investigating the demonstration of theorems.

From the nature of the subject, it must be at once obvious, that no general rules can be prescribed, which will be found applicable in all cases, and infallibly lead to the solution of every problem. The conditions of problems must suggest what constructions may be possible; and the consequences which follow from these constructions and the assumed solution, will shew the possibility or impossibility of arriving at some known property consistent with the data of the problem.

Though the data of a problem may be given in magnitude and position, certain ambiguities will arise, if they are not properly restricted. Two points may be considered as situated on the same side, or one on each side of a given line; and there may be two lines drawn from a given point making equal angles with a line given in position; and to avoid ambiguity, it must be stated on which side of the line the angle is to be formed.

A problem is said to be *determinate* when, with the prescribed conditions, it admits of one definite solution; the same construction which may be made on the other side of any given line, not being considered a different solution: and a problem is said to be *indeterminate* when it admits of more than one definite solution. This latter circumstance arises from the data not *absolutely fixing* but *merely restricting* the quæsita, leaving certain points or lines not fixed in one position only. The number of given conditions may be insufficient for a single determinate solution; or relations may subsist among some of the given conditions from which one or more of the remaining given conditions may be deduced.

If the base of a right-angled triangle be given, and also the difference of the squares of the hypothenuse and perpendicular, the triangle is indeterminate. For though apparently here are three things given, the right angle, the base, and the difference of the squares of the hypothenuse and perpendicular, it is obvious that these three apparent conditions are in fact reducible to two: for since in a right-angled triangle, the sum of the squares on the base and on the perpendicular are equal to the square on the hypothenuse, it follows that the difference of the squares of the hypothenuse and perpendicular is equal to the square of the base of the triangle, and therefore the base is known from the difference of the squares of the hypothenuse and perpendicular being known. The conditions therefore are insufficient to determine a right-angled triangle; an indefinite number of triangles may be found with the prescribed conditions, whose vertices will lie in the line which is perpendicular to the base.

If a problem relate to the determination of *a single point*, and the data be sufficient to determine the position of that point, the problem is *determinate*: but if one or more of the conditions be omitted, the data which remain may be sufficient for the determination of more than one point, each of which satisfies the conditions of the problem; in that case, the problem is *indeterminate*, and in general such points are found

to be situated in some line, and hence such line is called the locus of the point which satisfies the conditions of the problem.

If any two given points A and B (fig. Prop. 5, Book IV.) be joined by a straight line AB, and this line be bisected in D, then if a perpendicular be drawn from the point of bisection, it is manifest that a circle described with *any* point in the perpendicular as a centre, and a radius equal to its distance from one of the given points, will pass through the other point, and the perpendicular will be the locus of all the circles which can be described passing through the two given points.

Again, if a third point C be taken, but not in the same straight line with the other two, and this point be joined with the first point A; then the perpendicular drawn from the bisection E of this line will be the locus of the centres of all circles which pass through the first and third points A and C. But the perpendicular at the bisection of the first and second points A and B is the locus of the centres of circles which pass through these two points. Hence the intersection F of these two perpendiculars, will be the centre of a circle which passes through the three points, and is called the intersection of the two loci. Sometimes this method of solving geometrical problems may be pursued with advantage, by constructing the locus of every two points separately, which are given in the conditions of the problem. In the Geometrical Exercises which follow, only those local problems are given where the locus is either a straight line or a circle.

There is another class of Propositions called *Porisms*. The exact meaning of the term as applied to this class of propositions, appears to be as uncertain as the derivation of the word itself. In the original Greek of Euclid's Elements, the corollaries to the propositions are called porisms (*πορισματα*); but this scarcely explains the nature of *porisms*, as it is manifest that they are different from simple deductions from the demonstrations of propositions. Some analogy, however, we may suppose them to have to the porisms or corollaries in the Elements. Pappus (Coll. Matth. Lib. VII. pref.) informs us that Euclid wrote three books on Porisms, and gives an obscure account of them. He defines "a porism to be something between a problem and a theorem, or that in which something is proposed to be investigated." Dr. Simson, to whom is due the merit of having restored the porisms of Euclid, gives the following definition of that class of propositions: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem datis, sed quæ ad ea quæ data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam." Professor Dugald Stewart defines a porism to be "A proposition affirming the possibility of finding one or more of the conditions of an indeterminate theorem." Professor Playfair in a paper (from which the following account is taken) on Porisms, printed in the Transactions of the Royal Society of Edinburgh, for the year 1792, defines a porism to be "A proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate or capable of innumerable solutions."

It may without much difficulty be perceived that this definition represents a porism as almost the same with an indeterminate problem. There is a large class of indeterminate problems which are, in general, loci,

and satisfy certain defined conditions. Every indeterminate problem containing a locus may be made to assume the form of a porism, but not the converse. Porisms are of a more general nature than indeterminate problems which involve a locus.

It is highly probable that the ancient geometers arrived at all geometrical truths in their attempts at the solution of problems. The first mathematical enquiries must have occurred in the form of questions, where something was given, and something required to be done; and by reasonings necessary to answer these questions, or to discover the relations between the things that were given, and those that were to be found, many truths were suggested which came afterwards to be the subjects of separate demonstration.

The ancient geometers appear to have undertaken the solution of problems with a scrupulous and minute attention, which would scarcely allow any of the collateral truths to escape their observation. They never considered a problem as solved till they had distinguished all its varieties, and evolved separately every different case that could occur, carefully distinguishing whatever change might arise in the construction from any change that was supposed to take place among the magnitudes which were given. This cautious method of proceeding soon led them to see that there were circumstances in which the solution of a problem would cease to be possible; and this always happened when one of the conditions of the data was inconsistent with the rest. Such instances would occur in the simplest problems; but in the analysis of more complex problems, they must have remarked that their constructions failed, for a reason directly contrary to that assigned. Instances would be found where the lines, which, by their intersection, were to determine the thing sought, instead of intersecting one another, as they did in general, or of not meeting at all, would coincide with one another entirely, and consequently leave the question unresolved. The confusion thus arising would soon be cleared up, by observing, that a problem before determined by the intersection of two lines would now become capable of an indefinite number of solutions. This was soon perceived to arise from one of the conditions of the problem involving another, or from two parts of the data becoming one, so that there was not left a sufficient number of independent conditions to confine the problem to a single solution, or any determinate number of solutions. It was not difficult afterwards to perceive, that these cases of problems formed very curious propositions, of an indeterminate nature between problems and theorems, and that they admitted of being enunciated separately. It was to such propositions so enunciated that the ancient geometers gave the name of *Porisms*.

Besides, it will be found, that some problems are possible within certain limits, and that certain magnitudes increase while others decrease within those limits; and after having reached a certain value the former begin to decrease, while the latter increase. This circumstance gives rise to questions of *maxima* and *minima*, or the greatest and least values which certain magnitudes may admit of in indeterminate problems.

In the following collection of problems and theorems, many will be found to be of so simple a character, (being almost obvious deductions from propositions in the Elements) as scarcely to admit of the principle of the Geometrical Analysis being applied, in their solution.

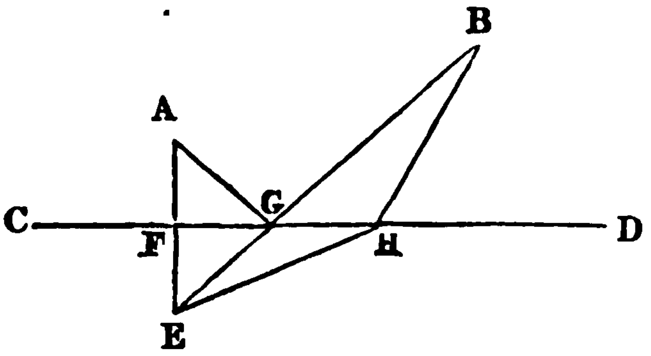
GEOMETRICAL EXERCISES ON BOOK I.

PROBLEM I.

From two given points on the same side of a straight line given in position, draw two straight lines which shall meet in that line, and make equal angles with it; also prove, that the sum of these two lines is less than the sum of any other two lines drawn to any other point in the line.

ANALYSIS. Let A, B be the two given points, and CD the given line.

Suppose G the required point in the line, such that AG and BG being joined, the angle AGF is equal to the angle BGD.



Draw AF perpendicular to CD and meeting BG produced in E.

Then, because the angle BGD is equal to AGF, (hyp.)
and also to the vertical angle FGE, (I. 15.)
therefore the angle AGF is equal to EGF;
also the right angle AFG is equal to the right angle EFG,
and the side FG is common to the two triangles AFG, EFG,
therefore AG is equal to EG, and AF to FE.

Hence the point E being known, the point G is determined by the intersection of CD and BE.

Synthesis. From A draw AF perpendicular to CD, and produce it to E, making FE equal to AF, and join BG cutting CD in G. Join also AG.

Then AG and BG make equal angles with CD.

For since AF is equal to FE, and FG is common to the two triangles AGF, CGF, and the included angles AFG, EFG are equal;
therefore the base AG is equal to the base EG, and the angle AGF to the angle EGF,
but the angle EGF is equal to the vertical angle BGD,
therefore the angle AGF is equal to the angle BGD;
that is, the straight lines AG and BG make equal angles with the straight line CD.

Also the sum of the lines AG, GB is a minimum.

For take any other point H in CD, and join EH, HB.

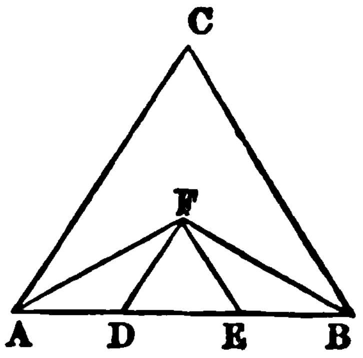
Then since any two sides of a triangle are greater than the third side,
therefore EH, HB are greater than EB in the triangle EHB.
But EG is equal to AG;
therefore EH, HB are greater than AG, GB.

That is, AG, GB are less than any other two lines which can be drawn from A, B, to any other point H in the line CD.

PROBLEM II.

To trisect a given straight line.

Analysis. Let AB be the given straight line, and suppose it divided into three equal parts in the points D, E.



On DE describe an equilateral triangle DEF,
then DF is equal to AD, and FE to EB.
On AB describe an equilateral triangle ABC,
and join AF, FB.
Then because AD is equal to DF,
therefore the angle DAF is equal to the angle DFA,
and the two angles DAF, DFA are double of one of them DAF.
But the angle FDE is equal to the angles DAF, DFA,
and the angle FDE is equal to DAC, each being an angle of an equilateral triangle;
therefore the angle DAC is double the angle DAF;
wherefore the angle DAC is bisected by AF.
Also because the angle FAC is equal to the angle FAD, and FAD to DFA;
therefore the angle CAF is equal to the alternate angle AFD:
and consequently FD is parallel to AC.

Synthesis. Upon AB describe an equilateral triangle ABC,
bisect the angles at A and B by the straight lines AF, BF, meeting in F;
through F draw FD parallel to AC, and FE parallel to BC.
Then AB is trisected in the points D, E.
For since AC is parallel to FD and FA meets them,
therefore the alternate angles FAC, AFD are equal;
but the angle FAD is equal to the angle FAC,
hence the angle DAF is equal to the angle AFD,
and therefore DA is equal to DF.
But the angle FDE is equal to the angle CAB, and FED to CBA; (I. 29.)
and therefore the remaining angle DFE is equal to the remaining angle ACB.
Hence the three sides of the triangle DFE are equal to one another,
and DF has been shewn to be equal to DA,
therefore AD, DE, EB are equal to one another.

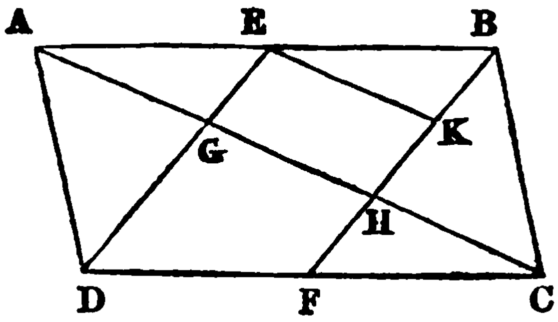
Hence the following theorem.

If the angles at the base of an equilateral triangle be bisected by two lines which meet at a point within the triangle; the two lines drawn from this point parallel to the sides of the triangle, divide the base into three equal parts.

THEOREM I.

If two opposite sides of a parallelogram be bisected, and two lines e drawn from the points of bisection to the opposite angles, these two ines trisect the diagonal.

Let $ABCD$ be a parallelogram of which the diagonal is AC.
Let AB be bisected in E and DC in F,
also let DE, FB be joined cutting the diagonal in G, H.
Then AC is trisected in the points G, H.

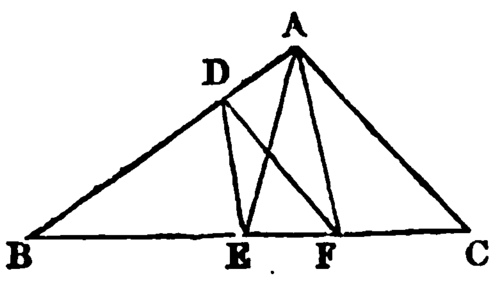


Through E draw EK parallel to AC and meeting FB in K.
Then because EB is the half of AB, and DF the half of DC;
therefore EB is equal to DF;
and these equal and parallel straight lines are joined towards the same parts by DE and FB;
therefore DE and FB are equal and parallel. (I. 33.)
And because AEB meets the parallels EK, AC,
therefore the exterior angle BEK is equal to the interior angle EAG.
For a similar reason, the angle EBK is equal to the angle AEG.
Hence in the triangles AEG, EBK, there are the two angles GAE, AEG in the one, equal to two angles KEB, EBK in the other, and one ide adjacent to the equal angles in each triangle, namely AE equal o EB:
therefore AG is equal to EK, (I. 26.)
but EK is equal to GH, (I. 34.)
therefore AG is equal to GH.
By a similar process, it may be shewn that GH is equal to HC.
Hence AG, GH, HC are equal to one another,
and therefore AC is trisected in the points G, H.

PROBLEM III.

To bisect a triangle by a line drawn from a given point in one of he sides. (Euclidis de Divisionibus.)

Analysis. Let ABC be the given triangle, and D the given point n the side AB.



Suppose DF the line drawn from D which bisects the triangle;
therefore the triangle DBF is half of the triangle ABC.

Bisect BC in E and join AE, DE, AF,
then the triangle ABE is half of the triangle ABC:
hence the triangle ABE is equal to the triangle DBF;
take away from these equals the triangle DBE,
therefore the remainder ADE is equal to the remainder DEF.

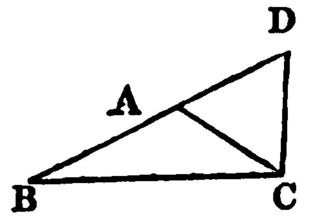
But ADE, DEF are equal triangles upon the same base DE, and on the same side of it,
they are therefore between the same parallels,
that is, AF is parallel to DE,
therefore the point F is determined.

Synthesis. Bisect the base BC in E, join DE,
from A, draw AF parallel to DE, and join DF.
Then because DE is parallel to AF,
therefore the triangle ADE is equal to the triangle DEF;
to each of these equals, add the triangle BDE,
therefore the whole triangle ABE is equal to the whole DBF,
but ABE is half of the whole triangle ABC;
therefore ABF is also half of the triangle ABC.

PROBLEM IV.

Given one angle, a side opposite to it, and the sum of the other two sides, construct the triangle.

Analysis. Suppose BAC the triangle required, having BC equal to the given side, BAC equal to the given angle opposite to BC, also BD equal to the sum of the other two sides.



Join DC.
Then since the two sides BA, AC are equal to BD,
by taking BA from these equals,
the remainder AC is equal to the remainder AD.

Hence the triangle ACD is isosceles, and therefore the angle ADC is equal to the angle ACD.

But the exterior angle BAC of the triangle ADC is equal to the two interior and opposite angles ACD and ADC:

Wherefore the angle BAC is double the angle BDC, and BDC is the half of the angle BAC.

Hence the synthesis.

At the point D in BD, make the angle BDC equal to half the given angle,
and from B the other extremity of BD, draw BC equal to the given side, and meeting DC in C,
at C in CD make the angle DCA equal to the angle CDA, so that CA may meet BD in the point A.

Then the triangle ABC shall have the required conditions.

PROBLEMS.

5. Given the base and one of the sides of an isosceles triangle, to describe the triangle.

6. Describe an isosceles triangle, each of the sides of which shall be double of the base.

7. In a given straight line, find a point equally distant from two given points; one in, and the other above or below, the given straight line.

8. AB, AC are straight lines cutting one another in A, D is a given point. Draw through D a straight line cutting off equal parts from AB and AC.

9. Draw through a given point, between two straight lines not parallel, a straight line, which shall be bisected in that point.

10. Divide a given right angle into three equal angles.

11. One of the acute angles of a right-angled triangle is three times as great as the other; trisect the smaller of these.

12. From a given point in a given straight line, it is required to erect a perpendicular by the help of straight lines only.

13. From a given point without a given straight line, to draw a line making an angle with the given line equal to a given rectilineal angle.

14. Through a given point draw a straight line which shall make equal angles with two straight lines given in position.

15. To determine that point in a straight line from which the straight lines drawn to two other given points shall be equal, provided the line joining the two given points is not perpendicular to the given line.

16. To place a straight line in a triangle (terminated by the two sides) which shall be equal to one straight line and parallel to another.

17. Determine the shortest path from one given point to another, subject to the condition, that it shall meet two given lines.

18. In the base of a triangle, find the point from which lines, drawn parallel to the sides of the triangle and limited by them, are equal.

19. From a given point in either of the equal sides of an isosceles triangle, to draw a straight line to the other side produced, which shall make with these sides a triangle equal to the given triangle. Prove that the line thus drawn will be greater than the base of the isosceles triangle.

20. From one of the obtuse angles of a rhomboid draw a straight line to the opposite side, which shall be bisected by the diagonal drawn through its acute angles.

21. Upon a straight line as a diagonal describe a parallelogram having an angle equal to a given angle.

22. On a given line to describe a parallelogram, having two of its opposite angles double of the other two, and all its sides equal. By means of this problem, trisect a right angle.

23. Describe a parallelogram equal to a given square, and having an angle equal to half a right angle.

24. To describe a rhombus which shall be equal to any given quadrilateral figure.

25. Describe a parallelogram equal in area and perimeter to a given triangle.

26. Describe on a given straight line a triangle which shall be equal to a given rectilineal figure, and have its vertical angle equal to a given angle.

27. Transform a given rectilinear figure into a triangle whose vertex shall be in a given angle of the figure and whose base be in one of the sides.

28. Straight lines are drawn from a fixed point to the several points of a straight line given in position, and on each base is described an equilateral triangle. Determine the locus of the vertices.

29. Upon a given base to describe an isosceles triangle, which shall be equal to a given triangle.

30. Given the base and one side of a triangle, to find the third side, so that the area may be the greatest possible.

31. Describe a right-angled triangle upon a given hypothenuse, so that the hypothenuse and one side shall be together double of the third side.

32. Having given two lines, which are not parallel, and a point between them; describe a triangle having two of its angles in the respective lines, and the third at the given point; and such that the sides shall be equally inclined to the lines which they meet.

33. Bisect a triangle by a straight line drawn parallel to one of its sides.

34. Bisect a triangle by a straight line drawn through a point within or without the triangle.

35. It is required to bisect any triangle by a line perpendicular to the base.

36. It is required to determine a point within a given triangle, from which lines drawn to the several angles, will divide the triangle into three equal parts.

37. Bisect a parallelogram, (1) by a line drawn from a point in one of its sides: (2) by a line drawn from a given point within or without it: (3) by a line perpendicular to one of the sides.

38. To bisect a trapezium (1) by a line drawn from one of its angular points: (2) by a line drawn from a given point in one side.

39. Divide a triangle into three equal parts, (1) by lines drawn from a point in one of the sides: (2) by lines drawn from the angles to a point within the triangle: (3) by lines drawn from a given point within the triangle. In how many ways can the third case be done?

40. To trisect a parallelogram by lines drawn from a given point in one of its sides.

41. To divide a parallelogram into four equal portions by straight lines drawn from a given point in one of its sides.

42. To divide a square into four equal portions by three straight lines drawn from any point in one of its sides.

43. From a given isosceles triangle, cut off a trapezium which shall have the same base as the triangle, and shall have its remaining three sides equal to each other.

44. Divide an equilateral triangle into nine equal parts.

45. To find a point in the side or side produced of any parallelogram, such that the angle it makes with the line joining the point and one extremity of the opposite side, may be bisected by the line joining it with the other extremity.

46. Find a point in the diagonal of a square produced, from which if a straight line be drawn parallel to any side of the square, and meeting another side produced, it will form together with the produced diagonal and produced side, a triangle equal to the square.

47. A trapezium is such, that the perpendiculars let fall on a diagonal from the opposite angles are equal. Divide the trapezium into four equal triangles, by straight lines drawn to the angles from a point within it.

48. Given one side of a right-angled triangle, and the difference between the hypothenuse and the sum of the other two sides, to construct the triangle.

49. In a right-angled triangle, given the sums of the base and the hypothenuse, and of the base and the perpendicular; to determine the triangle.

50. Given half the perimeter and the vertical angle of an isosceles triangle, it is required to find the sides.

51. Given the perimeter and the angles of a triangle, to construct it.

52. Given one of the angles at the base of a triangle, the base itself, and the sum of the two remaining sides, to construct the triangle.

53. Given the base, an angle adjacent to the base, and the difference of the sides of a triangle, to construct it.

54. Given one angle, a side opposite to it, and the difference of the other two sides; to construct the triangle.

55. Given the base, the perpendicular and the sum of the sides; to construct the triangle.

56. Given the base, the altitude, and the difference of the two remaining sides; construct the triangle.

57. To find a point in the base of a triangle, such that if perpendiculars be drawn from it upon the sides, their sum shall be equal to a given line.

58. Determine the locus of the vertices of all the equal triangles, which can be described on the same base, and upon the same side of it.

59. To describe a square upon a given straight line as a diameter.

60. Shew how the squares upon the sides of a right-angled triangle may be dissected, so as exactly to cover the square of the hypothenuse.

61. Find a square equal to 3, 5, or any number of squares.

62. Construct a square whose area shall be 8 or n times that of a given square.

63. Construct all right-angled triangles whose sides shall be rational, upon a given line as their base.

64. Describe a square which shall be equal to the difference between two given squares.

65. In a right-angled triangle, it is required to find analytically the base and perpendicular, their difference being 1, and the hypothenuse equal to 5.

66. Any two parallelograms having been described upon two sides of a given triangle, apply to the third side a parallelogram equal to their sum.

67. Given a square of one inch, shew how a rhombus may be constructed, whose area shall be equal to it, and each of its sides a mile long.

THEOREMS.

2. In the fig. I. 5. If FC and BG meet in H, then prove that AH bisects the angle BAC.

3. In the fig. I. 5. If the angle FBG be equal to the angle ABC, and BG, CF intersect in O; the angle BOF is equal to twice the angle BAC.

4. If a straight line drawn bisecting the vertical angle of a triangle, also bisects the base, the triangle is isosceles.

5. In the base BC of an isosceles triangle ABC take a point D, and in CA take CE equal to CD, let ED produced meet AB produced in F; then $3 . AEF = 2$ right angles $+ AFE$.

6. The difference between any two sides of a triangle is less than the third side.

7. ABC is a triangle right-angled at B, and having the angle A double the angle C; shew that the side AC is less than double the side AB.

8. The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

9. If from the right angle of a right-angled triangle two straight lines be drawn, one perpendicular to the base, and the other bisecting it, they will contain an angle equal to the difference of the two acute angles of the triangle.

10. If one angle at the base of a triangle be double of the other, the less side is equal to the sum or difference of the segments of the base made by the perpendicular from the vertex, according as the angle is greater or less than a right angle.

11. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

12. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines a line be drawn to the opposite angle of the triangle, it will bisect that angle.

13. In an obtuse-angled triangle if perpendiculars be drawn from the points bisecting the sides, prove that they all will pass through the same point.

14. In an obtuse-angled triangle, if perpendiculars be drawn from the angles to the opposite sides, produced if necessary, they will pass through the same point: required a proof.

15. Let ACB be a triangle; and let AD, CG, BE respectively bisect the exterior angles HAE, ECB, CBG, and meet BC, AB, AC produced in the points D, E, G. It is required to demonstrate that these three points are in the same straight line.

16. If two sides of a triangle be produced, the three straight lines which bisect the two exterior angles and the third interior angle shall all meet in the same point.

17. ABC is a triangle in which the angle ABC is twice the angle ACB; shew that if the point D in BC which divides it into segments, whose difference is equal to the side opposite to the angle ACD, be joined with the point A, AD is perpendicular to BC.

18. In Prop. 35, Book I, shew that the two parallelograms can be

divided into the same number of triangles which are actually equal, each to each, so that the divided figures may be superposed.

19. If from the vertex of a triangle, two straight lines be drawn to the base, one bisecting the vertical angle, and the other bisecting the base, prove that the latter is the greater of the two straight lines.

20. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, another bisecting the base, and the third perpendicular to the base, the first is always intermediate in magnitude and position to the other two.

21. In a right-angled isosceles triangle, the lines drawn from any of the angles to the opposite angle of the square described upon the opposite side are all equal.

22. Shew that the perimeter of the triangle, formed by joining the feet of the perpendiculars dropped from the angles upon the opposite sides of a triangle, is less than the perimeter of any other triangle, whose angular points are on the sides of the first.

23. From a given point there can be drawn only two equal straight lines to a given line, one on each side of the shortest line; and the shortest line is the perpendicular.

24. A, B are two fixed points: if two straight lines AC, BC be drawn making a given angle C, prove that the straight line bisecting C passes through a fixed point, and determine the point geometrically.

25. From every point of a given straight line, the straight lines drawn to each of two given points on opposite sides of the line are equal: prove that the line joining the given points will cut the given line at right angles.

26. From a given point without the angle contained by two straight lines given in position, draw a straight line in such a direction that the part of it intercepted between the given point and the nearest straight line, shall be equal to the part intercepted between the two straight lines.

27. If a straight line be drawn from a given point, and making a given angle with a given straight line, its length and the points of its intersection with the given line are given.

28. Shew that if there be two rectilinear figures on the same base, one of which wholly envelopes the other, the perimeter of the enveloping figure is greater than the perimeter of the other.

29. Shew that we may draw to a point within a triangle two straight lines which shall be greater than the sides of the triangle, if one of these two straight lines be *not* terminated in the extremity of the base.

30. If parallel lines be defined to be "lines in the same plane which make the same angle with any straight line which meets them," prove the following propositions respecting them.

(*a*) The alternate angles are equal.

(*b*) The two interior angles on the same side of the cutting line are equal to two right angles.

(*c*) Parallel lines never meet however far they are produced.

31. Can it be properly predicated of any two straight lines that they never meet if indefinitely produced either way, antecedently to our knowledge of some other property of such lines which makes the porperty first predicated of them a necessary conclusion from it?

32. Lines which are perpendicular to parallel lines are also parallel.

33. If the line joining two parallel lines be bisected, all the lines

drawn through the point of bisection and terminated by the parallel lines are also bisected in that point.

34. A quadrilateral figure whose sides are equal will be a parallelogram.

35. If the opposite angles of a quadrilateral figure be equal the opposite sides will be equal and parallel.

36. The diagonals of a parallelogram bisect each other.

37. The perimeter of a square is less than that of any other parallelogram of equal area.

38. Any straight line which bisects the diagonal of a parallelogram will also bisect the parallelogram; and no straight line can bisect a parallelogram unless it cut or meet the opposite sides.

39. The diagonals of a square and of a rhombus bisect each other at right angles.

40. If from any point in the diagonal of a parallelogram straight lines be drawn to the angles, the parallelogram will be divided into two pairs of equal triangles.

41. $ABCD$ is a parallelogram of which the angle C is opposite to the angle A. If through A any straight line be drawn, then the distance of C is equal to the sum or difference of the distances of B and of D from that straight line according as it lies without or within the parallelogram.

42. If in a parallelogram two lines be drawn parallel to adjacent sides, and meeting the other sides of the figure, the lines joining their extremities, if produced, will meet the diameter in the same point.

43. $ABCD$ is a parallelogram; draw the diagonal BC, and from D draw DE at right angles to BC, then if perpendiculars be drawn from B and C, they shall intersect in the line DE, produced if necessary.

44. If $ABCD$ be a parallelogram, and E any point in the diagonal AC, or AC produced; shew that the triangles EBC, EDC are equal.

45. If from a point without a parallelogram, lines be drawn to the extremities of two adjacent sides, and of the diagonal which they include. Of the triangles thus formed, that, whose base is the diagonal, is equal to the sum of the other two.

46. It is impossible to divide a quadrilateral figure (except it be a parallelogram) into equal triangles by lines drawn from a point within it to its four corners.

47. If of the four triangles into which the diagonals divide a trapezium, any two opposite ones are equal, the trapezium has two of its opposite sides parallel.

48. If two sides of a trapezium be parallel, the triangle contained by either of the other sides and the two straight lines drawn from its extremities to the bisection of the opposite sides is equal to half the trapezium.

49. The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles, from any point within the figure, except their intersection.

50. When the corner of a leaf of a book is turned down a second time, so that the lines of folding are parallel and equidistant, the space in the second fold is equal to three times that in the first.

51. If the sides of a quadrilateral figure be bisected and the points of bisection joined, the included figure is a parallelogram, and equal in area to half the original figure.

52. Along the sides of a parallelogram $ABCD$ taken in order, measure $AA' = BB' = CC' = DD'$; the figure $A'B'C'D'$ will be a parallelogram.

53. If the points of bisection of the sides of a triangle be joined, the triangle so formed shall be one-fourth of the given triangle.

54. Prove that two lines drawn to bisect the opposite sides of a trapezium will also bisect each other.

55. Upon stretching two chains AC, BD, across a field $ABCD$, I find that BD and AC make equal angles with DC, and that AC makes the same angle with AD, that BD does with BC; hence prove that AB is parallel to CD.

56. If a line intercepted between the extremity of the base of an isosceles triangle, and the opposite side (produced if necessary) be equal to a side of the triangle, the angle formed by this line and the base produced is equal to three times either of the equal angles of the triangle.

57. AD, BC are two parallel straight lines, cut obliquely by AB and perpendicularly by AC; BED is drawn cutting AC in E so that ED is equal to twice BA; prove that the angle DBC is equal to one-third of the angle ABC.

58. AB, BC, DE, EF are rods joined at B, F, E, and D, capable of angular motion in the same plane, and so placed that $FBDE$ is a parallelogram. If, when the rods are in any given position, points A, G and C be taken in the same line, shew that these points will always be in the same line, whatever be the angle the rods make with each other.

59. If upon the sides of a triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of the parallelograms will intersect in a point.

60. Prove that the perimeter of an isosceles triangle is greater than that of an equal rectangle of the same altitude.

61. If the areas of any triangle and of a square be equal, the perimeter of the triangle will be the greater.

62. If from the extremities A and B of the base of any triangle ABC, and on the same side of it, two straight lines AD, BE be drawn perpendicular to the base, each being double the altitude of the triangle, and straight lines DF, EG be drawn from D and E to the middle points of AC, BC; the sum or difference of the triangles ADF, BEG will be equal to the triangle ABC according as the angles, at the base of the latter, are acute or one of them is obtuse.

63. The perimeter of an isosceles triangle is less than that of any other equal triangle upon the same base.

64. Of all triangles having the same base and the same perimeter, that is the greatest which has the two undetermined sides equal.

65. Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in the given point.

66. If from the base to the opposite sides of an isosceles triangle three straight lines be drawn, making equal angles with the base, viz. one from its extremity, the other two from any other point in it, these two shall be together equal to the first.

67. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides, the angles made by them with the base are each equal to half the vertical angle.

68. If each of the equal angles of an isosceles triangle be one-fourth of the third angle, and from one of them a perpendicular be drawn to the base meeting the opposite side produced; then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

69. If two sides of a triangle be given, the triangle will be greatest when they contain a right angle.

70. The area of any two parallelograms described on the two sides of a triangle is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two sides of the former parallelograms produced to meet.

71. In the figure to Prop. 47, Book I,

(a) If BG and CH be joined, those lines will be parallel.

(b) If FC and BK be joined, they will cut off equal portions AD, AE.

(c) If perpendiculars be let fall from F and K on BC produced, the parts produced will be equal; and the perpendiculars together will be equal to BC.

(d) Shew that AL, BK, CF, intersect each other in the same point.

(e) Join GH, KE, FD, and prove that each of the triangles so formed equals the given triangle ABC.

(f) The sum of the squares of GH, KE, and FD will be equal to eight times the square of the hypothenuse.

(g) If the exterior angular points of the squares be joined, an irregular hexagon will be formed, whose area is equal to the area of the square described upon the hypothenuse of a right-angled triangle, one of whose sides is equal to the hypothenuse of the original triangle, and the other is equal to the sum of its sides.

72. A point is taken within a square, and straight lines drawn from it to the angular points of the square, and perpendicular to the sides; the squares on the first are double the sum of the squares on the last. Shew that these sums are least when the point is in the centre of the square.

73. If from the vertex of a plane triangle, a perpendicular fall upon the base or the base produced, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.

74. If from the middle point of one of the sides of a right-angled triangle a perpendicular be drawn to the hypothenuse, the difference of the squares of the segments into which it is divided, is equal to the square of the other side.

75. If a straight line be drawn from one of the acute angles of a right-angled triangle, bisecting the opposite side, the square upon that line is less than the square upon the hypothenuse by three times the square upon half the side bisected.

76. If one angle of a triangle be equal to a right angle, and another equal to two-thirds of a right angle, prove from Euclid, Book I, that the equilateral triangle described on the hypothenuse, is equal to the sum of the equilateral triangles described upon the sides which contain the right angle.

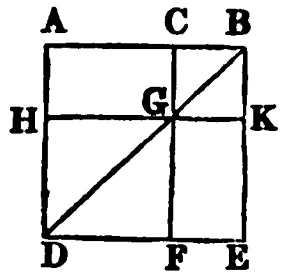
GEOMETRICAL EXERCISES ON BOOK II.

THEOREM I.

The square of the excess of one straight line above another is less than he squares of the two straight lines by twice their rectangle.

Let AB, BC be the two straight lines, whose difference is AC.

Then the square of AC is less than the squares of AB and BC by wice the rectangle contained by AB and BC.

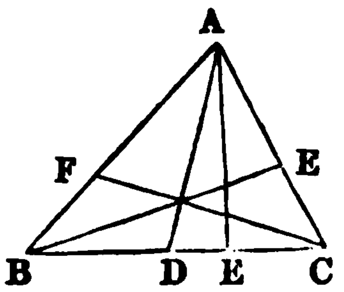


Constructing as in Prop. 4. Book II,
Because the complement AG is equal to GE,
add to each CK,
therefore the whole AK is equal to the whole CE;
and AK, CE together are double of AK;
but AK, CE are the gnomon AKF and CK,
and AK is the rectangle contained by AB, BC;
therefore the gnomon AKF and CK are equal to twice the rectangle AB, BC.
But AE, CK are equal to the squares of AB, BC;
hence taking the former equals from these equals,
therefore the difference of AE, and the gnomon AKF is equal to he difference between the squares AB, BC, and twice the rectangle AB, BC;
but the difference AE and the gnomon AKF is the figure HF which is equal to the square of AC.
Wherefore the square of AC is equal to the difference between the quares AB, BC, and twice the rectangle AB, BC.

THEOREM II.

If straight lines be drawn from each angle of a triangle bisecting the pposite side, four times the sum of the squares of these lines is equal to hree times the sum of the squares of the sides of the triangle.

Let ABC be any triangle, and let AD, BE, CF be drawn from A, B, C, to D, E, F, the bisections of the opposite sides of the triangle: draw AE perpendicular to BC.



Then the square of AB is equal to the squares of BD, DA together with twice the rectangle BD, DE, (II. 12.)

and the square of AC is equal to the squares of CD, DA diminished by twice the rectangle CD, DE; (II. 13.)

therefore the squares of AB, AC are equal to twice the square of BD, and twice the square of AD; for DC is equal to BD:

and twice the squares of AB, AC are equal to the square of BC, and four times the square of AD: for BC is twice BD.

Similarly, twice the squares of AB, BC are equal to the square of AC, and four times the square of BE:

and twice the squares of BC, CA are equal to the square of AB, and four times the square of FC:

hence, by adding these equals,

four times the squares of AB, AC, BC are equal to four times the squares of AD, BE, CF together with the squares of AB, AC, BC:

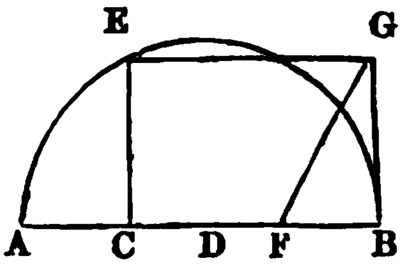
and taking the squares of AB, AC, BC from these equals,

therefore three times the squares of AB, AC, BC are equal to four times the squares of AD, BE, CF.

PROBLEM I.

Divide a given straight line into two such parts, that the rectangle contained by them may be three-fourths of the greatest which the case will admit.

Analysis. Let AB be the given line, and let AB be bisected in D: then the rectangle AD, DB, or the square of DB is the greatest possible rectangle.



Let C be the point required, such that the rectangle AC, CB is equal to three-fourths of the square of DB.

On AB describe a semicircle, and draw CE perpendicular to AB, then the square of CE is equal to the rectangle AC, CB.

Again, bisect DB in F; on FB describe the right-angled triangle FBG, having the hypothenuse FG equal to DB.

Then BG is the line, the square of which is three-fourths of the square of DB.

Hence GB is equal to EC: join GE.

Therefore the point E, and also the point C is found.

Synthesis. Bisect AB in D and DB in F: on FB make the right-angled triangle FBG having the hypothenuse FG equal to DB.

On AB describe a semicircle, and through G draw GE parallel to AB, and draw EC parallel to GB.

Then C is the point required, such that the rectangle contained by AC, CB is equal to three-fourths of the square of half the line AB.

PROBLEMS.

2. Divide a straight line into two parts, such, that their rectangle ay be equal to a given square; and determine the greatest square iat the rectangle can equal.

3. Divide a straight line into two such parts that the difference the squares of the two parts shall be equal to twice the rectangle ntained by them.

4. Divide a straight line into two parts such, that the rectangle ntained by them may be equal to the square of their difference.

5. To divide a straight line so that the rectangle under its segents may equal a given rectangle.

6. Divide a given straight line so that the rectangle under the irts may be equal to a given square, and point out the limit which the de of the given square must not exceed so that the problem may be ossible.

7. Divide a given straight line into two parts, such that the uares of the whole line and one of the parts shall be equal to twice ie square of the other part.

8. Divide a given line, so that the square of the greater part ay equal twice the rectangle of the whole and the less part.

9. Divide algebraically a given line (a) into two parts, such that ie rectangle contained by the whole and one part may be equal to the uare of the other. Deduce Euclid's construction from one solution, id explain the other.

10. Divide a given straight line into three parts, such that the uare of the whole line may be equal to the squares of the extreme arts together with twice the rectangle contained by the whole and the iiddle part.

11. To produce a straight line AB to C, so that the rectangle coniined by the sum and difference of AB and AC may be equal to a given uare.

12. If a straight line be divided into any two parts, produce it so iat the rectangle contained by the whole line produced and the part roduced, may be equal to the rectangle contained by the given line id one segment.

13. Construct a rectangle that shall be equal to a given square id the difference of whose adjacent sides shall be equal to a given line.

14. Construct a rectangle equal to a given square, and having the um of its sides equal to a given line.

15. Find a square which shall be equal to the sum of two given ectilineal figures.

16. Shew how to divide a given rectangle into parts which ogether will form a rectangle of any proposed length.

17. A given line AB is divided into two parts in the point C. 'ind the position of any point D above the line, so that the sum of he squares on the lines DA, DC, DB may be equal to the square on AB, diminished by the rectangle AC, AB.

18. Construct a triangle with three sides a, b and c, such that $b=c^2$, and $a+b=4c$.

THEOREMS.

3. The area of a rhombus is equal to half the rectangle contained by the diagonals.

4. Prove that the area of a trapezium whose bases are parallel is half of the rectangles contained by each of the bases, and the perpendicular distance.

5. The area of any right-angled triangle is equal to the rectangle of the semiperimeter and excess of the semiperimeter above the hypothenuse. Required proof.

6. Any rectangle is the half of the rectangle contained by the diameters of the squares on its two sides.

7. The sum of the squares of two lines is never less than twice their rectangle.

8. If a straight line be divided into two equal and into two unequal parts, the squares of the two unequal parts are equal to twice the rectangle contained by the two unequal parts, together with four times the square of the line between the points of section.

9. If the points C, D be equidistant from the extremities of the straight line AB, shew that the squares constructed on AD and AC exceed twice the rectangle AC, AD by the square constructed on CD.

10. In a right-angled triangle, the square on that side which is the greater of the two containing the right angle is equal to the rectangle by the sum and difference of the other sides.

11. Shew that the first of the algebraical propositions,

$$(a+x)(a-x)+x^2=a^2,$$

$$(a+x)^2+(a-x)^2=2a^2+2x^2,$$

is equivalent to the two Propositions v and vi, and the second of them to the two Propositions ix and x of the second book of Euclid.

12. Shew how in all the possible cases, a straight line may be *geometrically* divided into two such parts, that the sum of their squares shall be equal to a given square.

13. $ABCD$ is a rectangular parallelogram, of which A, C are opposite angles, E any point in BC, F any point in CD. Prove that the area of the triangle AEF together with the rectangle BE, DF is equal to the parallelogram AC.

14. A, B, C, D are four points in the same straight line, E a point in that line equally distant from the middle of the segments AB, CD; F is any other whatever in AD; then $AF^2+BF^2+CF^2+DF^2 = AE^2+BE^2+CE^2+DE^2+4EF^2$.

15. The sum of the perpendiculars let fall from any points within an equilateral triangle, will be equal to the perpendicular let fall from one of its angles upon the opposite side.

Is this proposition true when the point is in one of the sides of the triangle?

In what manner must the proposition be enunciated when the point is without the triangle?

16. If a line AB be divided into two parts AC and CB in the point C (Prop. 11, Book II.), so that the rectangle $AB\times BC=AC^2$; and if AC be divided in D, so that $CD=BC$, prove that $AC\times AD=BC^2$.

17. All plane rectilineal figures admit of quadrature. Point out the succession of steps by which Euclid establishes the truth of this proposition.

18. The hypothenuse (c) of a right-angled triangle ABC is trisected in the points D, E; prove that if CD, CE be joined, the sum of the squares of the sides of the triangle $CDE = \frac{2}{3} \cdot c^2$.

19. If from the right angle C of a right-angled triangle ABC, straight lines be drawn to the opposite angles of the square described on the hypothenuse AB; shew that the difference of the squares described on these lines is equal to the difference of the squares described on the two sides AC, BC.

20. If the sides of the triangle be as the numbers 2, 4, 5, shew whether it will be acute or obtuse-angled.

21. If an angle of a triangle be two-thirds of two right angles, shew that the square of the side subtending that angle is equal to the squares of the sides containing it, together with the rectangle contained by those sides.

22. In any triangle the squares of the two sides are together double of the two squares of half the base and of the straight line joining its bisection with the opposite angle.

23. The square described on a straight line drawn from one of the angles at the base of a triangle to the middle point of the opposite side, is equal to the sum or difference of the square of half the side bisected and the rectangle contained between the base and that part of it, or of it produced, which is intercepted between the same angle and a perpendicular drawn from the vertex.

24. If perpendiculars be drawn from the extremities of the base of a triangle on a straight line which bisects the angle opposite to the base, the area of the triangle is equal to the rectangle contained by either of the perpendiculars and the segment of the bisecting line between the angle and the other perpendicular.

25. Upon the sides AB, BC, CA of the triangle ABC, or upon these produced, let fall the perpendiculars DE, DF, DG, from the point D within or without the triangle. Then $AE^2 + BF^2 + GC^2 = BE^2 + CF^2 + AG^2$. Required a demonstration.

26. If from the three angles of a triangle, lines be drawn to the points of bisection of the opposite sides, the squares of the distances between the angles and the common intersection are together one-third of the squares of the sides of the triangle.

27. Prove that the square of any straight line drawn from the vertex of an isosceles triangle to the base, is less than the square of a side of the triangle by the rectangle contained by the segments of the base.

28. If from one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, the rectangle contained by that side and the segment of it intercepted between the perpendicular and base is equal to half the square described upon the base.

29. If in an isosceles triangle a perpendicular be let fall from one of the equal angles to the opposite side, the square of the perpendicular is equal to the square of the line intercepted between the other equal angle and the perpendicular, together with twice the rectangle contained by the segments of that side.

30. The square on the base of an isosceles triangle whose ver-

tical angle is a right angle, is equal to four times the area of the triangle.

31. If ABC be an isosceles triangle, and CD be drawn perpendicular to AB; the sum of the squares of the three sides $= AD^2 + 2.BD^2 + 3.CD^2$.

32. If ABC be an isosceles triangle, and DE be drawn parallel to the base BC, and EB joined; prove that $BE^2 = BC \times DE + CE^2$.

33. If ABC be an isosceles triangle of which the angles at B and C are each double of A; then the square of AC is equal to the square of BC together with the rectangle contained by AC and BC.

34. Shew that in a parallelogram the squares of the diagonals are equal to the sum of the squares of all the sides.

35. If $ABCD$ be any rectangle, A and C being opposite angles, and O any point either within or without the rectangle: $OA^2 + OC^2 = OB^2 + OD^2$.

36. In any quadrilateral figure, the sum of the squares of the diagonals together with four times the square of the line joining their middle points is equal to the sum of the squares of all the sides.

37. In any trapezium, if the opposite sides be bisected, the sum of the squares of the two other sides, together with the squares of the diagonals, is equal to the sum of the squares of the bisected sides together with four times the square of the line, joining the points of bisection.

38. The squares of the diagonals of a trapezium are together double the squares of the two lines joining the bisections of the opposite sides.

39. In any trapezium two of whose sides are parallel, the squares of the diagonals are together equal to the squares of its two sides which are not parallel, and twice the rectangle contained by the sides which are parallel.

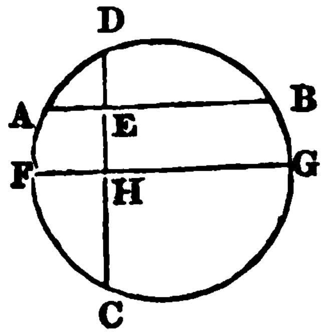
40. If squares be described on the sides of any triangle and the angular points of the squares be joined; the sum of the squares of the sides of the hexagonal figure thus formed is equal to four times the sum of the squares of the sides of the triangle.

GEOMETRICAL EXERCISES ON BOOK III.

THEOREM I.

If AB, CD *be chords of a circle at right angles to each other, prove ɩat the sum of the arcs* AC, BD *is equal to the sum of the arcs* AD, BC. ⁴rchimedis, Lemm. Prop. 9.)

Draw the diameter FHG parallel to AB, and cutting CD in H.

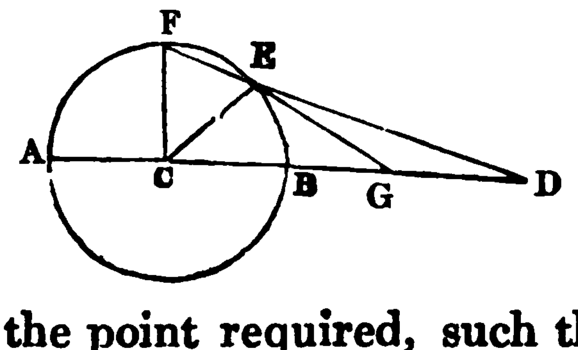


Then the arcs FDG and FCG are each half the circumference.
Also since CD is bisected in the point H,
the arc FD is equal to the arc FC,
ıd the arc FD is equal to the arcs FA, AD, of which, AF is equal to BG,
therefore the arcs AD, BG are equal to the arc FC;
add to each CG,
ıerefore the arcs AD, BC are equal to the arcs FC, CG which make up the half circumference.
Hence also the arcs AC, DB are equal to half the circumference.
Wherefore the arcs AD, BC are equal to the arcs AC, DB.

PROBLEM I.

The diameter of a circle having been produced to a given point, it ɩ required to find in the part produced a point, from which if a tanȝent be drawn to the circle, it shall be equal to the segment of the part ɹroduced, that is, between the given point and the point found.

Analysis. Let AEB be a circle whose centre is C and whose ŀiameter AB is produced to the given point D.



Suppose that G is the point required, such that the segment GD is ·qual to the tangent GE drawn from G to touch the circle in E.
Join DE and produce it to meet the circumference again in F
join also CE and CF.
Then in the triangle GDF, because GD is equal to GE,
therefore the angle GED is equal to the angle GDE;
and because CE is equal to CF,
the angle CEF is equal to the angle CFE;

therefore the angles *CEF*, *GED* are equal to the angles *CFE*, *GDE*:
but since *GE* is a tangent at *E*,
therefore the angle *CEG* is a right angle, (III. 18.)
hence the angles *CEF*, *GEF* are equal to a right angle,
and consequently, the angles *CFE*, *EDG* are also equal to a right angle,
wherefore the remaining angle *FCD* of the triangle *CFD* is a right angle.
and therefore *CF* is perpendicular to *AD*.

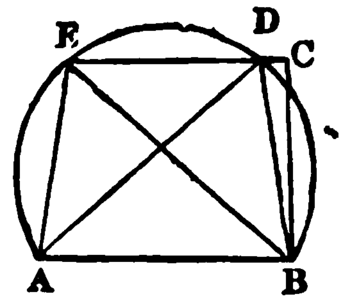
Synthesis. From the centre *C*, draw *CF* perpendicular to *AD* meeting the circumference of the circle in *F*:
join *DF* cutting the circumference in *E*,
join also *CE*, and at *E* draw *EG* perpendicular to *CE* and intersecting *BD* in *G*.
Then *G* will be the point required.

For in the triangle *CFD*, since *FCD* is a right angle, the angles *CFD*, *CDF* are together equal to a right angle;
also since *CEG* is a right angle,
therefore the angles *CEF*, *GED* are together equal to a right angle;
therefore the angles *CEF*, *GED* are equal to the angles *CFD*, *CDF*;
but because *CE* is equal to *CF*,
the angle *CEF* is equal to the angle *CFD*,
wherefore the remaining angle *GED* is equal to the remaining angle *CDF*,
and the side *GD* is equal to the side *GE* of the triangle *EGD*,
therefore the point *G* is determined according to the required conditions.

PROBLEM II.

Given the base, the vertical angle, and the perpendicular in a plane triangle, to construct it.

Upon the given base *AB* describe a segment of a circle containing an angle equal to the given angle. (III. 33.)



At the point *B* draw *BC* perpendicular to *AB*, and equal to the altitude of the triangle. (I. 11, 3.)

Through *C* draw *CDE* parallel to *AB*, and meeting the circumference in *D* and *E*. (I. 31.)
Join *DA*, *DB*; also *EA*, *EB*.
Then *EAB* or *DAB* is the triangle required.

It is also manifest, that if *CDE* touch the circle, there will be only one triangle which can be constructed on the base *AB* with the given altitude.

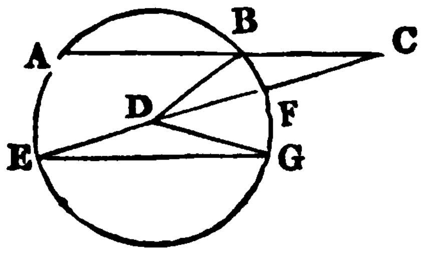
THEOREM II.

If a chord of a circle be produced till the part produced be equal to the radius, and if from its extremity a line be drawn through the centre and meeting the convex and concave circumferences, the convex is one third of the concave circumference. (Archimedis, Lemm. Prop. 8.)

Let AB any chord be produced to C, so that BC is equal to the radius of the circle:

and let CE be drawn from C through the centre D, and meeting the convex circumference in F, and the concave in E.

Then the arc BF is one third of the arc AE,



Draw EG parallel to AB, and join DB, DG.

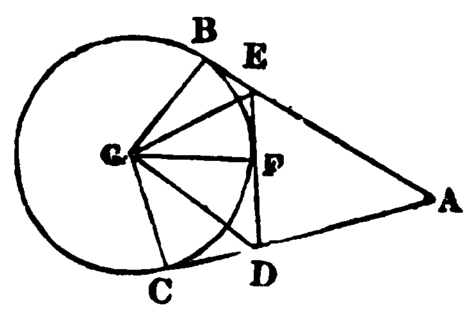
Since the angle DEG is equal to the angle DGE; (I. 5.)
and the angle GDF is equal to the angles DEG, DGE, (I. 32.)
therefore the angle GDC is double of the angle DEG.
But the angle BDC is equal to the angle BCD, (I. 5.)
and the angle CEG is equal to the alternate angle ACE; (I. 29.)
therefore the angle GDC is double of the angle CDB,
add to these equals the angle CDB,
therefore the whole angle GDB is treble of the angle CDB,
but the angles GDB, CDB at the centre D, are subtended by the arcs BF, BG, of which BG is equal to AE.
Wherefore the circumference AE is treble of the circumference BF, and BF is one third of AE.

Hence may be solved the following problem:

AE, BF are two arcs of a circle intercepted between a chord and a given diameter. Determine the position of the chord, so that one arc shall be triple of the other.

THEOREM III.

AB, AC *and* ED *are tangents to the circle* CFB; *at whatever point between* C *and* B *the tangent* EFD *is drawn, the three sides of the triangle* AED *are equal to twice* AB *or twice* AC: *also the angle subtended by the tangent* EFD *at the centre of the circle, is a constant quantity.*

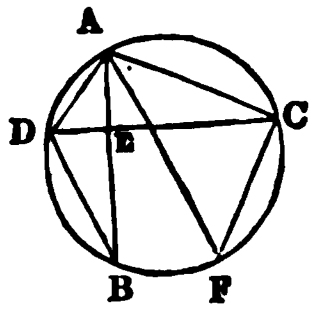


Take G the centre of the circle, and join GB, GE, GF, GD, GC.
Then EB is equal to EF, and DC to DF; (III. 37.)
therefore ED is equal to EB and DC;
to each of these add AE, AD,
wherefore AD, AE, ED are equal to AB, AC;
and AB is equal to AC,
therefore AD, AE, ED are equal to twice AB, or twice AC;
or the perimeter of the triangle AED is a constant quantity.
Again, the angle EGF is half of the angle BGF,
and the angle DGF is half of the angle CGF,
therefore the angle DGE is half of the angle CGB,
or the angle subtended by the tangent ED at G, is half of the angle contained between the two radii which meet the circle at the points where the two tangents AB, AC meet the circle.

THEOREM IV.

If two chords of a circle intersect each other at right angles, the sum of the square described upon the four segments is equal to the square described upon the diameter. (Archimedis, Lemm. Prop. 11.)

Let the chords AB, CD intersect at right angles in E.



Draw the diameter AF, and join AC, AD, CF, DB.
Then the angle ACF in a semicircle is a right angle, (III. 31.)
and equal to the angle AED: also the angle ADC is equal to the angle AFC. (III. 21.)
Hence in the triangles ADE, AFC, there are two angles in the one respectively equal to two angles in the other;
consequently, the third angle CAF is equal to the third angle DAB,
therefore the arc DB is equal to the arc CF, (III. 26.)
and therefore also the chord DB is equal to the chord CF. (III. 29.)
Because AEC is a right-angled triangle,
the squares of AE, EC are equal to the square of AC; (I. 47.)
similarly, the squares of DE, EB are equal to the square of DB;
therefore the squares of AE, EC, DE, EB, are equal to the squares of AC, DB;
but DB was proved equal to FC,
and the squares of AC, FC are equal to the square of AF,
wherefore the squares of AE, EC, DE, EB, are equal to the square of AF, the diameter of the circle.

PROBLEMS.

3. Given the centre of a circle; find its diameter by means of the compasses alone.

4. Through a given point within a circle, to draw a chord which shall be bisected in that point.

5. Through a point in a circle which is not the centre, to draw the least chord.

6. To draw that diameter of a given circle which shall pass at a given distance from a given point.

7. Draw through one of the points in which any two circles cut one another, a straight line which shall be terminated by their circumferences and bisected in their point of section.

8. Determine the distance of a point from the centre of a given circle, so that if tangents be drawn from it to the circle, the concave part of the circumference may be double of the convex.

9. Find two points in a given straight line from each of which if two tangents be drawn to two given points on the same side of the given circle, they shall make an angle equal to a given angle.

Is any limitation required?

10. Find a point without a given circle, such that the sum of the two lines drawn from it touching the circle, shall be equal to the line drawn from it through the centre to meet the circle.

11. From a given point without a circle, a straight line is drawn cutting a circle. Draw from the same point another line so as to intercept two arcs which together shall subtend an angle equal to a given angle.

12. In a chord of a circle produced, it is required to find a point, from which if a straight line be drawn touching the circle, the line so drawn shall be equal to a given straight line.

13. Determine the point without a circle, from which, if two straight lines be drawn touching the circle, they may form an equilateral triangle with the chord which joins the points of contact.

14. *A, B, C,* are three given points, find the position of a circle such that all the tangents to it drawn from the points *A, B, C* shall be equal to one another. What is that circle which is the superior limit to those that satisfy the above condition?

15. Two parallel chords in a circle are respectively six and eight inches in length, and are one inch apart; how many inches is the diameter in length?

16. The radius of the circle *ABDE* whose centre is *C*, is equal to five inches. The distance of the line *AB* from the centre is four inches. The distance of the line *DE* from the centre is three inches. Required the lengths of the straight lines *AB, DE.*

17. Required the locus of the vertices of all triangles upon the same base, having the sum of the squares of their sides equal to a given square.

18. Draw a straight line which shall touch a given circle, and make a given angle with a given straight line.

19. Draw a straight line which shall touch two given circles: (1) on the same side; (2) on the alternate sides.

20. Through a given point without a given circle, draw a straight line which shall cut off a quadrantal arc of that circle.

21. Three given straight lines are in the same straight line; find a point from which lines drawn to their extremities shall contain equal angles.

22. Draw through a given point in the diameter of a circle a chord, which shall form with the lines joining its extremities with either extremity of the diameter, the greatest possible triangle.

23. ADB, ACB, are the arcs of two equal circles cutting one another in the straight line AB, draw the chord ACD cutting the inner circumference in C and the outer in D, such that AD and DB together may be double of AC and CB together.

24. From a given point without a given circle a line is drawn cutting the circle. It is required to draw from the same point another line also cutting the circle, so that the sum of the arcs intercepted between these two lines shall be equal to a given arc.

25. A given straight line being divided in a given point, to find a point at which each segment of the given straight line shall subtend an angle equal to half a right angle.

26. Divide a circle into two parts such that the angle contained in one segment shall equal twice the angle contained in the other.

27. Any segment of a circle being described on the base of a triangle; to describe on the other sides segments similar to that on the base.

28. Through a given point within or without a circle, it is required to draw a straight line cutting off a segment containing a given angle.

29. Through two given points to describe a circle bisecting the circumference of a given circle.

30. A segment of a circle being described upon AB, it is required to draw a chord AC, such that CK being drawn perpendicular to AB, $AC + CK$ shall be a maximum.

31. In the circumference of a given circle, to determine a point to which two straight lines drawn to two given points shall contain an angle equal to a given angle, pointing out the limitations within which the problem is possible.

32. One side of a trapezium capable of being inscribed in a given circle is given, the sum of the remaining three sides is given; and also one of the angles opposite to the given side: construct it.

33. To find a point P, so that tangents drawn from it to the outsides of two equal circles which touch each other, may contain an angle equal to a given angle.

34. Given two circles: it is required to find a point from which tangents may be drawn to each, equal to two given straight lines.

35. Between two given circles to place a straight line terminated by them, such that it shall equal a given straight line, and be inclined at a given angle to the straight line joining their centres.

36. Two circles being given in position and magnitude, draw a straight line cutting them, so that the chords in each circle may be equal to a given line not greater than the diameter of the smaller circle.

37. Describe two circles with given radii which shall cut each other, and have the line between the points of section equal to a given line.

38. If two circles cut each other; to draw from one of the points of intersection a straight line meeting the circles, so that the part of

it intercepted between the circumferences may be equal to a given line.

39. Three points being in the same plane, find a fourth, where lines drawn from the former three shall make given angles with one another.

40. Two given circles touch each other internally. Find the semichord drawn perpendicularly to the diameter passing through the point of contact, which shall be bisected by the circumference of the inner circle.

41. The circumference of one circle is wholly within that of another. Find the greatest and the least straight lines that can be drawn touching the former and terminated by the latter.

42. Draw a straight line through two concentric circles, so that the chord terminated by the exterior circumference may be double that terminated by the interior. What is the least value of the radius of the interior circle for which the problem is possible?

43. To draw a straight line cutting two concentric circles, so that the part of it which is intercepted by the circumference of the greater may be triple the part intercepted by the circumference of the less.

44. Find the greatest of all triangles having the same vertical angle and equal distances between that angle and the bisection of the opposite sides.

45. If a string of a given length be fixed at each end to two given points A and B, and be pulled downwards, so as to form a triangle with the line joining A and B, determine the lowest point that it may be made to reach.

46. From a given point without a circle, at a distance from the circumference of the circle not greater than its diameter, draw a straight line to the concave circumference which shall be bisected by the convex circumference.

47. Find a point in the circumference of a circle, from whence a line drawn, making a given angle with a given radius, may be equal to a given straight line.

48. To find within an acute-angled triangle a point from which, if straight lines be drawn to the three angles of the triangle, they shall make equal angles with each other.

49. From two lines, including a given angle, cut off by a line of given length, a triangle equal to a given rectilineal figure.

50. From the extremities of the diameter of a given semicircle, draw two chords to meet in the circumference, which shall intercept a given length on a given oblique chord.

51. The positions of three stations, A, B, and C, have been laid down on a map, and an observer at D (a station in the same horizontal plane as A, B, and C,) determines the angles ADB and BDC; give a geometrical construction for laying down D on the map.

52. In an acute-angled triangle, to find a point from which if three lines be drawn to the three angles, the sum of these lines shall be the least possible.

53. Draw lines from the angles of a triangle to the points of bisection of the opposite sides and terminated in these points. If from the extremities of any one of them, lines be drawn parallel to the remaining two and produced to meet, a triangle will be formed whose sides are equal to the three lines first drawn.

54. It it required within an isosceles triangle to find a point such,

that its distance from one of the equal angles may be double its distance from the vertical angle.

55. To construct an isosceles triangle equal to a scalene triangle and having an equal vertical angle with it.

56. Given the angle at the base of an isosceles triangle, and the perpendicular from it on the opposite side, to construct the triangle.

57. Given the base, the vertical angle, and the differences of the sides, to construct the triangle.

58. Describe a triangle with a given vertical angle, so that the line which bisects the base shall be equal to a given line, and the angle which the bisecting line makes with the base shall be equal to a given angle.

59. Given the perpendicular height, the vertical angle and the sum of the sides, to construct the triangle.

60. Construct a triangle in which the vertical angle and the difference of the two angles at the base shall be respectively equal to two given angles, and whose base shall be equal to a given straight line.

61. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex; construct the triangle.

62. On a given straight line to describe a triangle having its vertical angle equal to a given angle, and the difference of its sides equal to a given line.

63. Given the vertical angle, and the lengths of two lines drawn from the extremities of the base to the points of bisection of the sides, to construct the triangle.

64. Given the base, and vertical angle, to find the triangle whose area is a maximum.

65. Find a triangle of which the vertical angle, the sum of the squares of the two sides containing it and the area are given.

66. The base, vertical angle, and rectangle under the sum of the other sides and one of them are given. Construct the triangle.

67. Describe a circle the circumference of which shall pass through a given point and touch a given circle in a given point.

68. Describe a circle which shall pass through a given point and which shall touch a given straight line in a given point.

69. Describe a circle to touch two right lines given in position and such that a tangent drawn to it from a given point may be equal to a given line.

70. Let AB, AC be any two lines given in position; DE a line of given length; find the position of that circle which is touched by both the lines AB, AC and whose diameter is equal to DE.

71. Describe a circle to touch two right lines given in position, so that lines drawn from a given point to the points of contact shall contain a given angle.

72. Describe a circle through a given point, and touching a given straight line, so that the chord joining the given point and point of contact may cut off a segment containing a given angle.

73. To describe a circle through two given points to cut a straight line given in position, so that a diameter of the circle drawn through the point of intersection shall make a given angle with the line.

74. The straight lines drawn from the same point, and touching the same circle, are equal. Having proved this, exhibit a construction

that shall include all the triangles which can be described with a given perimeter and given vertical angle.

75. A flag-staff of a given height is erected on a tower whose height is also given: at what point on the horizon will the flag-staff appear under the greatest possible angle.

76. Find a point from which, if straight lines be drawn to touch three given circles, none of which lies within the other, the tangents so drawn shall be equal.

77. The centres of three circles (A, B, C,) are in the same straight line, B and C touch each other externally and A internally, if a line be drawn through the point of contact of B and C, making any angle with the common diameter, then the portion of this line intercepted between C and A, is equal to the portion intercepted between B and A.

78. If P be a point without a circle whose centre is O, and AOB a diameter perpendicular to PO: draw a line $PMEC$ cutting the circle in M and C and the diameter in E, so that the rectangle PM, PC, may be four times the rectangle AE, EB.

79. From a given point without a circle draw a straight line cutting the circle, so that the rectangle contained by the part of it without and the part within the circle shall be equal to a given square.

80. Let AP be a tangent to any circle, and AB a diameter. To determine the point P, so that PCB being drawn, cutting the circumference in C, the rectangle contained by PC, CB, shall be equal to a given square; and shew in what cases this is impossible.

81. The diameter ACD of a circle, whose centre is C, is produced to P, determine a point F in the line AP such that the rectangle $PF.PC$ may be equal to the rectangle $PD.PA$.

82. Through a given point draw a line terminating in two lines given in position, so that the rectangle contained by the two parts may be equal to a given rectangle.

83. A ladder is gradually raised against a wall; find the locus of its middle point.

84. A, B, C, D are four points in order in a straight line, find a point E between B and C, such that $AE.EB = ED.EC$ by a geometrical construction.

85. Determine the locus of the extremities of any number of straight lines drawn from a given point, so that the rectangle contained by each, and a segment cut off from each by a line given in position, may be equal to a given rectangle.

86. Find a point without a given circle from which if two tangents be drawn to it, they shall contain an angle equal to a given angle, and shew that the locus of this point is a circle concentric with the given circle.

87. Find the locus of the vertex of a triangle described on a given base; (1) when the sum of the angles at the base is given; (2) when one of them is always double of the other.

88. Find the locus of the centres of all circles which cut off from the directions of two sides of a triangle, chords equal to two given straight lines.

Hence describe a circle that shall cut off from the direction of three sides of a triangle, chords respectively equal to three given straight lines.

89. In a given straight line to find a point at which two other straight lines, being drawn to two given points, shall contain a right angle. Shew that if the distance between the two given points be *greater* than the sum of their distances from the given line, there will be two such points; if *equal*, there may be only one; if *less*, the problem may be impossible.

90. Produce a given straight line so that the rectangle under the given line, and the whole line produced may equal the square of the part produced.

91. To produce a given straight line, so that the rectangle contained by the whole line thus produced, and the part of it produced, shall be equal to a given square.

92. To determine a right-angled triangle whose hypothenuse may be equal to a given straight line, and the rectangle contained by whose sides may be equal to the square of their difference.

93. Given the lengths of the three lines drawn from the angles of a triangle to the points of bisection of the opposite sides, construct the triangle.

94. Describe a triangle whose sides shall be bisected by three given straight lines, and one of whose sides shall pass through a given point.

95. Find the locus of a point, such that if straight lines be drawn from it to the four corners of a given square, the sum of the squares shall be invariable.

THEOREMS.

5. The arcs intercepted between any two parallel chords in a circle are equal. Also the sum of the arcs subtending the vertical angle made by any two chords that intersect, is the same, as long as the angle of intersection remains the same.

6. From the extremities A and C of a given circular arc AC, equal arcs AB, CD are measured in opposite directions: prove that the chords AC, BD are parallel.

7. Two circles cut each other, and from the points of intersection straight lines are drawn parallel to one another, the portions intercepted by the circumferences are equal.

8. A, B, C, A', B', C' are points on the circumference of a circle; if the lines AB, AC be respectively parallel to $A'B'$, $A'C'$, shew that BC' is parallel to $B'C$.

9. C, C' are the centres of two circles of unequal radii, CR, $C'R'$ any pair of parallel radii, join RR': then shall all such lines produced meet in one point. Prove this property, and from it deduce a method of drawing a common tangent to two circles.

10. If from a given point a straight line be drawn touching a circle given in position, the straight line is given in position and magnitude.

11. DF is a straight line touching a circle, and terminated by AD, BF, the tangents at the extremities of the diameter AB, shew that the angle which DF subtends at the centre is a right angle.

12. The circles described on the sides of any triangle as diameters will intersect in the sides, or sides produced, of the triangle.

13. The circles which are described upon the sides of a right-angled triangle as diameters, meet the hypothenuse in the same point; and the line drawn from the point of intersection to the centre of either of the circles will be a tangent to the other circle.

14. If on the sides of a triangle circular arcs be described containing angles whose sum is equal to two right angles, the triangle formed by the lines joining their centres has its angles equal to those in the segments.

15. If AO, BO be the bounding radii of a quadrant, and in OB any point Q be taken, and QC drawn meeting the circumference in C, and making $\angle BQC = \angle AQO$, shew that $2 . \angle QAO + \angle COB = \angle AOB$.

16. Any two chords of a circle which cut a diameter in the same point and at equal angles, are equal to each other.

17. If two equal chords be drawn in a circle, and another chord be drawn through their middle points, the portions of this last chord intercepted between the middle points and the circumference are equal.

18. If through the middle point of any chord of a circle two chords be drawn, the lines joining their extremities shall intersect the first chord at equal distances from its extremities.

19. If from any point in a circular arc, perpendiculars be let fall on its bounding radii, the distance of their feet is invariable.

20. Of two circular segments upon the same base, the larger is that which contains the smaller angle.

21. $ABCD$ is a semicircle; AC, BD are chords drawn from the extremities of the diameter AD, cutting each other in F; join AB, DC, and produce them till they meet in E; join EF, and produce it to meet the diameter in G. Prove that EG is perpendicular to AD.

22. Let N be any point in the diameter of a circle, whose centre is S, PNQ a chord drawn through N, and join SP; shew geometrically and analytically that PQ is a minimum, and the angle SPQ a maximum, when PQ is perpendicular to the diameter.

23. If the diameter of a circle be one of the equal sides of an isosceles triangle, the base will be bisected by the circumference.

24. If two diameters to a circle be drawn, and through their extremities any other circles pass, the portions of the diameters lying between any two of these circles are equal to each other.

25. Two diameters AOA', BOB' of a circle are at right angles to each other; P is a point in the circumference, the tangent at P meets BOB' produced in Q, and AP, $A'P$ meet the same line in C, C' respectively. Prove that CQ' is equal to $C'Q$.

26. ADB, ACB are the arcs of two *equal* circles cutting one another in the straight line AB; draw any chord ACD cutting both arcs, and join CB, DB; CB is equal to DB.

27. If from a point without a circle lines be drawn touching the circle, the angle included is measured by half the difference of the concave and convex parts of the circumference.

28. If the chord AB be produced to D, the angle which DB makes with the chord BC, is equal to half the sum of the angles which the arcs AB, BC subtend at the centre of the circle.

29. The perpendiculars let fall from the three angles of any triangle upon the opposite sides, intersect each other in the same point.

30. Through a given point within a circle two chords are drawn, and the extremities of one joined to different extremities of the other; shew that two triangles are formed which are equiangular.

31. The lines, which bisect the vertical angles of all triangles on the same base and with the same vertical angle, all intersect in one point.

32. A number of triangles are described upon the same base and having the same vertical angle; and from a given point, at the same distance from the two extremities of the base, lines are drawn to the vertices of the triangles; shew that if this point be so chosen that one of these lines shall bisect the angle at the vertex to which it is drawn, all the lines will bisect the angles at the vertices to which they are drawn.

33. Of all triangles on the same base, having equal perimeters, the equilateral has the greatest area.

34. The middle point of the hypothenuse of a right-angled triangle is equally distant from the three angular points.

35. Of all triangles on the same base and between the same parallels, the isosceles has the greatest vertical angle.

36. Of all straight lines which can be drawn from two given points to meet in the convex circumference of a given circle, the sum of those two will be the least, which make equal angles with the tangent at the point of concourse.

37. If from any point without a circle, lines be drawn cutting the circle and making equal angles with the longest line, they will cut off equal segments.

38. From a point without a circle two straight lines are drawn cutting the convex and concave circumferences, and also respectively parallel to two radii of the circle. Prove that the difference of the concave and convex arcs intercepted by the cutting lines, is equal to twice the arc intercepted by the radii.

39. If round any point within the circumference of a circle, (not being the centre) equal adjoining angles be described; of the circumferences on which they stand, that which is nearer the diameter passing through the point is less than the more remote.

40. If two equal chords of a circle cut one another either within or without a circle, the segments of the one between the point of intersection and the circumference, shall be equal to the segments of the other, each to each.

41. If in a given circle a diameter be drawn, and in it a point be taken which is not the centre of the circle, and a line be drawn through it to meet the circumference both ways, only one line equal to it can pass through that point and meet the circle both ways, and of all other lines which pass through the same point, those which make smaller angles with the diameter are greater than those which make a greater.

42. Two straight lines which at one point make equal angles with the diameter of a circle or the diameter produced, are equally distant from the centre.

43. If from a point without a circle two tangents be drawn; the straight line which joins the points of contact will be bisected at right angles by a line drawn from the centre to the point without the circle.

44. Tangents to a circle at the extremities of any chord, contain

an angle which is twice the angle contained by the same chord and a diameter drawn from either of the extremities.

45. If tangents be drawn at the extremities of any two diameters of a circle, and produced to intersect one another; the straight lines joining the opposite points of intersection will both pass through the centre.

46. If any chord of a circle be produced equally both ways, and tangents to the circle be drawn, on opposite sides of it, from its extremities, the line joining the points of contact bisects the given chord.

47. When the vertical angle and the sum of the sides of a plane triangle are given, prove that the locus of the middle of the base is a line given in position.

48. If a tangent be drawn at any point of a circle, and from the point of contact a perpendicular be let fall on a diameter of the circle, and from the extremities of the diameter perpendiculars be let fall on the tangent, these perpendiculars will be respectively equal to the segments of the diameter.

49. If from two fixed points in the circumference of a circle, straight lines be drawn intercepting a given arc and meeting without the circle, the locus of their intersections is a circle.

50. Two straight lines stand at right angles to each other, one of which passes through the centre of a given circle, and from any point in the other, tangents are drawn to the circle. Prove that the chord joining the points of contact cuts the first line in the same point, whatever be the point in the second from which the tangents are drawn.

51. If from any point without a circle two lines be drawn touching the circle, and from the extremities of any diameter lines be drawn to the point of contact cutting each other within the circle, the line drawn from the point without the circle to the point of intersection shall be perpendicular to the diameter.

52. If from a point without a circle two lines be drawn touching the circle, and from one of the points of contact a perpendicular be let fall on the diameter passing through the other, it will be bisected by a straight line drawn from the point without the circle to the farthest extremity of the diameter.

53. If from any point without a circle, lines be drawn touching it, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through any one of them.

54. Let ACB be a quadrant of a circle, of which the centre is C, and the terminating radii CA, CB; join AB, and on it describe a semicircle: from the point A draw a straight line AED cutting the quadrant in E, and terminated by the semicircle in D; and join DB; it is required to prove that DE is always equal to DB, and that the greater only of the two lines AD, DB can cut the quadrant.

55. With any point A in the arc of the circle ABF as a centre, and any radius AB less than the diameter, and greater than half the radius of ABF, describe a circle cutting ABF in F; in it place the chords BD, DE, EC each equal to the chord AB; with radius CF, and centres A and C, describe two circles intersecting in G; with the same radius and centre G, describe a circle intersecting the circle BDF in H; the chord HB is equal to the radius of the circle ABF.

56. If from a point in the circumference of a circle any number

of chords be drawn, the locus of their points of bisection will be a circle.

57. In a circle with centre O, any two chords AB, CD are drawn cutting in E, and OA, OB, OC, OD are joined. Prove that the angles $AOC+BOD=2.AEC$, and $AOD+BOC=2.AED$.

58. Any two points A, B being taken within the circle CDE, it is required to find a point D so that the difference of the angles BDE, ADC may equal a given angle.

59. If from the point where a common tangent to two circles meets the line joining their centres, any line be drawn cutting the circles, it will cut off similar segments.

60. ACB, ADB are two segments of circles on the same base AB, take any point C in segment ACB; join AC, BC, and produce them to meet the segment ADB in D and E respectively: shew that the arc DE is constant.

61. If there be two circles in the same plane, not cutting each other, and two lines be drawn (one of them meeting the line joining the centres) to touch both circles and terminate at the points of contact; prove that the difference of the squares of these lines will be equal to the rectangle of the diameters of the circles.

62. If on two lines containing an angle segments of circles be described containing angles equal to it, the lines produced will touch the segments.

63. If two circles touch each other externally, and two parallel lines be drawn, so touching the circles in points A and B respectively that neither circle is cut, then a straight line AB will pass through the point of contact of the circles.

64. A common tangent is drawn to two circles which touch each other externally; if a circle be described on that part of it which lies between the points of contact, as diameter, this circle will pass through the point of contact of the two circles, and will touch the line which joins their centres.

65. If two circles do not touch each other, and a segment of the line joining their centres be intercepted between the convex circumferences, any circle whose diameter is not less than that segment may be so placed as to touch both the circles.

66. If lines be drawn from a given point in the circumference of a circle, such that the rectangles between the whole lines and the parts of them within the circle be constant, the locus of their extremities will be a straight line.

67. C, c, are the centres of two unequal circles; CA, CB, CD, &c., any radii in the one, and ca, cb, cd, &c., radii in the other respectively parallel to the former, and on the same side of the line Cc. Join Aa, Bb, &c.; these lines being produced all meet in the same point.

68. If two equal circles cut each other, and if through one of the points of intersection a line be drawn terminated by the circles, the lines joining its extremities with the other point of intersection are equal.

69. Shew that, if two circles cut each other, and from any point in the straight line produced which joins their intersections two tangents be drawn, one to each circle, they shall be equal to one another.

70. If any two circles, the centres of which are given, intersect

each other, the greatest line which can be drawn through either point of intersection and terminated by the circle, is independent of the diameters of the circles.

71. If two circles intersect each other in A, B; any two parallel lines CD, EF, drawn through A, B, respectively, and cutting the circles in C, D; E, F; are equal. Required proof.

72. If two circles cut each other, and from their points of intersection, lines be drawn to any point in the circumference of one of them, the arc of the other circle intercepted by these lines, (produced if necessary) is invariable.

73. If two circles intersect, the common chord produced bisects the common tangent.

74. Two circles intersect in the points A and B; through A and B any two straight lines CAD, EBF, are drawn cutting the circles in the points C, D, E, F; prove that CE is parallel to DF.

75. If two circles intersect, the straight line joining their centres bisects the intercepted circumferences.

76. If two circles cut each other, the straight line joining their centres, will bisect their common chord at right angles.

77. If two equal circles cut each other, and from either point of intersection a circle be described cutting them, the points where this circle cuts them and the other point of intersection of the equal circles, are in the same straight line.

78. Two circles intersect in the points A and C, the centre of one being in the circumference of the other; draw any chord COH common to both circles, and join AH, then shall $AH = HO$.

79. If two circles touch each other internally, and from any two points in the circumference of the interior circle, straight lines be drawn to the point of contact, they will contain a greater angle than any other two lines drawn from the same points to any other point in the exterior circumference.

80. Let one circle touch another internally, and let straight lines touch the inner circle, and be terminated by the outer: shew that the greatest of these lines is the one parallel to the common tangent at the point of contact.

81. Two circles touch each other externally, the diameter of one being double of the diameter of the other; through the point of contact any line is drawn to meet the circumferences of both; shew that the part of the line which lies in the larger circle is double of that in the smaller.

82. If two circles touch each other externally and parallel diameters be drawn, the straight line joining the extremities of these diameters will pass through the point of contact.

83. If two circles touch each other internally, and any circle be described touching both, prove that the sum of the distances of its centre from the centres of the two given circles will be invariable.

84. If two circles touch each other externally, and a straight line which touches both of them intersect another straight line passing through their centres, at a point whose distance from the nearer circle is equal to its diameter, the radius of one of the circles will be twice as great as that of the other.

85. If two circles touch each other, any straight line passing through the point of contact cuts off similar parts of their circumferences.

86. If a straight line be drawn cutting any number of concentric circles, shew that the segments so cut off are similar.

87. If from any point in the circumference of the exterior of two concentric circles, two straight lines be drawn touching the interior and cutting the exterior; the distance between the points of contact will be half that between the points of intersection.

88. If three equal circles have a common point of intersection, prove that a straight line joining any two of the points of intersection will be perpendicular to the straight line joining the other two points of intersection.

89. If three circles touch each other in one point, and if from a point external to all, a pair of tangents be drawn to each circle, the three chords joining the points of contact will all pass through one point.

90. Two equal circles cut one another, and a third circle touches each of these two equal circles externally; the straight line which joins the points of section will, if produced, pass through the centre of the third circle.

91. If on any three chords drawn through the same point in the circumference of a circle, as diameters, three circles be described: the points of intersection of these circles two and two lie in the same straight line.

92. Three equal circles whose centres are in one straight line are so placed that the two extreme circles touch, and that point of contact is the centre of the middle one: shew that by joining the points of intersection of the circles, a rectangle will be formed whose diagonals intersect in the centre of the middle circle.

93. A number of circles touch each other at the same point, and a straight line is drawn from it cutting them: the straight lines joining each point of intersection with the centre of the circle will be all parallel.

94. A fixed circle is cut by a series of circles, all of which pass through two given points: shew that the lines which join the points of intersection of the fixed circle with each circle of the series, all converge to one point.

95. A series of circles touching each other at one point are cut by a fixed circle; shew that the intersections of the pairs of tangents to the latter, at the points where it is cut by each of the other circles, lie in a straight line.

96. If a common tangent be drawn to any number of circles, which touch each other internally; and from any point in this tangent as a centre, a circle be described cutting the others, and from this centre lines be drawn through the intersections of the circles respectively; the segments of those within each circle will be equal.

97. If in the diameter or diameter produced of a circle, two points be taken equally distant from the centre; the sum of the squares of the distances of any point in the circumference from these two points is constant.

98. If from any point in the diameter of a circle, straight lines be drawn to the extremities of a parallel chord, the squares of these lines are together equal to the squares of the segments into which the diameter is divided.

99. If any two chords be drawn in a circle perpendicular to each

other, the sum of their squares is equal to twice the square of the diameter diminished by four times the square of the line joining the centre with their point of intersection.

100. Let $ABCD$ be any quadrilateral figure, let the line joining the middle points of the diagonals be bisected in E, and with centre E any circle be described. Prove that for all points P in this circle,

$$PA^2 + PD^2 + PC^2 + PD^2$$

is the same, and that it $= EA^2 + EB^2 + ED^2 + 4 . EP^2$.

101. $ABCD$ is a quadrilateral figure having the angles at A and B right angles; CD is bisected in E, and with centre E and distance EC a circle is drawn cutting AD and BC, or these lines produced in H, C. AF is a tangent to the circle, and FEG a diameter. Join AG, and shew that the squares on the four sides of the quadrilateral figure are together equal to twice the square on AG.

102. If two circles be concentric, and any chord in the interior circle be produced either way to meet the circumference of the exterior circle, the rectangle contained by the whole line thus produced and the part produced, is a constant quantity.

103. If there be two concentric circles, and any chord of the greater circle cut the less in any point, this point will divide the chord into two segments whose rectangle is invariable.

104. Prove that if a straight line could be drawn from any assumed point in the curve of a semi-circle to meet the diameter produced, so that the part of the line without the curve should be equal to the radius, any angle might be trisected.

105. If AOB be a chord to the outer, and a tangent to the inner of two concentric circles, whose diameters are D and d; and if any two chords EOF, COD be drawn through O: and E, D, and C, F be joined, cutting AB in M and N respectively, then

$$4 . CN . NF + MN^2 = D^2 - d^2.$$

106. If a rectangle $ABKI$ be described on the diameter of the circle AB, cutting the perimeter in C, and from C, CA, CB be drawn, and AD, BD parallel to BC, AC, the rectangle $ADBC$ is equal to the rectangle $ABKI$.

107. Two points are taken in the diameter of a circle at any equal distances from the centre; through one of these draw any chord, and join its extremities and the other point. The triangle so formed has the sum of the squares of its sides invariable.

108. If chords drawn from any fixed point in the circumference of a circle, be cut by another chord which is parallel to the tangent at that point, the rectangle contained by each chord, and the part of it intercepted between the given point and the given chord, is constant.

109. ABC is a triangle whose acute vertex is A and base BC, shew that BC^2 is less than $AC^2 + AB^2$ by twice the square of a line drawn from A to touch the circle of which BC is the diameter.

110. A chord POQ cuts the diameter of a circle in O, in an angle equal to half a right angle;

$$PO^2 + OQ^2 = 2\,(\text{rad.})^2.$$

111. Let $ACDB$ be a semicircle whose diameter is AB; and AD, BC any two chords intersecting in P; prove that

$$AB^2 = DA . AP + CB . BP.$$

112. The centre of the circle ACE is divided into six equal parts in the points A, B, C, D, E, F; G is the centre of ACE; with radius AC, and centres A, D, describe two circles intersecting in H; with radius AC, and centres C, E, describe two circles intersecting in K;

$$AK \cdot AG = KG^2.$$

113. If on the radius AC of a circle another semicircle be described and a perpendicular BDE be drawn to the diameter cutting the circles in B and D, and AB, AD be joined, then the square on AB is double of the square on AD.

114. If $ABDC$ be any parallelogram, and if a circle be described passing through the point A, and cutting the sides AB, AC, and the diagonal AD, in the points F, G, H respectively, shew that

$$AB \cdot AF + AC \cdot AG = AD \cdot AH.$$

115. ACB is an isosceles triangle having a right angle at C; with centre C and distance CA, describe a circle; if, from a point Q in the circumference of this circle, QRr be drawn parallel to AB meeting AC, BC, in R, r, respectively, prove that

$$QR^2 + Qr^2 = AB^2.$$

116. If any two circles touch each other in the point O, and lines be drawn through O at right angles to each other, the one line cutting the circles in P, P', the other in Q, Q': and if the line joining the centres of the circles cut them in A, A',

$$PP'^2 + QQ'^2 = AA'^2.$$

117. If from the intersection of any two tangents to a circle, any straight line be drawn cutting the chord which joins the two points of contact and again meeting the circumference, it shall be divided by the circumference and the chord into three segments, such that the rectangle contained by the whole and the middle point shall be equal to the rectangle contained by the extreme parts.

118. The straight line AB joining A and B, the centres of two circles, whose radii are R and r respectively, is divided in C, so that $AC^2 - BC^2 = R^2 - r^2$, and a straight line is drawn from C perpendicular to AB; prove that the tangents drawn to both circles from any point in this line are equal.

119. Two circles whose centres are A and B, touch one another in the point C; shew that if D be a point without the circles such that DA, DB make equal angles with DC, the rectangle contained by the tangents drawn from D is equal to the square of DC.

120. If to any point C in a circle, a tangent ACB be drawn, and from A, C, B perpendiculars be drawn to the diameter DE, and CF be produced to I; the rectangle contained by AB, CI is equal to that contained by DE, GH.

121. If A be a point within a circle, BC the diameter, and through A, AD be drawn perpendicular to the diameter, and BAE meeting the circumference in E, then $BA \cdot BE = BC \cdot BD$.

122. FEI is a common tangent to two circles, meeting the line which joins the centres produced in the point I; $IDCGH$ is any straight line cutting the circles. Prove that

$$IE \times IF = ID \times IH = IC \times IG.$$

123. If the sides of a quadrilateral figure inscribed in a circle be produced to meet, and from each of the points of intersection a straight

ine be drawn touching the circle, the squares of these tangents are ogether equal to the square of the straight line joining the points of ntersection.

124. If from a point without a circle two straight lines be drawn o touch or cut the circle, and from the same point a secant be drawn hrough the line joining the points of contact or intersection, and erminating in the circumference, it will be so divided, that the ectangle under the whole line and its middle segment shall be equal o the rectangle under the extreme segments.

125. If two chords of a circle intersect each other without the ircle at right angles, the sum of the squares of the segments is equal o the square of the diameter.

126. If a circle roll within another of twice its size, any point in ts circumference will trace out a diameter of the first.

127. If from a given point S, a perpendicular Sy be drawn to the angent Py at any point P of a circle whose centre is C, and in the ine MP drawn perpendicular CS, or in MP produced, a point Q be lways taken such that $MQ = Sy$, then the locus of Q is a straight line.

128. If from a fixed point P, any two straight lines be drawn cutting a circle, and BD, AC be drawn intersecting in O, prove that O is lways in the straight line joining the points where the tangents from P meet the circle.

129. If a rod move betwixt two fixed straight lines CP, CQ; and he perpendiculars from P, Q upon CP, CQ respectively meet in R, whilst those from P, Q upon CQ, CP respectively meet in S: the loci f R and S will be two circles having their common centre in C.

130. In all triangles upon the same base and with equal angles at he vertex, the perpendiculars drawn from the angles at the base to the pposite sides in each triangle shall intersect each other in the circumference of a circle; prove this, and describe the circle.

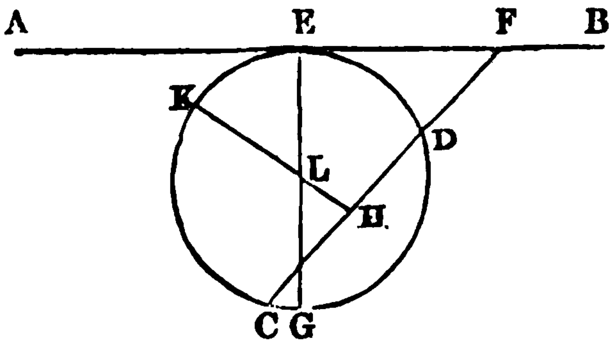
GEOMETRICAL EXERCISES ON BOOK IV.

PROBLEM I.

To describe a circle which shall touch a straight line given in position, and pass through two given points. (Apollonii de Tactionibus, Prob. 7.)

Analysis. Let AB be the given straight line, and C, D the two given points.

Suppose the circle required which passes through the points C, D, to touch the line AB in the point E.



Join C, D, and produce DC to meet AB in F,
and let the circle be described having the centre L,
join also LE, and draw LH perpendicular to CD.
Then CD is bisected in H, and LE is perpendicular to AB.

Also, since from the point F without the circle, are drawn two straight lines, one of which FE touches the circle, and the other FDC cuts it; the rectangle contained by FC, FD, is equal to the square of FE. (III. 36.)

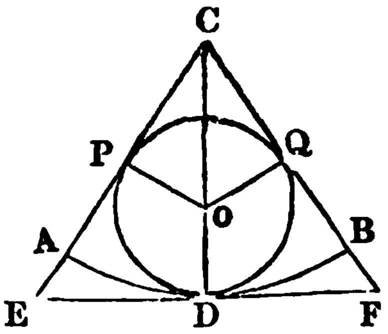
Synthesis. Join C, D, and produce CD to meet AB in F,
take the point E in FB, such that the square of FE, shall be equal to the rectangle FD, FC.
Bisect CD in H, and draw HK perpendicular to CD;
then HK passes through the centre. (III. 1. Cor. 1.)
At E draw EG perpendicular to FB,
then EG passes through the centre, (III. 19.)
consequently L, the point of intersection of these two lines is the centre of the circle.

It is also manifest, that another circle may be described passing through C, D, and touching the line AB on the other side of the point F; and this circle will be equal to, greater than, or less than the other circle, according as the angle CFB is equal to, greater than, or less than the angle CFA.

PROBLEM II.

Inscribe a circle in a given sector of a circle.

Analysis. Let CAB be the given sector, and let the required circle whose centre is O, touch the radii in P, Q, and the arc of the sector in D.

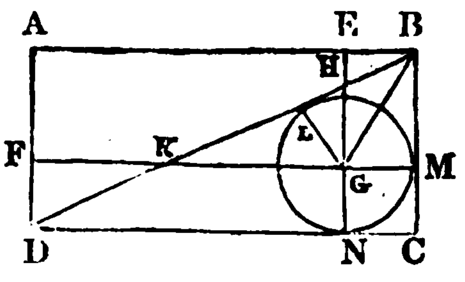


Join OP, OQ, these lines are equal to one another.
Join also CO.
Then in the triangles CPO, CQO, the two sides PC, CO are equal to QC, CO, and the base OP is equal to the base OQ;
therefore the angle PCO is equal to the angle QCO;
and the angle ACB is bisected by CO:
also CO produced will bisect the arc AB in D. (III. 26.)
If a tangent EDF be drawn to touch the arc AB in D;
and CA, CB be produced to meet it in E, F:
the inscription of the circle in the sector is reduced to the inscription of a circle in a triangle. (IV. 4.)

PROBLEM III.

ABCD *is a rectangular parallelogram. Required to draw* EG, FG *parallel to* AD, DC, *so that the rectangle* EF *may be equal to the figure* EMD, *and* EB *equal to* FD.

Analysis. Let EG, FG be drawn, as required, bisecting the rectangle $ABCD$.
Draw the diagonal BD cutting EG in H and FG in K.
Then BD also bisects the rectangle $ABCD$;
and therefore the area of the triangle KGH is equal to that of the two triangles EHB, FKD.



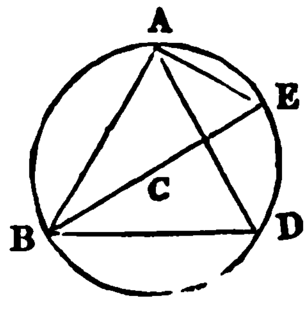
Draw GL perpendicular to BD, and join GB,
also produce FG to M, and EG to N.
If the triangle LGH be supposed equal to the triangle EHB, by adding HGB to each,
the triangles LGB, GEB are equal, and they are upon the same base GB, and on the same side of it;
therefore they are between the same parallels,
that is, if LE were joined, LE would be parallel to GB;

and if a semicircle were described on GB as a diameter, it would pass through the points E, L; for the angles at E, L are right angles:
also LE would be a chord parallel to the diameter GB;
therefore the arcs intercepted between the parallels LE, GB are equal,
and consequently the chords EB, LG are also equal;
but EB is equal to GM, and GM to GN;
wherefore LG, GM, GN, are equal to one another;
hence G is the centre of the circle inscribed in the triangle BDC.
Synthesis. Draw the diagonal BD.
Find G the centre of the circle inscribed in the triangle BDC;
through G draw EGN parallel to BC, and FKM parallel to AB.
Then EG and FG bisect the rectangle $ABCD$.
Draw GL perpendicular to the diagonal BD.
In the triangles GLH, EHB, the angles GLH, HEB are equal, each being a right angle, and the vertical angles LHG, EHB, also the side LG is equal to the side EB;
therefore the triangle LHG is equal to the triangle EHB.
Similarly, it may be proved, that the triangle GLK is equal to the triangle KFD;
therefore the whole triangle KGH is equal to the two triangles EHB, KFD;
and consequently EG, FG bisect the rectangle $ABCD$.

THEOREM I.

If an equilateral triangle be inscribed in a circle, the square of the side of the triangle is triple the square of the radius, or of the side of the regular hexagon inscribed in the same circle. (Euclid's Elements, XIII. 12.)

Let ABD be an equilateral triangle inscribed in the circle ABD, of which the centre is C.



Join BC, and produce BC to meet the circumference in E, also join AE.
And because ABD is an equilateral triangle inscribed in the circle;
therefore AED is one third of the whole circumference,
and therefore AE is one sixth of the circumference,
and consequently, the straight line AE is the side of a regular hexagon (IV. 15.), and is equal to EC.
And because BE is double of EC or AE,
therefore the square of BE is quadruple of the square of AE,
but the square of BE is equal to the squares of AB, AE;
therefore the squares of AB, AE are quadruple of the square of AE,
and taking from these equals the square of AE,
therefore the square of AB is triple the square of AE.

PROBLEMS.

4. In a given circle, place a straight line equal and parallel to a given straight line not greater than the diameter of the circle.

5. From two given points in the circumference of a given circle, inflect straight lines to a point in the opposite circumference, such as to intercept on a given chord, a segment which may be equal to a given line.

6. In a given circle to place a straight line equal to a given straight line, not greater than the diameter of the circle, which shall pass through a given point, either within or without the circle.

7. Trisect a given circle by dividing it into three equal sectors.

8. In a given circle place a straight line cutting two radii which are perpendicular to each other, in such a manner, that the line itself may be trisected.

9. Draw through the angles of an equilateral triangle three straight lines which shall form by their intersection another equilateral triangle; shew that there may be an infinite number of such triangles, and describe the greatest.

10. Describe, about a given circle, a triangle having each of the angles at the base double the vertical angle.

11. Inscribe three circles in an isosceles triangle touching each other, and each of them touching two of the three sides of the triangle.

12. In a given triangle inscribe two isosceles triangles of equal areas, having one of the equal sides of the one equal to one of the equal sides of the other, and each of these equal sides equal to half one of the sides of the given triangle.

13. In a given segment of a circle inscribe an isosceles triangle, such that its vertex may be in the middle of the chord, and the base and perpendicular together equal to a given line.

14. Describe a circle which shall pass through three given points. What condition is necessary in order that it may pass through four given points?

15. Describe a circle about a given triangle, and the sides of the triangle being 6, 8, 10, find the radius of the circle.

16. Inscribe in a circle a triangle whose sides or sides produced shall pass through three given points in the same plane.

17. In a given triangle having inscribed a circle, inscribe another circle in the space thus intercepted at one of the angles.

18. Inscribe a square in a given square. Shew that an infinite number can be inscribed, and find the least.

19. Inscribe a circle in a rhombus.

20. In a given circle inscribe a rectangle equal to a given rectilineal figure not exceeding half of the square upon the diameter.

21. Inscribe the greatest quadrilateral figure in a given circle. Can a circle always be inscribed in a proposed quadrilateral figure, or described about one?

22. Let two straight lines be drawn from any point within a circle to the circumference: describe a circle, which shall touch them both, and the arc between them.

23. From a given point without a given circle two tangents are

drawn to the circle; describe another circle which shall touch the first circle and its tangents.

24. Having given the distances of the centres of two equal circles which cut one another, inscribe a square in the space included between the two circumferences.

25. If two circles cut each other, inscribe in the space between them a parallelogram of which one side is given.

26. Find a point which is equally distant from three given points, the three points not being in the same straight line.

27. Describe a circle about a figure formed by constructing an equilateral triangle upon the base of an isosceles triangle, the vertical angle of which is four times the angle at the base. State also in what case it is possible to circumscribe any trapezium with a circle.

28. To describe a circle about the quadrilateral figure formed by two tangents drawn from the same points, and the lines drawn from the circumference to the centre.

29. The centre of the circle which touches the two semicircles described on the sides of a right-angled triangle is the middle point of the hypothenuse.

30. If a straight line AB is bisected in the point C, and upon the lines AB, AC, and CB, and towards the same side of AB, three semicircles are described; it is required to describe a circle which shall touch each of these semicircles, and to compare its diameter with the line AB.

31. If on one of the bounding radii of a quadrant a semicircle be described, and on the other, another semicircle be described, so as to touch the former and the quadrantal arc; find the centre of the circle inscribed in the figure bounded by the three curves.

32. In a given triangle inscribe a parallelogram which shall be equal to one half the triangle. Is there any limit to the number of such parallelograms?

33. In a given triangle to inscribe a triangle, the sides of which shall be parallel to the sides of a given triangle.

34. Inscribe an equilateral triangle in a square. (1) When the vertex of the triangle is in an angle of the square. (2) When the vertex of the triangle is in the point of bisection of a side of the square.

35. By means of the compasses alone, it is required to inscribe in a square an equilateral triangle having one angle in an angle of the square.

36. Given a circle and two points A, B, exterior to it, find a point X in the circle such that if XA, XB, be drawn cutting the circle in P, Q, PQ shall be parallel to AB.

37. Two points being given without a given circle, determine a point in the circumference, from which lines being drawn to the two given points shall contain the greatest possible angle.

38. Given two points A, B, within a circle. Determine the point P in the circumference such that the angle APB shall be a maximum.

39. Determine that point in the arc of a quadrant from which two lines being drawn, one to the centre and the other bisecting the radius, the included angle shall be the greatest possible.

40. The given straight line AC is inclined at any angle to the straight line CD; to find a point D in the straight line CD, such, that

the angle subtended by the given part AB of AC may be the greatest possible.

41. Find a point in a triangle from which two straight lines drawn to the extremities of the base shall contain an angle equal to twice the vertical angle of the triangle. Within what limitations is this possible?

42. Given the base of a triangle, and the point from which the perpendiculars on its three sides are equal; construct the triangle. To what limitation is the position of this point subject in order that the triangle may lie on the same side of the base?

43. Describe a circle touching three straight lines.

44. Draw a circle touching two given straight lines, and passing through a given point.

45. To describe a circle which shall touch a given circle in a given point, and also a given straight line.

46. To describe a circle touching a given circle in a given point, and passing through a given point not in the circumference of the given circle. In what case is this impossible?

47. Describe a circle touching a given straight line, and also two given circles.

48. Describe a circle which shall touch a given circle, and each of two given straight lines.

49. Two points are given, one in each of two given circles; describe a circle passing through both points and touching one of the circles.

50. Shew how to describe a circle that shall have its centre in a given straight line, which shall pass through a given point, and also touch another given straight line.

51. Describe a circle which shall touch two sides and pass through one angle of a given square.

52. If two circles touch each other externally, describe a circle which shall touch one of them in a given point, and also touch the other. In what case does this become impossible?

53. Describe a circle touching a straight line in a given point, and also touching a given circle. When the line cuts the given circle, shew that your construction will enable you to obtain six circles touching the given circle and the given line, but not necessarily in the given point.

54. Describe three circles touching each other and having their centres at three given points. In how many different ways may this be done?

55. Describe three equal circles touching each other, and each passing through the angle of a given equilateral triangle.

56. Having given the hypothenuse of a right-angled triangle, and the radius of the inscribed circle, to construct the triangle.

57. Having given the vertical angle of a triangle, and the radii of the inscribed and circumscribed circles, to construct the triangle.

58. Given the base and vertical angle of a triangle, and also the radius of the inscribed circle, required to construct it.

59. Given the three angles of a triangle, and the radius of the inscribed circle, to construct the triangle.

60. A straight line is drawn from the extremity of the diameter ($2R$) of a circle, and making an angle with it equal to one third of a

right angle; another circle whose radius is $\frac{3}{4}.R$, touches the former circle externally and the straight line. Find its position.

61. If the base and vertical angle of a plane triangle be given, prove that the locus of the centres of the inscribed circle is a circle, and find its position and magnitude.

62. To describe two circles, each having a given semidiameter, which shall touch the same straight line, and also each other.

63. In a given circle inscribe three equal circles touching each other and the given circle.

64. Find the centre and diameter of a circle that touches three given circles, each of which touches the other two.

65. Describe in a given circle three circles which shall touch one another and the given circle.

66. If there be three concentric circles, whose radii are 1, 2, 3; determine how many circles may be described round the interior one, having their centres in the circumference of the circle, whose radius is 2, and touching the interior and exterior circles, and each other.

67. Shew how a right angle may be divided into five equal parts.

68. Upon a given straight line to construct a regular pentagon. Would an equilateral stellated pentagon answer the conditions of Euclid's definition?

69. Describe a triangle which shall be equal to a given equilateral and equiangular pentagon, and of the same altitude.

70. Inscribe a regular pentagon in a given square so that four angles of the pentagon may touch respectively the four sides of the square.

71. Inscribe a regular decagon in a given circle.

72. Express the side of a regular decagon inscribed in a circle, in terms of the radius.

73. Having an equilateral triangle inscribed in a given circle, shew how a regular hexagon may be described about the same circle.

74. Compare the area of an equilateral and equiangular hexagon inscribed in a circle, with that of one circumscribing the same circle.

75. Inscribe a regular hexagon in a given equilateral triangle.

76. Inscribe a hexagon equilateral, but not equiangular in a given square.

77. To inscribe a regular dodecagon in a given circle, and shew that its area is equal to the square of the side of an equilateral triangle inscribed in the circle.

78. Determine the distance between the opposite sides of an equilateral and equiangular hexagon inscribed in a circle; and compare the area of the hexagon with the area of the rectangle formed by joining corresponding extremities of the opposite sides.

79. In a given pentagon describe an equilateral and equiangular hexagon, so that five of its angles may touch the five sides of the pentagon.

80. On a given straight line describe an equilateral and equiangular octagon.

81. To inscribe an equilateral and equiangular octagon in a circle.

82. Find the area of an equilateral and equiangular octagon.

83. Compare the area of the regular hexagon inscribed in a circle with the area of the circumscribing octagon.

84. Describe a regular octagon equal to a given regular pentagon.

85. Find the value of the interior angle of a regular decagon, and : a regular quindecagon.

86. The angles of a ten-sided figure are in arithmetic progression, ıd the common difference is 10°. Required the angles.

87. The interior angles of a rectilineal figure are in arithmetic rogression; the least angle is 120°, and the common difference 5°, equired the number of sides.

88. The angles of a polygon of 16 sides are in arithmetic proression, the smallest angle is 30°, required the common difference.

89. If the angles of a rectilinear figure be in arithmetic progression ıd the least angle be 30°, and the common difference 15°, find the umber of sides.

90. The angles of an octagon are in arithmetic progression, and ıe common difference is 10°, find the angles.

91. How many sides has each of the regular polygons when one : the interior angles of each polygon is 104°, 135°, 156° respectively?

92. Shew that a regular polygon can be described which has its des each equal to a given line, and each angle equal to 140°, and onstruct it.

93. Find the centre of the circle which will circumscribe a given gular polygon of any number of sides.

94. Given a polygon traced upon a plane, describe a triangle that ıall have an equivalent area.

95. To convert a given regular polygon into another which shall ave the same perimeter, but double the number of sides.

96. Produce the sides of a given heptagon both ways, till they ıeet, forming seven triangles; required the sum of their vertical angles; so the sum of the vertical angles when the figure has (n) sides.

97. Shew what regular figures will fill up the plane space round point. What advantage arises from the honey-comb consisting of exagonal cells?

98. Into what number of equal parts may a right angle be divided eometrically? What connection has the solution of this problem with ıe possibility of inscribing regular figures in circles?

99. Find the least triangle which can be circumscribed about a iven circle.

100. On the sides of an equilateral triangle three squares are deribed. Compare the area of the triangle formed by joining the centres f these squares with the area of the equilateral triangle.

101. If three equal circles touch each other; to compare the area f the triangle formed by joining their centres with the area of the triıgle formed by joining the points of contact.

102. Four equal circles touch each other and a given circle exrnally: compare the radius of one of the four circles with the radius : the given circle.

103. A circle having a square inscribed in it being given, to find a rcle in which a regular octagon of a perimeter equal to that of the uare, may be inscribed.

104. What plane rectilineal figure is that in which a similar rectineal figure can be inscribed, whose area shall be one n^{th} part of the 'ea of the former? Ex. $n=2$, $n=4$.

THEOREMS.

2. The centres of the circles inscribed in, and circumscribed about, an equilateral triangle coincide; and the diameter of one is twice the diameter of the other.

3. An equilateral triangle is inscribed in a circle, and through the angular points another is circumscribed: compare the sides and areas of the two triangles, and shew that the line joining any two points of contact is parallel to the remaining side.

4. If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected, the line joining the points of bisection shall be trisected by the sides.

5. If perpendiculars Aa, Bb, Cc be drawn from the angular points of a triangle ABC upon the opposite sides, shew that they will bisect the angles of the triangle abc, and thence prove that the perimeter of abc will be less than that of any other triangle which can be inscribed in ABC.

6. ABC is an equilateral triangle, and upon BC, AC, AB, equilateral triangles BaC, AbC, AcB, are described; prove that the triangle acb is equilateral.

7. If on the sides of any triangle, three equilateral triangles be described, and circles inscribed in each of these triangles, the straight lines joining the centres of the circles will form an equilateral triangle.

8. If an equilateral triangle be inscribed in a circle, and a straight line be drawn from the vertical angle to meet the circumference, it will be equal to the sum or difference of the straight lines drawn from the extremities of the base to the point where the line meets the circumference, according as the line does or does not cut the base.

9. ABC is an equilateral triangle; AF, DE, are drawn perpendicular to the sides BC, AC, intersecting each other in D: shew that if FS be drawn to the middle point of AB, it will be a tangent to the circle described about $CEDF$.

10. The perimeter of an equilateral triangle inscribed in a circle is greater than the perimeter of any other isosceles triangle inscribed in the same circle.

11. If an equilateral triangle be turned about its centre in its own plane, any two positions of the altitude will always make the same angle as those of the sides; and there will be three positions of coincidence of the triangle.

12. If an equilateral triangle be inscribed in a circle, any of its sides will cut off one-fourth part of the diameter drawn through the opposite angle.

13. The perpendicular from the vertex on the base of an equilateral triangle, is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base. Required proof.

14. If a circle be inscribed in a right-angled triangle, the excess of the sides containing the right angle above the hypothenuse is equal to the diameter of the inscribed circle.

15. If two circles be drawn one within, and the other about, a given right-angled triangle, the sum of their diameters will equal the sum of the sides containing the right angle.

16. ABC is a triangle inscribed in a circle, the line joining the middle points of the arcs AB, AC, will cut off equal portions of the two contiguous sides measured from the angle A.

17. If in a given triangle a circle be inscribed, and tangents to it be drawn parallel to the sides; the sum of the perimeters of the three small triangles cut off by these tangents will equal the perimeter of the given triangle.

18. Let the three perpendiculars from the angles of a triangle ABC on the opposite sides meet in P, a circle described so as to pass through P and any two of the points A, B, C, is equal to the circumscribing circle of the triangle.

19. If from the centre of the circle circumscribing any triangle, perpendiculars be let fall upon the sides, the sum of these perpendiculars is equal to the sum of the radii of the inscribed and circumscribed circles.

20. The line joining the centres of the inscribed and circumscribed circles of a triangle, subtends at any one of the angular points an angle equal to the semi-difference of the other two angles.

21. If ACB be any plane triangle, GCF its circumscribing circle, and GEF a diameter perpendicular to the base AB, then if CF be joined, the angle GFC is equal to half the difference of the angles at the base of the triangle.

22. The straight line which bisects any angle of a triangle inscribed in a circle, cuts the circumference in a point which is equidistant from the extremities of the side opposite to the bisected angle, and from the centre of a circle inscribed in the triangle.

23. In a triangle ABC let AD bisecting the angle A meet BC in D: from O the centre of the inscribed circle draw OE perpendicular to BC; then is the angle BOE equal to the angle DOC.

24. Any number of triangles having the same base and the same vertical angle, will be circumscribed by one circle.

25. From any point B in the radius CA of a given circle whose centre is C, a straight line is drawn at right angles to CA meeting the circumference in D; the circle described round the triangle CBD touches the given circle in D.

26. The diameter of a semicircle is divided into two parts, on each of which as diameter a semicircle is described, and on the same side as the given one. If a circle be described touching each of these three semicircles, the distance of its centre from their common diameter is equal to twice its radius.

27. The locus of the centres of the circles, which are inscribed in all right-angled triangles on the same hypothenuse, is the quadrant described on the hypothenuse.

28. If a circle be described, touching the base of a triangle and the sides produced, and a second circle be inscribed in the triangle; prove that the points where the circles touch the base are equidistant from its extremities, and that the distance between the points where they touch either one of the sides, is equal to the base.

29. If a circle be described about a triangle ABC, and perpendiculars be let fall from the angular points A, B, C, on the opposite sides, and produced to meet the circle in D, E, F, respectively, the circumferences EF, FD, DE, are bisected in the points A, B, C.

30. If from the angles of a triangle, lines be drawn to the points

where the inscribed circle touches the sides; these lines shall intersect in the same point.

31. A square is inscribed in another, the difference of the areas is twice the rectangle contained by the segments of the side which are made at the angular point of the inscribed square.

32. The sides of a given square are bisected, and by joining the bisections another is inscribed; another in this, and so on, ad infinitum. Shew that the sum of all the inscribed squares is equal to the area of the original one.

33. The square inscribed in a circle is equal to half the square described about the same circle.

34. Shew that no parallelogram can be inscribed in a circle except a rectangle.

35. The square is greater than any oblong inscribed in the same circle.

36. If two triangles have one side of the one equal to one side of the other, and the angles opposite these sides equal, the circumscribing circles are equal.

37. If a quadrilateral rectilineal figure be described about a circle, the angles subtended at the centre of the circle by any two opposite sides of the figure, are together equal to two right angles.

38. If a quadrilateral figure be described about a circle, the sums of the opposite sides are equal; and each sum equal to half the perimeter of the figure.

39. Shew that a circle may be inscribed within any quadrilateral figure if the sums of its opposite sides be equal.

40. If two opposite sides of a quadrilateral figure about or in a circle be at equal distances from the centre, the other sides will be parallel.

41. Of all quadrilateral figures contained by four given straight lines, the greatest is that which is inscriptible in a circle.

42. If any number of quadrilaterals inscribed in a circle, have a common side, and the sides adjacent to this be produced to meet, the lines joining the point of concourse with the intersection of the diagonals of the quadrilateral, shall all meet in the same point.

43. Four circles are drawn, of which each touches one side of a quadrilateral figure, and the adjacent sides produced; shew that the centres of these four circles will all lie in the circumference of a circle.

44. Let AB, AC, be the bounding radii of a quadrant; complete the square $ABDC$ and draw the diagonal AD; then the part of the diagonal without the quadrant will be equal to the radius of a circle inscribed in the quadrant.

45. If the sides of a quadrilateral figure circumscribing a circle, touch the circle at the angular points of an inscribed quadrilateral figure; all the diagonals will intersect at the same point.

46. If a circle be inscribed in a square, an equilateral triangle in the circle, and a circle in the triangle, the lines drawn from the vertices of the triangle to meet in any point of the inner circle, and the radius of that circle, have their squares together equal to the given square.

47. If any number of parallelograms be inscribed in a given parallelogram, the diameters of all the figures shall cut one another in the same point.

48. If equilateral parallelograms successively decreasing in area

by one half be inscribed one in the other, the sides of the circumscribing figures must throughout be bisected.

49. Shew that if two circles be inscribed in a third to touch one another, the tangents of the points of contact will all meet in the same point.

50. Shew that nine equal circles may be placed in contact, so that a square whose side is three times the diameter of one of them will circumscribe them.

51. $ABCP$, and $A'B'C'P'$ are two concentric circles, ABC, $A'B'C'$ are any two equilateral triangles inscribed in them. If P, P' be any two points in the circumferences of these circles, shew that

$$A'P^2 + B'P^2 + C'P^2 = AP'^2 + BP'^2 + CP'^2.$$

52. In the fig. Prop. 10, Book IV, shew that the base BD is the side of a regular decagon inscribed in the larger circle, and the side of a regular pentagon inscribed in the smaller circle.

53. In the fig. Prop. 10, Book IV, produce DC to meet the circle in F, and draw BF; then the angle ABF shall be equal to three times the angle BFD.

54. If the alternate angles of a regular pentagon be joined, the figure formed by the intersection of the joining lines will itself be a regular pentagon.

55. If $ABCDE$ be any pentagon inscribed in a circle, and AC, BD, CE, DA, EB be joined, then are the angles ABE, BCA, CDB, DEC, EAD, together equal to two right angles.

56. If the sides of an equilateral and equiangular pentagon be produced to intersect; the straight lines joining the points of intersection will form another equilateral and equiangular pentagon.

57. AB, AC are the sides of a regular pentagon and a regular decagon inscribed in a circle, and drawn on different sides of the point A, shew that if BC be joined, the difference between BC and AC is equal to the radius of the circle.

58. If the sides of an equilateral pentagon be produced, the stellated figure which results is equilateral and equiangular likewise. What is the sum of its re-entrant angles? What is the sum of the exterior angles which are not in contact with the primitive pentagon?

59. If the sides of the pentagon $ABCDE$ be produced both ways, so as to meet in the points $\alpha, \beta, \gamma, \delta, \epsilon$; prove that the sum of the angles at these points is equal to two right angles.

60. A watch-ribbon is folded up into a flat knot of five edges, shew that the sides of the knot form an equilateral pentagon.

61. Shew that perpendiculars from each of the angles of a regular pentagon upon the opposite side, pass through the same point. Find also the value of the angle contained by any two of them, and compare it with an exterior angle of the pentagon.

62. The difference between the side and the diagonal of a regular pentagon is equal to the side of another regular pentagon, whose diagonal is equal to the side of the first.

63. If AB, CD are two diameters drawn at right angles to each other in the circle whose centre is O, if the radius OB is bisected in E and on EA there be taken EF equal to EC, shew that CF will be the side of the inscribed pentagon.

64. If from the extremities of the side of a regular pentagon inscribed in a circle, straight lines be drawn to the middle of the arc

subtended by the adjacent side, their difference is equal to the radius; the sum of their squares to three times the square of the radius; and the rectangle contained by them is equal to the square of the radius.

65. If all the sides of a regular hexagon be produced both ways till they meet, the angles thus formed are together equal to four right angles.

66. If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

67. Prove that the area of a regular hexagon is greater than that of an equilateral triangle of the same perimeter.

68. If the alternate sides of a regular hexagon be produced to meet, the figure so formed will be regular, and have an area equal to twice the area of the hexagon.

69. If two equilateral triangles be inscribed in a circle so as to have the sides of one parallel to the sides of the other, the figure common to both will be a regular hexagon, whose area and perimeter will be equal to the remainder of the area and perimeter of the two triangles.

70. If from the centre of an equilateral hexagon and each of its angles, perpendiculars be drawn to any line, the sum of those drawn from the angles will be equal to six times that which is drawn from the centre.

71. If a regular hexagon be inscribed in a circle, six circles equal to it may be described, every one touching the original circle, and two of the others. If the centres of these circles be joined successively, a regular hexagon will be formed whose area is four times the area of the former: and if the *outermost* points of the six circles be joined, another hexagon will be formed whose area is seven times the area of the first.

72. If the alternate sides of a regular hexagon be produced to meet one another, and the angular points of the triangles thus formed be joined, a regular hexagon will be formed, the area of which is equal to three times the area of the original hexagon.

73. If in any circle the side of an inscribed hexagon be produced till it becomes equal to the side of an inscribed square, a tangent drawn from the extremity, without the circle, shall be equal to the side of an inscribed octagon.

74. A regular octagon inscribed in a circle is equal to the rectangle contained by the sides of the squares inscribed in, and circumscribed about the circle.

75. If a straight line bisect at right angles any side AB of a regular polygon of an odd number of sides, shew that it will pass through the point of intersection of the two sides of the polygon which are most remote from AB.

76. Shew that every equilateral figure inscribed in a circle is also equiangular, and that the converse of this is true when the number of sides is odd, but not necessarily, when the number of sides is even.

77. Prove that the opposite sides of any equiangular rectilineal figure are parallel when the number of sides is even.

78. If equal straight lines be placed similarly round a circle just without it, the loci of their extremities will be concentric circles.

79. If any number of equal straight lines be placed in the circumference of a circle, the points of bisection will be in the circumference of a circle having the same centre.

80. If a polygon has $(n+4)$ sides, prove that the angles formed at the points of concourse of these sides produced, are together equal to $(2n)$ right angles.

81. The interior angles of all polygons are equal, provided the number of sides be the same in all.

82. Prove that of the polygons of a given number of sides, which can be inscribed in a given circle, the greatest is that which is equilateral.

83. Of all polygons formed with given sides, the greatest is that which may be inscribed in a circle.

84. Of all polygons having equal perimeters, and the same number of sides, the equilateral polygon has the greatest area.

85. In any polygon of an even number of sides, inscribed in a circle, the sum of the 1st, 3rd, 5th, &c. angles is equal to the sum of the 2nd, 4th, 6th, &c.

86. In any plane equilateral and equiangular polygon of (n) sides, where (n) is any odd number, if lines be drawn from one of the angles to the extremities of the opposite side, the angle contained by these lines is equal to one n^{th} of two right angles.

87. If a polygon be inscribed in a circle, and from each angle be drawn lines to the next angle but one, the lines so drawn will include a polygon of the same number of sides, and the sum of the angles of the included polygon at the extremities of any one of its sides equals the sum of the angles of the original polygon through which that side produced, passes: if the original polygon be regular, the included one will also be regular.

88. If any two sides of a regular polygon taken *alternately*, be produced to meet, the figure contained by the straight line made up of one side and the part produced, the part produced of the other, and the distances of their extremities to the centre, will be a quadrilateral about which a circle may be described.

89. What is the difficulty of inscribing geometrically an equilateral heptagon or undecagon in a given circle? Assuming the demonstrations of Euclid in his 4th book, shew that any equilateral figure of 2^{n+2}, 3×2^n, 5×2^n, or 15×2^n sides, (where n is any of the numbers 0, 1, 2, 3, &c.) may be inscribed in a circle.

90. If a point be assumed in a regular polygon of n sides, from which perpendiculars are drawn to each of the sides, or the sides produced, the sum of these perpendiculars is n times the radius of the inscribed circle.

91. If from any point within a polygon, straight lines be drawn perpendicular to the several sides, the sum of these perpendiculars shall be equal to the sum of the perpendiculars drawn from the centre of the polygon.

92. If from a point whose distance is d from the centre of a circle whose radius is r, perpendiculars be drawn on all the sides of a regular polygon of n sides circumscribing the circle, the sum of the squares of the perpendiculars $= n\left(\frac{d^2}{2}+r^2\right)$.

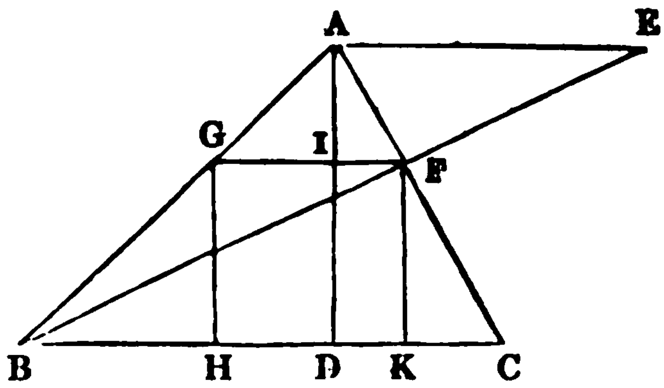
93. If two circles have the same centre, and a regular polygon be described about the outer circle, and from any point in the circumference of the inner circle perpendiculars be drawn to the sides of the polygon, the figure formed by joining the bottom of the perpendiculars will have an invariable magnitude.

GEOMETRICAL EXERCISES ON BOOK VI.

PROBLEM I.

To inscribe a square in a given triangle.

Analysis. Let ABC be the given triangle, the base BC, and the perpendicular AD of which are given.



Let $FGHK$ be the required inscribed square.
Then BHG, BDA are similar triangles,
and GH is to GB, as AD is to AB,
but GF is equal to GH;
therefore GF is to GB as AD is to AB.

Let BF be joined and produced to meet a line drawn from A parallel to the base BC in the point E.
Then the triangles BGF, BAE are similar,
and AE is to AB as GF is to GB,
but GF is to GB as AD is to AB.
Wherefore AE is to AB as AD is to AB;
hence AE is equal to AD.

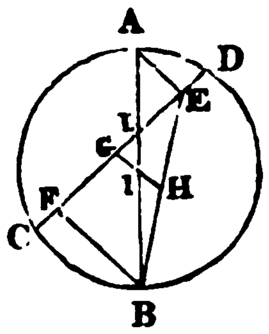
Synthesis. Through the vertex A, draw AE parallel to BC the base of the triangle,
make AE equal to AD,
join EB cutting AC in F,
through F, draw FG parallel to BC, and FK parallel to AD;
also through G draw GH parallel to AD.
Then $GHKF$ is the square required.

The different cases may be considered when the triangle is equilateral, scalene, or isosceles, and when each side is taken as the base.

THEOREM I.

If from the extremities of any diameter of a given circle, perpendiculars be drawn to any chord of the circle, they shall meet the chord, or the chord produced in two points which are equidistant from the centre. (Archimedis Lemm. Prop. 13.)

First, let the chord CD intersect the diameter AB in L, but not at right angles; and from A, B, let AE, BF be drawn perpendicular to CD. Then the points F, E are equidistant from the centre of the chord CD.



Join EB, and from I the centre of the circle, draw IG perpendicular to CD, and produce it to meet EB in H.

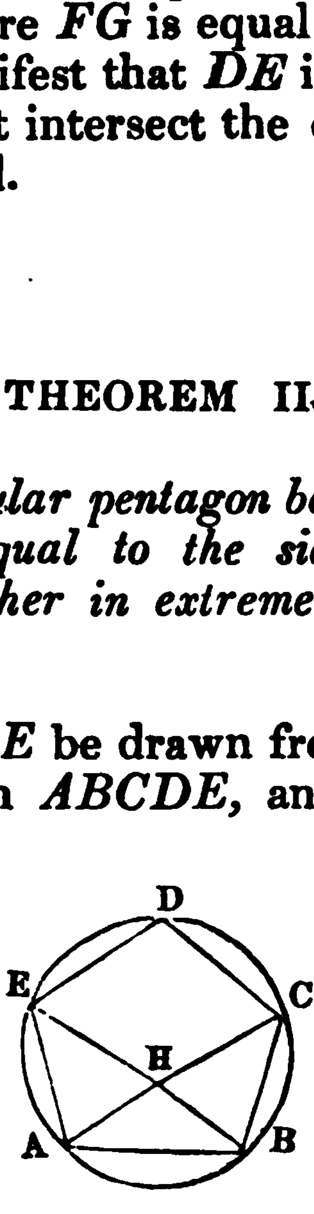
Then IG bisects CD in G; (III. 2.)
and IG, AE being both perpendicular to CD, are parallel. (I. 29.)
Therefore BI is to BH as IA is to HE; (VI. 2.)
and BH is to FG as HE is to GE;
therefore BI is to FG as IA is to GE;
but BI is equal to IA;
therefore FG is equal to GE.
It is also manifest that DE is equal to CF.

When the chord does not intersect the diameter, the perpendiculars intersect the chord produced.

THEOREM II.

If two diagonals of a regular pentagon be drawn to cut one another, the greater segments will be equal to the side of the pentagon, and the diagonals will cut one another in extreme and mean ratio. (Euclid's Elements, XIII. 8.)

Let the diagonals AC, BE be drawn from the extremities of the side AB of the regular pentagon $ABCDE$, and intersect each other in the point H.



Then BE and AC are cut in extreme and mean ratio in H, and the greater segment of each is equal to the side of the pentagon.

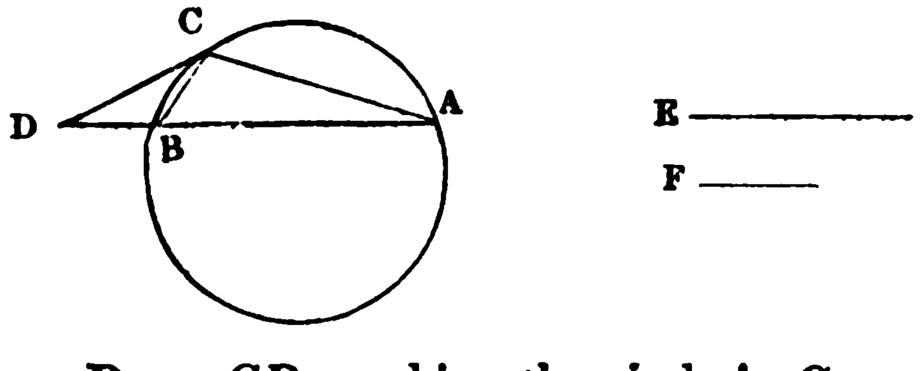
Let the circle $ABCDE$ be described about the pentagon. (IV. 14.)
Because EA, AB are equal to AB, BC, and they contain equal angles;
therefore the base EB is equal to the base AC, (I. 4.)
and the triangle EAB equal to the triangle CBA,
and the remaining angles will be equal to the remaining angles,
each to each, to which the equal sides are opposite.
Therefore the angle BAC is equal to the angle ABE;
and the angle AHE is double of the angle BAH, (I. 32.)
but the angle EAC is also double of the angle BAC, (VI. 33.)
therefore the angle HAE is equal to AHE,
and consequently HE is equal to EA (I. 6.) or to AB.
And because BA is equal to AE,

the angle ABE is equal to the angle AEB;
but the angle ABE has been proved equal to BAH;
therefore the angle BEA is equal to the angle BAH;
and ABE is common to the two triangles ABE, ABH;
therefore the remaining angle BAE is equal to the remaining angle AHB:
and consequently the triangles ABE, ABH are equiangular;
therefore EB is to BA as AB to BH: but BA is equal to EH,
therefore EB is to EH as EH is to BH,
but BE is greater than EH; therefore EH is greater than HB;
therefore BE has been cut in extreme and mean ratio in H.
Similarly, it may be shewn, that AC has also been cut in extreme and mean ratio in H, and that the greater segment of it CH is equal to the side of the pentagon.

PROBLEM II.

Divide a given arc of a circle into two parts which shall have their chords in a given ratio. (Pappi Math. Coll. VII. 155.)

Analysis. Let A, B be the two given points in the circumference of the circle, and C the point required to be found, such that when the chords AC and BC are joined, the lines AC and BC shall have to one another the ratio of E to F.



Draw CD touching the circle in C;
join AB and produce it to meet CD in D.
Since the angle BAC is equal to the angle BCD, (III. 32.)
and the angle CDB is common to the two triangles DBC, DAC;
therefore the third angle CBD in one, is equal to the third angle DCA in the other, and the triangles are similar,
therefore AD is to DC as DC is to DB; (VI. 4.)
hence also the square of AD is to the square of DC as AD is to BD. (VI. 20. Cor.)
But AD is to AC as DC is to CB, (VI. 4.)
and AD is to DC as AC to CB, (V. 16.)
also the square of AD is to the square of DC as the square of AC is to the square of CB;
but the square of AD is to the square of DC as AD is to DB;
wherefore the square of AC is to the square of CB as AD is to BD;
but AC is to CB as E is to F, (constr.)
therefore AD is to DB as the square of E is to the square of F.
Hence the ratio of AD to DB is given,
and AB is given in magnitude, because the points A, B in the circumference of the circle are given.
Wherefore also the ratio of AD to AB is given, and also the magnitude of AD.

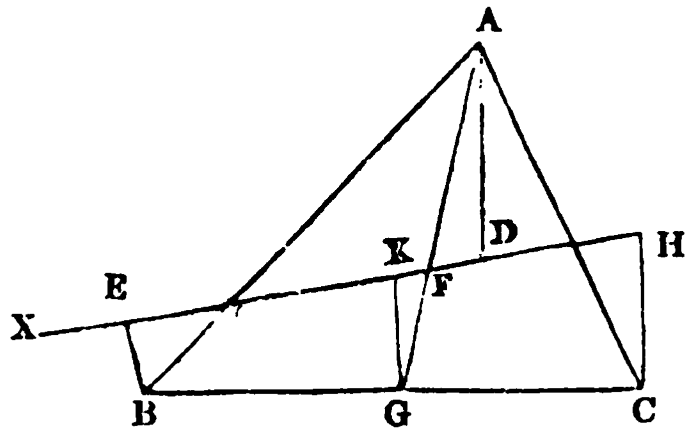
Synthesis. Join AB and produce it to D, so that AD shall be to BD as the square of E to the square of F.

From D draw DC to touch the circle in C, and join CB, CA.
Since AD is to DB as the square of E is to the square of F, (constr.)
and AD is to DB as the square of AC is to the square of BC;
therefore the square of AC is to the square of BC as the square of E is to the square of F,
and AC is to BC as E is to F.

PROBLEM III.

A, B, C *are given points. It is required to draw through any other point in the same plane with* A, B, *and* C, *a straight line, such that the sum of its distances from two of the given points, may be equal to its distance from the third.* (Phil. Trans. Edin. 1792.)

Analysis. Suppose F the point required, such that the line XFH being drawn through any other point X, and AD, BE, CH perpendiculars on XFH, the sum of BE and CH is equal to AD.



Join AB, BC, CA, then ABC is a triangle.
Draw AG to bisect the base BC in G, and draw GK perpendicular to EF.
Then since BC is bisected in G,
the sum of the perpendiculars CH, BE is double of GK;
but CH and BE are equal to AD, (hyp.)
therefore AD must be double of GK;
but since AD is parallel to GK,
the triangles ADF, GKF are similar,
therefore AD is to AF as GK is to GF;
but AD is double of GK, therefore AF is double of GF;
and consequently, GF is one third of AG the line drawn from the vertex of the triangle to the bisection of the base.
But AG is a line given in magnitude and position,
therefore the point F is determined.

Synthesis. Join AB, AC, BC, and bisect the base BC of the triangle ABC in G; join AG and take GF equal to one third of GA;
the line drawn through X and F will be the line required.

It is also obvious, that while the relative position of the points A, B, C, remains the same, the point F remains the same, wherever the point X may be. The point X may therefore coincide with the point F, and when this is the case, the position of the line FX is left undetermined. Hence the following *porism*.

A triangle being given in position, a point in it may be found, such, that any straight line whatever being drawn through that point, the perpendiculars drawn to this straight line from the two angles of the triangle, which are on one side of it, will be together equal to the perpendicular that is drawn to the same line from the angle on the other side of it.

PROBLEMS.

4. Given the perimeter of a triangle: describe it so that it may be similar to a given triangle.

5. On a given base to describe a triangle equal to a given triangle.

6. Given the perimeter of a right-angled triangle, it is required to construct it,

(1) If the sides are in arithmetical progression.

(2) If the sides are in geometrical progression.

7. Given the vertical angle, the perpendicular drawn from it to the base, and the ratio of the segments of the base made by it to construct the triangle.

8. Find the locus of intersection of two lines drawn from two given points, so that their lengths are in a given ratio.

9. In the triangle ABC, let D be the middle point of AB, and E the foot of the perpendicular let fall from C on AB; then prove that, if $CB - CA = DE$, the three sides CA, AB, BC are in arithmetical progression. Also shew how such a triangle may be constructed.

10. Apply (VI. C.) to construct a triangle; having given the vertical angle, the radius of the inscribed circle, and the rectangle contained by the straight lines drawn from the centre of the circle to the angles at the base.

11. Having given the difference of the angles at the base, the perpendicular altitude, and the ratio of the sides, to construct the triangle.

12. Given the vertical angle and the base of a triangle, and also a line drawn from either of the angles at the base cutting the opposite side in a given ratio, to construct the triangle.

13. Upon a given straight line as an hypothenuse, describe a right-angled triangle which shall have its three sides in continued proportion.

14. In any triangle inscribe a triangle similar to a given triangle.

15. Given the base, the ratio of the sides containing the vertical angle, and the distance of the vertex from a given point in the base; to construct the triangle.

16. Given the base, the vertical angle, and the ratio of the parts that the line bisecting the base is divided into, by the line joining the centres of the circumscribed and inscribed circles, to construct the plane triangle.

17. The vertical angle and the base of a triangle are given, and the ratio of the segments into which one of the sides adjacent to the vertical angle is divided by a perpendicular drawn from the opposite angle, construct the triangle.

18. Upon the given base AB construct a triangle having its sides in a given ratio and its vertex situated in the given indefinite line CD.

19. Describe an equilateral triangle equal to a given triangle.

20. Given the hypothenuse of a right-angled triangle, and the side of an inscribed square. Required the two sides of the triangle.

21. Given the perimeter of a right-angled triangle, and the perpendicular drawn from the right angle upon the opposite side; to find the three sides (1) geometrically, (2) algebraically.

22. Given one of the angles and the perimeter of a plane triangle, to find the sides, when the area is the greatest possible.

23. To make a triangle, which shall be equal to a given triangle, and have two of its sides equal to two given straight lines; and shew that if the rectangle contained by the two straight lines be less than twice the given triangle, the problem is impossible.

24. From the obtuse angle of a triangle, it is required to draw a line to the base, which shall be a mean proportional between the segments of the base. How many answers does this question admit of?

25. To find, by a geometrical construction, an arithmetic, geometric, and harmonic mean between two given lines.

26. Find two arithmetic means between two given straight lines.

27. Draw a straight line such that the perpendiculars let fall from any point in it on two given lines may be in a given ratio.

28. In the figure to Proposition 47, Book I. draw through G a right line, so that the sum of the perpendiculars falling on it from B and C may be equal to BK.

29. Find a point, such that the perpendiculars let fall from it on three straight lines given in position may be in a given ratio.

30. If a perpendicular be drawn from a given point of a given line; find a point in it, such that lines drawn from it to the extremities of the given line, may be in a given ratio.

31. To find a point P in the base BC of a triangle produced, so that PD being drawn parallel to AC, and meeting AC produced to D, $AC : CP :: CP : PD$.

32. Determine the point in the produced side of a triangle, from which a straight line being drawn to a given point in the base, shall be intersected by the other side of the triangle in a given ratio.

33. A point and a straight line being given in position, to draw a line parallel to the given line, so that all the lines drawn through the given point may be divided at it in a given ratio.

34. It is required to cut off a part of a given line so that the part cut off may be a mean proportional between the remainder and another given line.

35. Find a straight line which shall have to a given straight line the ratio of $1 : \sqrt{5}$.

36. It is required to divide a given finite straight line into two parts, the squares of which shall have a given ratio to each other.

37. The sum of two straight lines and another straight line, a mean proportional between them being given, to find the two lines.

38. To divide a given line in harmonical proportion.

39. Prove both geometrically and algebraically, that an arithmetic mean between two quantities is greater than a geometric mean. Also having given the sum of two lines, and the excess of their arithmetic above their geometric mean, find by a construction the lines themselves.

40. Determine that point in the base produced of a right-angled triangle from which the line drawn to the angle opposite the base, shall have the same ratio to the base produced, which the perpendicular has to the base itself.

41. On two given straight lines similar triangles are described. Required to find a third, on which, if a triangle similar to them be described, its area shall equal the difference of their areas.

42. C is a given point in a given line AB, it is required to determine a point P, so that AP and BP together may be equal to a given line, and the angle APB be bisected by PC.

43. From the vertex of a triangle to the base, to draw a straight line which shall be an arithmetic mean between the sides containing the vertical angle.

44. To draw a line from the vertex of a triangle to the base, which shall be a mean proportional between the whole base and one segment.

45. The distances of three of the angles of a square from a given point are given, to construct the square. Shew that the same data will afford two solutions of the problem, and that the assumed distance of the interjacent angle must have to the sum of the other two, a less ratio than that of the side of a square to its diagonal.

46. From two straight lines cut off two parts having a given ratio, so that the sum of the squares of the remainders may be equal to a given square.

47. If a given straight line AB be divided into any two parts in the point C, it is required to produce it, so that the whole line produced may be harmonically divided in C and B.

48. Two straight lines and a point between them being given in position, to draw two lines from the given point to terminate in the lines given in position, so that they shall contain a given angle, and have a given ratio.

49. If a straight line be divided in two given points, determine a third point such that its distances from the extremities may be proportional to its distances from the given points.

50. Through a given point between two indefinite straight lines not parallel to each other, to draw a straight line which shall be terminated by them, so that the rectangle contained by its segments shall be less than the rectangle contained by the segments of any other straight line drawn through the same point and terminated by the same straight lines.

51. A rectangular leaf $ABCD$ of which the side $AB = 34a$, and the side $BC = 13a$ has its two corners, A and B, equally bent up at right angles to the rest of the leaf, and the crease is inclined at the same angle to each side. From the points A and B straight lines are drawn to the middle point of CD, these are found to be at right angles to each other, find how much of the leaf is bent up.

52. ABC is a right-angled triangle; find the point P in AC, so that the sum of the distances from A and AC is the least possible.

53. ABC is a right-angled triangle, having a right angle at C, find a point P in the hypothenuse, so situated, that PA may be half of the perpendicular dropped from P upon BC the base.

54. Find three points in the sides of a triangle, such that they being joined, the triangle shall be divided into four equal triangles.

55. Determine two straight lines, such that the sum of their squares may equal a given square, and their rectangle equal a given rectangle.

56. From the vertex of any triangle ABC, draw a straight line meeting the base produced in D, so that the rectangle $DB \cdot DC = AD^2$.

57. Describe a circle passing through two given points and touching a given circle.

58. Describe a circle which shall pass through a given point and touch a given straight line and a given circle.

59. Through a given point draw a circle touching two given circles.

60. With a given radius to describe a circle, touching two given circles.

61. Describe a circle which shall have its centre in a given line, and shall touch a circle and a straight line given in position.

62. In a given circle place a straight line parallel to a given straight line, and having a given ratio to it; the ratio not being greater than that of the diameter to the given line in the circle.

63. If any rectangle be inscribed in a given triangle, required the locus of the point of intersection of its diagonals.

64. In the diameter BB' of a circle produced, a point A is taken such that $AB = \frac{1}{4}$ of the radius: from A a tangent AC is drawn to the circle; find a point D in the circle such that AD shall be a mean proportional between AB and AC.

65. Through a given point within a given circle, to draw a straight line such that the parts of it intercepted between that point and the circumference, may have a given ratio.

66. From a given point without a circle, it is required to draw a straight line to the concave circumference, which shall be divided in a given ratio at the point where it intersects the convex circumference.

67. Find a point external to two circles in the same plane that do not meet, such that if straight lines be drawn through it cutting both circles, the portions of all such straight lines intercepted within the circles, shall be proportional to their radii.

Hence draw a pair of common tangents to two circles, and determine within what limits a point must be situated, so that a straight line may be drawn from it cutting both.

68. Given the magnitude and position of two circles; to draw a chord AB of the greater to touch the less at C, so that CA shall have to CB a given ratio.

69. From what point in a circle must a tangent be drawn, so that a perpendicular on it from a given point in the circumference may be cut by the circle in a given ratio?

70. If a straight line be divided into any two parts, to find the locus of the point in which these parts subtend equal angles.

71. A given rectangular field is to be laid out as a grass-plot (having the form of a double square,) surrounded by a walk of uniform breadth. Find the breadth of the walk, and give a construction for it.

72. Given the sides of a quadrilateral figure inscribed in a circle, to find the ratio of its diagonals.

73. If in a given equilateral and equiangular hexagon another be inscribed, to determine its ratio to the given one.

74. From a given point in the side of a triangle, to draw lines to the sides which shall divide the triangle into any number of equal parts.

75. Any two triangles being given, to draw a straight line parallel to a side of the greater, which shall cut off a triangle equal to the less.

76. To divide a given triangle into two parts, having a given ratio to one another, by a straight line drawn parallel to one of its sides.

77. Inscribe a square in a given right-angled isosceles triangle.

78. Of the two squares which can be inscribed in a right-angled triangle, which is the greater?

79. The area of a rectangle inscribed in a given triangle being given, it is required to find its four sides.

80. Inscribe a square in a sector of a circle, so that the angular points shall be one on each radius, and the other two in the circumference.

81. Inscribe in a square a triangle of which one side and the area are given.

82. Determine which is the greatest, and which the least, of the three squares which may be inscribed in a given triangle.

83. Inscribe a square in a given equilateral and equiangular pentagon.

84. Inscribe a parallelogram in a given triangle similar to a given parallelogram.

85. Inscribe the greatest parallelogram in a given triangle.

86. Apply (VI. 25.) to describe a triangle whose sides are as 3, 4, 5, and its area equal to a square of side 6; what are the sides of the triangle?

87. Inscribe a square in a segment of a circle.

88. Inscribe the greatest parallelogram in a given semicircle.

89. In a given rectangle inscribe a parallelogram of given area and containing an angle equal to a given angle; and point out the limitations of the problem.

90. Describe a rectangular parallelogram which shall be equal to a given square, and have its sides in a given ratio.

91. Through a given point, either within or without a given triangle, to draw a straight line, which shall cut off from the triangle any part required.

92. Two similar rectilineal figures being given, to find a third figure, which shall be similar to them, and also have its area a mean proportional between them.

93. In a given rectangle inscribe another, whose sides shall bear to each other a given ratio.

94. To describe a figure similar to each of two other figures, and equal to their sum or difference.

95. It is required to cut off from a rectangle a similar rectangle which shall be any required part of it.

96. Inscribe in a given circle an isosceles triangle whose base shall be equal to its altitude.

97. Inscribe in a circle two isosceles triangles, the area of one of which shall be one-fourth the area of the other.

98. In a given segment of a circle to inscribe a similar segment.

99. An equilateral triangle and an equilateral and equiangular pentagon have the same perimeter: compare the radii of the circles inscribed in them.

100. A series of circles is described all touching two given straight lines, and also touching each other externally in succession; find the law of their diameters.

THEOREMS.

3. If two triangles are to each other as their bases: prove that ıey have the same altitude.

4. Triangles and parallelograms of unequal altitudes are to each her in the ratio compounded of the ratios of their bases and altitudes.

5. By the construction in Props. 2, A, Book VI, the line made up 'the base and the part produced is divided harmonically.

6. If, in similar triangles, from any two equal angles to the ›posite sides, two straight lines be drawn making equal angles with ıe homologous sides, these straight lines will have the same ratio as the des on which they fall, and will also divide those sides proportionally.

7. From Prop. 2, Book VI, shew how a line may be drawn on ıe ground through a given point parallel to a given straight line by eans of a piece of string.

8. If two systems of parallel straight lines at equal distances cut ıe another, the figures between them will be all similar parallelograms.

9. Let AB, AC be two given straight lines, take $AD = \frac{1}{m}$th of B, and $AE = \frac{1}{m}$th of AC; join BE, CD to meet in F; shew that DF $\frac{DC}{m+1}$.

10. ABC is a triangle; Ab, Ac are taken $= \frac{AB}{n}, \frac{AC}{n}$ respectively, b, Bc meet in D, and AD produced meets BC in E: shew that $.AE=(n+1)AD$; and E is the middle point of AB.

11. Any three lines being drawn making equal angles with the ıree sides of any triangles towards the same parts, and meeting one ıother, will form a triangle similar to the original triangle.

12. If the original triangle, Prop. 8, Book VI, be isosceles, and ›rpendiculars be continually drawn from the foot of the last to the ›posite side, the sum of all the triangles so formed is equal to the area ' the original triangle.

13. If a, b, c represent respectively the sides and hypothenuse of right-angled triangle, and h be the length of the line drawn from the ght angle perpendicular to the opposite side: prove that the triangle hose sides are $a+b$, h, and $c+h$, is also right-angled.

14. Shew geometrically that the diagonal and side of a square are commensurable.

15. If from any point in the circumference of a circle perpendiculars be drawn to the sides, or sides produced, of an inscribed iangle: shew that the three points of intersection will be in the same raight line.

16. If the sides AB, AC of a given triangle be cut proportionally . any two points D and F, and perpendiculars to the sides from those ›ints intersect in L: prove that all the points L corresponding to ifferent positions of D and F will be in a straight line passing ırough A.

17. Shew that the locus of the vertices of all the triangles conructed upon a given base, and having their sides in a given ratio, is a ircle.

18. BD, CD are perpendicular to the sides AB, AC of a triangle ABC, and CE is drawn perpendicular to AD, meeting AB in E: shew that the triangles ABC, ACE are similar.

19. If one side of a triangle be produced, and the other shortened by equal quantities, the line joining the points of section will be divided by the base in the inverse ratio of the sides.

20. CAB, CEB are two triangles having a common angle CBA, and the sides opposite to it CA, CE, equal: if BAE be produced to D, and ED be taken a third proportional to BA, AC, then shall the triangle BDC be similar to the triangle BAC.

21. In any right-angled triangle, one side is to the other, as the excess of the hypothenuse above the second, to the line cut off from the first between the right angle and the line bisecting the opposite angle.

22. If from the extremities of the base of a triangle, two straight lines be drawn, each of which is parallel to one of the sides, and equal to the other, the straight lines joining their other extremities with the other extremities of the base, will cut off equal segments from the sides, and each of these will be a mean proportional between the other two segments.

23. If a line AB be divided in extreme and mean ratio in C, and produced to D so that BD is equal to BC the larger segment, then AD is similarly divided in B. Also if a point E be taken in BC, such that BE is equal to AC, AE is similarly divided in C.

24. In any triangle, if a perpendicular be let fall upon the base from the vertical angle, the base will be to the sum of the sides, as the difference of the sides to the difference or sum of the segments of the base made by the perpendicular, according as it falls within or without the triangle.

25. From the vertex of an isosceles triangle two straight lines drawn to the opposite angles of the square described on the base, cut the diagonals of the square in E and F: prove that the line EF is parallel to the base.

26. If an n^{th} part be cut off from any straight line, the rectangle contained by the whole line and the remaining part, shall be equal to n times the rectangle contained by the two parts.

27. If two triangles are equal, and the sides about one angle of the one are reciprocally proportional to the sides about one angle of the other, then two angles are either equal or supplements to each other.

28. If the vertical angle CAB of a triangle ABC be bisected by AD, to which the perpendiculars CE, BF, are drawn from the remaining angles: bisect the base BC in G, join GE, GF, and prove these lines equal to each other.

29. If perpendiculars be drawn from the extremities of the base of a triangle on a straight line which bisects the angle opposite to the base, the area of the triangle is equal to the rectangle contained by either of the perpendiculars, and the segment of the bisecting line between the angle and the other perpendicular.

30. If lines be drawn from the angles of a triangle bisecting the opposite sides, shew that the line joining the points of bisection is parallel to the third side of the triangle.

31. If the three sides of a triangle be bisected, the lines which join the points of bisection will divide the triangle into four equal triangles, each of them similar to the whole triangle.

32. If in the figure, Prop. 10, Book IV, the straight lines DC, BA be produced to meet the circle again in E, F, and EF be joined: shew that the triangle CEF is to the triangle ABD as $3+\sqrt{5} : 2$; and that the triangle ABD is a mean proportional between CEF and BCD.

33. The square of the line bisecting the vertical angle of any triangle is a mean proportional between the differences of the squares of each side containing that angle, and the square of the adjacent segment of the base.

34. If the triangle ABC has the angle at C a right angle, and from C a perpendicular be dropped on the opposite side intersecting it in D, then $AD : DB :: AC^2 : CB^2$.

35. If the triangle ACB has a right angle C, and AD be drawn bisecting the angle A, and meeting CB in D, prove that

$$AC^2 : AD^2 :: BC : BC+BD.$$

36. If through the point of bisection of the base of a triangle any line be drawn, intersecting one side of the triangle, the other produced, and a line drawn parallel to the base from the vertex, this line shall be cut harmonically.

37. If the interior angle BAC, and the exterior angle DAC of any triangle ABC be bisected by lines AE, AF, which also cut BC in E, F: shew that BF, BC, BE are in harmonic progression.

38. ACB is a triangle whose base AB is divided in E, and produced to F, so that $AE : EB$, and also $AF : FB$, as $AC : CB$. Join CE, CF, and shew that the angle ECF is a right angle.

39. The square described upon the side of a regular pentagon in a circle, is equal to the square of the side of a regular hexagon, together with the square upon the side of a regular decagon in the same circle.

40. From B the right angle of a right-angled triangle ABC, Bp is let fall perpendicular to AC, from p, pq is let fall perpendicular to BA, &c.: prove that

$$Bp+pq+\&c. : AB :: AB+AC : BC.$$

41. If CD be drawn from the vertex C to any point D in the base of the triangle ABC, then

$$AC^2 . BD + BC^2 . AD = CD^2 . AB + AB . AD . BD.$$

42. If on the two sides of a right-angled triangle squares be described, the lines joining the acute angles of the triangle and the opposite angles of the squares, will cut off equal segments from the sides; and each of these equal segments will be a mean proportional between the remaining segments.

43. ABC is an equilateral triangle; E any point in AC; in BC produced take $CD=CA$, $CF=CE$, AF, DE intersect in H;

$$\frac{HC}{EC}=\frac{AC}{AC+EC}.$$

44. ABC is an isosceles triangle; draw CE perpendicular to the base AB; draw ADF intersecting CE in D, and CB in F;

$$\frac{DE}{CE}=\frac{CA-CF}{CA+CF}.$$

45. If in a triangle ABC, CA be produced to D, and from CB, BE be cut off equal to AD; then the line ED joining the points of section will divide the base in the inverse ratio of $CD : CE$.

46. In any right-angled triangle ABC, (whose hypothenuse is AB) bisect the angle A by AD meeting CB in D, and prove that

$$2AC^2 : AC^2 - CD^2 :: BC : CD.$$

47. ABC is a right-angled triangle, ADE is an isosceles triangle, and AF is perpendicular to DE: shew that AF is a mean proportional between AC and $\frac{1}{2}(AB+AC)$.

48. If from the angles of a triangle straight lines be drawn bisecting the opposite sides, the sum of the squares of the sides : sum of the squares of the bisecting lines :: 4 : 3.

49. From any two angles of a triangle, straight lines are drawn to the points of bisection of the opposite sides: prove that the point of their intersection is distant from either angle, two-thirds of the line drawn from it.

50. If the exterior angle CAD of a triangle BAC be bisected by a straight line AE which likewise cuts the base BC produced in E: shew that the rectangle BE, EC is equal to the rectangle BA, AC together with the square of AE.

51. If any point be taken in the plane of a parallelogram from which perpendiculars are let fall on the diagonal, and on the sides which include it, the rectangle of the diagonal and the perpendicular on it, is equal to the sum or difference of the rectangles of the sides and the perpendiculars on them.

52. BE, AC are parallel lines; F, G, H, &c. a series of equidistant points in AC; draw Bfh cutting BE in B, and EF, EG, EH, &c. in f, g, h, &c. Bf, Bg, Bh, &c. are in harmonic progression.

53. If diverging lines cut a straight line, so that the whole is to one extreme, as the other extreme is to the middle part, they will intersect every other intercepted line in the same ratio.

54. A straight line CD is drawn bisecting the vertical angle C of a triangle ACB, and cutting the base AB in D; also on AB produced a point E is taken equidistant from C and D: prove that $AE.BE = DE^2$.

55. From an angle of a triangle a line is drawn to the middle point of the opposite side, and through the point of bisection of this line another is drawn from either angle to the side subtending it. Prove that the latter line divides this side into segments which are as 2 to 1.

56. No triangle can be inscribed in a square, greater than half the square.

57. If the opposite sides of a hexagon inscribed in a circle be produced till they meet, the three points of intersection will be in the same straight line.

58. If a hexagon be described about a circle, the three lines joining the opposite angular points, intersect in a point.

59. The square inscribed in a circle is to the square inscribed in the semicircle :: 5 : 2.

60. If a square be inscribed in a right-angled triangle of which one side coincides with the hypothenuse of the triangle, the extremities of that side divide the base into three segments that are continued proportionals.

61. The square inscribed in a semicircle is to the square inscribed in a quadrant of the same circle :: 8 : 5.

62. Shew that if a triangle inscribed in a circle be isosceles, having each of its sides double the base, the squares described upon

the radius of the circle and one of the sides of the triangle shall be to each other in the ratio of 4 : 15.

63. If on the diameter of a semicircle, two equal semicircles be described, and in the curvilinear space a circle be inscribed touching the three semicircles, the radius of this circle shall be to the radius of either less semicircle :: 2 : 3.

64. If a straight line be drawn through the points of bisection of any two sides of a triangle, it will divide the triangle into two parts which are to each other as 1 to 3.

65. If A_1, A_2 be the areas of the squares inscribed in, and circumscribed about a given circle, A that of a regular octagon inscribed in the same circle, shew that $A_1 : A :: A : A_2$.

66. The diagonals of a trapezium, two of whose sides are parallel, cut one another in the same ratio.

67. If from any two points within or without a parallelogram, straight lines be drawn perpendicular to each of two adjacent sides and intersecting each other, they form a parallelogram similar to the former.

68. $ABCD$ is a trapezium of which the opposite sides AD, BC are parallel, shew that

$$AC^2 \sim BD^2 : AB^2 \sim DC^2 :: BC + AD : BC \sim AD.$$

69. The rectangle contained by two lines is a mean proportional between their squares.

70. If AD is drawn bisecting the angle BAC, and AE is drawn at right angles to AD; if $BDCE$ be any straight line cutting AB, AD, AC, AE respectively in the points B, C, D, E: shew that

$$\frac{1}{BE} + \frac{1}{BD} = \frac{2}{BC}.$$

71. If from a point to a straight line there be drawn four lines in harmonical proportion, two of which terminate in its extremities; the whole line and the three parts into which it is divided, will be in harmonical proportion also.

72. If through a given point within a triangle lines be drawn from the angles to the opposite sides, and the points of section be joined, the three first drawn lines will be harmonically divided.

73. HKL, PQR are two triangles; prove that if the straight lines HP, KQ, LR meet in one point, the intersections of KL, QR; LH, RP; HK, PQ lie in a straight line.

74. Four lines AB, CD, EF, GH, drawn in any direction, intersect in the same point P; then if from any point m in one of these lines, another be drawn parallel to the next in order, cutting the remaining two in p and q; the ratio of $mp : pq$ is the same in whichever line the point m is taken.

75. If a straight line be harmonically divided, and the angle contained by lines drawn from the extremity and one point, be bisected by a line drawn from the other point; that line shall be at right angles to the line which is drawn from the other extremity.

76. Through the point of intersection D of two straight lines drawn from two angles of a triangle to bisect the opposite sides, a straight line $EDFG$ is made to pass, meeting two sides in E and F and the remaining side produced in G: prove that

$$\frac{1}{DE} - \frac{1}{DF} = \frac{1}{DG}.$$

77. From A, the lines AB, AC, AD, AE, are drawn intersecting in B, C, D and E; and F, G, H and K; the lines BE and FK which are inclined to each other at any angle, shew that if

$$BE : ED :: BC : CD,$$

$$\text{then } FK : KH :: FG : GH.$$

78. If two lines intersecting each other, are each terminated by two unlimited lines given in position, the ratio of the rectangles contained by their respective segments will be the same with the ratio of the rectangles made by the segments of any other two lines intersecting each other which are similarly terminated, and are respectively parallel to the former.

79. If from any point four straight lines be drawn, so that a line parallel to one of them falling upon the other three may be bisected by the middle one of these, then will the four lines cut any line in harmonic proportion.

80. Let $DACEB$ be points in a straight line such that

$$AC : CB :: AD : DB, \text{ and } AE = EB;$$

shew that rectangle DCE equals rectangle ACB.

81. APB is a quadrant, SPT a straight line touching it at P, PM perpendicular to CA; prove that triangle SCT : triangle ACB :: triangle ACB : triangle CMP.

82. ABC is a triangle, right-angled at A, and having the angle ABC twice the angle ACB, draw BD, bisecting the angle ABC, and meeting AC in D; AE, DF are perpendiculars on BC, meeting it respectively in the points E and F. Shew that

$$\frac{1}{BE\,.\,DF} - \frac{1}{AE\,.\,BF} = \frac{1}{BE\,.\,AE}.$$

83. In the triangle ABC, $AC = 2\,.\,BC$. If CD, CE respectively bisect the angle C, and the exterior angle formed by producing AC; prove that the triangles CBD, ACD, ABC, CDE, have their areas as 1, 2, 3, 4.

84. D, E, are the middle points of the sides CA, CB of a triangle; join D and E, and draw AE, BD, intersecting in O; then shall the areas of the triangles DOE, EOB, BOA, be in geometrical progression.

85. If triangles AEF, ABC have a common angle A, triangle ABC : triangle AEF :: $AB\,.\,AC$: $AE\,.\,AF$.

86. If ABC be a right-angled triangle having the right angle at B, and squares $ABDE$, $CBFG$, be described on the sides AB, CB; and DE, FG be produced to meet AC produced in the points H and K, the triangle ABC will be a mean proportional between the triangles AEH, CGK.

87. If there be three equal rectangles contained by AB, AG; CD, CH; EF, EK. Then if $EK - CH = CH - AG$, $AB - CD$ will be to $CD - EF$ :: AB : EF. Also if this latter proportion hold, $EK - CH = CH - AG$. Prove this, and explain how this is really a geometrical proof of the common propositions of harmonic progression.

88. If about an equilateral triangle another be described, and about this latter another, and so on. Prove that the areas of the triangles are in geometrical progression.

89. If a tangent to two circles be drawn cutting the straight line which joins their centres, the chords are parallel which join the points of contact and the points where the line through the centres cuts the circumferences.

90. Given three unequal circles which do not intersect, and let pairs of double tangents be drawn internally to each pair of them, the three intersections will be in one right line.

91. If one chord in a circle bisect another, and tangents drawn from the extremities of each be produced to meet, the line joining their points of intersection will be parallel to the bisected chord.

92. If corresponding points at which the tangents are parallel, be taken in the circumferences of two circles which cut each other, the straight lines joining them will cut the line passing through both centres in the same point; and if δ be the distance of that point from the centre of the circle whose radius is (r), $\delta = \dfrac{ra}{R \mp r}$, where ($a$) is the distance between the centres of the circles, and R the radius of the other circle.

93. If two lines AP, BP, drawn from two fixed points A, B, have a constant ratio, the locus of P is a circle: prove this and find the radius of the circle.

94. In any isosceles triangle having the angles at the base each double of the vertical angle, if r, r' be the radii of the inscribed and circumscribed circles, (a) either side, shew that

$$r \, . \, r' : \tfrac{1}{2} \, . \, a^2 :: \sqrt{5} - 1 : \sqrt{5} + 3.$$

95. Let ACB be the diameter of any circle whose centre is C, P any point in it, Q a point so taken that $CP : CA :: CA : CQ$; take M any point in the circumference, join QM, AM, PM. Prove that AM always bisects the angle QMP.

96. The rectangle contained by the sides of any triangle is to the rectangle by the radii of the inscribed and circumscribed circles, as twice the perimeter is to the base.

97. In any triangle, the rectangle contained by the excess of half the perimeter above each of the two sides including an angle, is equal to the rectangle contained by the radius of the inscribed circle, and the radius of the circle which touches the base and the two sides produced.

98. If a straight line touch a circle, and a perpendicular be drawn from the point of contact upon any diameter; and if from the extremities of the same diameter, and from the centre, perpendiculars be raised upon the diameter and produced to meet the line which touches the circle, these four perpendiculars are proportional.

99. A point is taken in the diameter of a circle, and another point in the diameter produced, so that the radius is a mean proportional between their distances from the centre: prove that the lines drawn from these points to any, the same point in the circumference, are always in the same ratio.

100. If from the extremity of a diameter of a circle tangents be drawn, any other tangent to the circle terminated by them is so divided at its point of contact that the radius of the circle is a mean proportional between its segments.

101. If through any point in the arc of a quadrant whose radius is R,

two circles be drawn touching the bounding radii of the quadrant, and r, r' be the radii of these circles: shew that $rr' = R^2$.

102. If a circle be inscribed in an equilateral triangle, another similar triangle in the circle, another circle in the last triangle, and so on for ever; the radius of the first circle shall be equal to the sum of the radii of all the others.

103. Let the two circles, radii R, r, which touch (1) the three sides of a triangle ABC, and (2) one side, BC, and the other two produced, touch AB in D_1, D_2, AC in E_1, E_2; shew that

$$BD_1 . BD_2 = CE_1 . CE_2 = R . r.$$

104. ACB is the quadrant of a circle, AD, BD, lines drawn from the extremities A and B, so that AD is always equal to CD. Prove that the point D is in the circumference of a semicircle whose diameter is to the diameter of the circle to which the quadrant belongs, as 1 to $\sqrt{2}$.

105. Let the two diameters AB, CD, of the circle $ADBC$ be at right angles to each other, draw any chord EF, join CE, CF, meeting AB in G and H; prove that the triangles CGH and CEF are similar.

106. Let P be a given point within a circle upon the radius AC, and let a point Q be taken externally upon the same radius produced, so that $CP : CA :: CA : CQ$; if from any point M in the circumference, straight lines MP, MQ be drawn to the two points P and Q, these straight lines will every where have the same ratio, or $MP : MQ :: AP : AQ$.

107. If any triangle ABC be inscribed in a circle, and at the points A, B, C, tangents be drawn, and produced to meet the opposite sides of the triangle produced in the points D, E, F; these points will be in a straight line.

108. Prove that there may be two, but not more than two, similar triangles in the same segment of a circle.

109. If the rectangles under the segments of the diagonals of a quadrilateral figure be equal, a circle may be described about it.

110. By help of VI. D. it may be shewn, that if three of the angular points of a quadrilateral figure in a circle, be equally distant from each other, then the distance of the fourth from the middle one equals the sum of the distances from the other two.

111. $ABCD$ is any trapezium inscribed in a circle. Its sides are produced to meet in P and Q. Prove the following proportion; $PD . DQ : AD . DC :: PB . BQ : AB . BC$.

112. If a straight line drawn from the vertex of an isosceles triangle cutting the base, be produced to the circumference of a circle described about the triangle, the rectangle contained by the whole line produced and the part of it between the vertex and the base, is equal to the square of either of the sides of the triangle.

113. A circle may be described about any quadrilateral figure, if, when any two of its sides are produced to meet, the rectangle by one side produced and the part produced, shall be equal to the rectangle by the other side produced and the part produced.

114. In the figure to Prop. 47, Book I, the points A, D, L, E, are in the circumference of a circle.

115. If the line bisecting the vertical angle of a triangle be divided

into parts which are to one another as the base to the sum of the sides, the point of division is the centre of the inscribed circle.

116. In a plane triangle, if the line joining the centre of the circumscribed circle and the point of intersection of the perpendiculars, be bisected; the point so determined is distant from the vertical angle, two thirds of the line drawn from that angle to the bisection of the base.

117. If two circles touch each other externally and also touch a straight line, the part of the line between the points of contact is a mean proportional between the diameters of the circles.

118. If a perpendicular be drawn from the right angle to the hypothenuse of a right-angled triangle, and circles be inscribed within the two smaller triangles into which the given triangle is divided, their diameters will be to each other as the sides containing the right angle.

119. If from a point without a circle there be drawn three straight lines, two of which touch the circle, and the other cuts it, the line which cuts the circle will be divided harmonically by the convex circumference, and the chord which joins the points of contact.

120. The perpendicular drawn from the centre of a circle on the chord of any arc, is a mean proportional between half the radius and the line made up of the radius and the perpendicular drawn from the centre on the chord of double the arc.

121. About a triangle ABC describe a circle, and through the points ABC draw tangents DAF, FBE, ECD, forming a triangle DFE; shew that the rectangle DE, AF, is to the rectangle EC, DF, in the duplicate ratio of AB to CB.

122. If two straight lines be drawn from the same point, and divided into parts reciprocally proportional to the whole lines, a circle may be described through the extremities and the points of division.

123. Let S be any point in the radius of a circle whose centre is O and radius OA. In OA produced, take OL a third proportional to the two OS, OA, draw any straight line LE to the circumference, and join SE, AE; AE bisects the angle LES. Required a demonstration.

124. If a straight line AB be bisected in C, and upon AB and AC semicircles be described, and from B a straight line be drawn touching the inner semicircle and terminated by the other in D;

$$\text{then } BD : AC :: 4\sqrt{2} : 3.$$

125. If through the vertex, and the extremities of the base of a triangle, two circles be described, intersecting one another in the base or its continuation, their diameters are proportional to the sides of the triangle.

126. A straight line $ABCD$ is terminated and trisected at the points where it meets the circumferences of two circles that cut each other, and it is intersected in E by FG which joins their centres: shew that $AE : ED :: 2EF + EG : 2EG + EF$.

127. If a triangle be inscribed in a circle, and from its vertex, lines be drawn parallel to the tangents at the extremities of its base, they will cut off similar triangles.

128. If a semicircumference be divided into any number of equal parts AB, BC, CD, DE, &c., and the straight lines AB, AC, AD, AE, &c., be drawn, then $AB : AC :: AC : AB + AD$ and $AC : AD + AB ::$ $AD : AC + AE$.

129. If $BEHN$ be a line cutting a circle whose centre is H, and points in the circumference E and N; also if $BFIK$ be a line touching it where EF, NK, are perpendicular on BI; then $BI^2 = \dfrac{EH^2}{FI^2} \cdot BF \cdot BK$.

130. BEF is a circle inscribed in the triangle ABC touching the sides BC, AC, AB, in D, E, F; HIK is a circle touching AB in K and CB, CA, produced in H, I; in CH take $CL = CA$, and in CI take $CM = CB$; then $FK = AM$ and $4AF.FB = AL.MB$.

131. If the sides of a triangle be in arithmetic progression, and a and a_1 be the longest and shortest sides, and r and r_1 the radii of the inscribed and circumscribed circles, $6rr_1 = aa_1$.

132. ABC is a triangle inscribed in a circle; AB, AC, BC, produced, cut a line given in position, in points m, n, p, respectively. If t_m, t_n, t_p, be the lengths of the lines drawn from m, n, p, touching the circle, shew that $t_m . t_n . t_p = An . Bm . Cp = Am . Bp . Cn$.

133. If two circles cut each other so that the circumference of one may pass through the centre of the other, and from that centre, lines be drawn cutting both circumferences, the segments of the lines intercepted between the circumferences, will be to each other as the perpendiculars let fall from the intersections with the inner circumference upon the line which joins the points of intersection of the circles.

134. If one point be taken in the radius of a circle and another in the radius produced, such that the radius is a mean proportional between their distances from the centre, and from the nearer point, a perpendicular be erected from any point in which tangents are drawn to the circle, the line joining the points of contact will always pass through the farther point.

135. If A be the area of any triangle, prove that the area of a triangle whose angular points divide the sides of the former in the ratio of n to 1, is equal to

$$\frac{n^2 - n + 1}{(n+1)^2} . A.$$

136. If two chords in a circle intersect each other, and the two segments of one chord have the same ratio to each other as the two segments of the other chord; a straight line bisecting the angle contained by the two segments which form corresponding terms of the ratios, will pass through the centre of the circle.

137. The product of all the sides of any triangle is equal to four times its area multiplied into the radius of its circumscribed circle.

138. If as in Euclid VI. 3, the vertical angle BAC of the triangle BAC be bisected by AD, and BA be produced to meet CE drawn parallel to AD in E; shew that AD will be a tangent to the circle described about the triangle EAC.

139. If a triangle ABC be inscribed in a circle, and a straight line BD be drawn parallel to the tangent at A, meeting AC or AC produced in D, then will $AD : AB :: AB : AC$.

140. If a circle be inscribed in a right-angled triangle and another be described touching the side opposite to the right angle and the produced parts of the other sides, shew that the rectangle under the radii is equal to the triangle, and the sum of the radii equal to the sum of the sides which contain the right angle.

141. ACB is an isosceles triangle inscribed in a semicircle, of which the diameter is AB; from any point in AC produced, DE is drawn perpendicular to AB, and cutting the semicircle in F, and CB in G. Prove that EF is a mean proportional between EG and ED.

142. If two circles cut each other, and a common chord $ABCDE$ be drawn, meeting one circle in A and D, the other in B and E, and

the line joining these points of intersection in C; then the square of BD shall be to the square of AE as the rectangle BC, CD, is to the rectangle AC, CE.

143. If a circle be inscribed in a triangle, and another be described about it; the distance between their centres is a mean proportional between the radius of the circumscribed circle, and its excess above twice the radius of the inscribed circle.

144. The diagonals AC, BD, of a trapezium inscribed in a circle cut each other at right angles in the point E; the rectangle $AB.BC$: the rectangle $AD.DC :: BE : ED$.

145. If two circles touch each other internally in a point A, from which a straight line ABC is drawn cutting the circles in B, C; then if another straight line drawn through either of these points perpendicular to the line which joins the centres, cut the other circle in D, AD shall be a mean proportional between AB and AC.

146. Two circles whose radii are as $2:3$ touch each other internally; through the centre of the smaller circle a straight line is drawn perpendicular to their common diameter; and from the points where this straight line meets the circumference of the larger circle, tangents are drawn to the smaller circle; shew that these tangents will be perpendicular to each other.

147. If two circles cut each other in any two points, and from either point diameters be drawn, the extremities of these diameters and the other point of intersection are in the same straight line. Also, if on the line joining the points of intersection, as a diameter, a circle be described, and any line be drawn from one extremity of this diameter cutting the two circles, the parts of this line intercepted between the circumference of the circle, and the circumferences of the other two circles shall be proportional to the perpendiculars drawn in these circles from the other extremity of the diameter.

148. If two circles can be described so that each shall touch the other and three sides of a quadrilateral figure, one fourth of the difference of the sums of the opposite sides is a mean proportional between the radii. Express the area of the quadrilateral figure in terms of the sides and the radii of the circles.

149. Two circles intersect in A and B; AD, AD', are diameters; AC, AC', are chords, each of which touches the circle of which it is not a chord; the line AEE' bisects the angle DAD' and cuts the circles in E and E'; then the common tangent to the circles is a mean proportional between the chords DE, $D'E'$; and their common chord AB is a mean proportional between the chords BC, BC'.

150. The sides containing a given angle are in a given ratio and the vertex is fixed; shew that if the extremity of one of the sides moves in a given line, so does the extremity of the other.

151. Two equal circles are drawn intersecting in the points A and B, a third circle is drawn with centre A and any radius not greater than AB intersecting the former circles in D and C. Shew that the three points B, C, D, lie in one and the same straight line.

152. Two equal circles are drawn intersecting in the points A and B, a third circle is drawn with centre A and any radius not greater than AB intersecting the former circles in D and C; join BA and upon it as a diameter, describe a semicircle BEA; then if from the point B any line $BFEG$ be drawn cutting ABD in F, AEB in E, and ACB in G, FE equals EG.

153. S is a fixed point in OA the radius of a circle, centre O; in OA produced, OL is taken a third proportional to OS and OA, and from the point L, any line LPQ is drawn cutting the circle in P and Q, and if the line LPQ cut the line drawn from S at right angles to AO in the point E, prove that LQ is divided harmonically in the points E and P, that is, $LQ : LP :: QE : EP$.

154. S is a fixed point in OA the radius of a circle, centre O; in OA produced, OL is taken a third proportional to OS and OA, and from the point L any line LPQ is drawn cutting the circle in P and Q; it is required to demonstrate the following properties:

(1) If PS be joined, the ratio of LP to PS is a constant ratio.

(2) If QS be also joined, the rectangle contained by PS, SQ is a constant rectangle.

(3) If through S a line be drawn at right angles to AO, tangents to the circle at P and Q will cut this line in one and the same point.

N.B. By means of the first property, it may be shewn that the following proposition inserted in Simson's edition of the "Data" is not true. (Prop. 80.) "If the sides about an angle of a triangle have a given ratio to each other, and if the perpendicular drawn from that angle to the base has a given ratio to the base; the triangle is given in species."

155. A circle, a straight line, and a point being given in position, required a point in the line, such that a line drawn from it to the given point may be equal to a line drawn from it touching the circle. What must be the relation among the data, that the problem may become porismatic, i. e. admit of innumerable solutions?

156. If a point (C') be taken in any one (as AB) of three indefinite straight lines that intersect in A, B, and C; and lines (as $C'B'A'$) be drawn from C' cutting AC, BC (as in B', A'); then all the intersections of each pair of lines (as BB', AA') drawn from B and A to the points of section (B', A'), lie in a line that passes through C.

157. In the sides AB, AC of a given triangle ABC, take two points M, N, and in the line joining them take a point P, such that

$$\frac{MB}{AM}=\frac{AN}{NC}=\frac{MP}{PN}:$$

prove that if PB, PC, be joined, the triangle BPC is twice the triangle AMN; and that the circle described about the triangle AMN will always pass through a fixed point.

158. Lines are drawn from the angular points A, B, C of a triangle through any point, meeting the opposite sides in a, b, c; a circle is described through these three points cutting the same sides in a', b', c'. Shew that Aa', Bb', Cc' meet in one point.

Assuming that $Ab \cdot Bc \cdot Ca = Ba \cdot Cb \cdot Ac$, and conversely, that when this relation holds, the lines pass through one point.

159. In the figure annexed to the 47th Proposition, Book I, if the points G, H; F, K; be joined, the lines GH, FK, and BC, if produced, meet in one and the same point.

160. If any point D be taken in the interior of a triangle, and AD, BD, and CD be respectively produced to meet BC, AC, and AB in E, F, G: then $AF \cdot BG \cdot CE = AG \cdot BE \cdot CF$.

161. If a straight line intersect the two sides AC, BC of a plane triangle in the points b, a, and the base AB produced in c; prove that $Ab \times Bc \times Ca = Ac \times Ba \times Cb$.

162. If perpendiculars be raised upon the middle points of the sides of a triangle, and respectively equal to half of those sides, and the extremities of those perpendiculars be joined; the sum of the squares of these last lines is equal to the sum of the squares of the sides of the triangle together with six times its area.

163. If R be the radius of the circle inscribed in a right-angled triangle ABC, right-angled at A; and a perpendicular be let fall from A on the hypothenuse BC, and if r, r' be the radii of the circles inscribed in the triangles ADB, ACD: prove that $r^2 + r'^2 = R^2$.

164. If from the extremities of the base of a triangle, perpendiculars be let fall on the opposite sides, and likewise straight lines drawn to bisect the same, the intersection of the perpendiculars, that of the bisecting lines, and the centre of the circumscribing circle will be in the same straight line.

165. If a circle be inscribed in any plane triangle, and a right line drawn from the vertex to the point where the circle touches its base: prove that the middle point of that line, the centre of the circle, and the middle point of the base of the triangle, are in the same straight line.

166. A circle is inscribed in a triangle; and a second triangle is formed, whose sides are equal to the distances of the points of contact from the angles of the triangle. If r be the radius of the circle inscribed in the first triangle; and ρ, ρ' the radii of the inscribed and circumscribed circles of the second triangle, then will $r^2 = 2\rho\rho'$.

167. The side of an equilateral triangle inscribed in a circle : side of a square inscribed in the same :: $\sqrt{3} : \sqrt{2}$; and the area of the triangle : area of a square :: $3\sqrt{3} : 4\sqrt{4}$.

168. If two equilateral polygons be drawn, one within, and the other about a circle, and another of half the number of sides be inscribed in the circle, this shall be a mean proportional between the other two.

169. One of the sides of a rectilinear figure is divided into any number of parts, and figures described upon the parts, similar and similarly situated to the rectilinear figure, shew that the sum of the peripheries of the smaller figures is equal to the periphery of the larger; and if the parts be equal and n in number, that the areas of the smaller figures are together equal to $\frac{1}{n}$th of the area of the larger.

170. Area of inscribed pentagon: area of circumscribing pentagon :: the square of the radius of the circle inscribed within the greater pentagon : the square of the radius of circle circumscribing it.

171. If the sides of any regular figure be bisected and each two adjacent points of bisection be joined, a similar figure will be formed, and its area will be to that portion of the area of the original figure which is without it as the square on the diameter of the circle circumscribing it is to the square on one of its sides.

172. A regular hexagon inscribed in a circle is a mean proportional between an inscribed and circumscribed equilateral triangle.

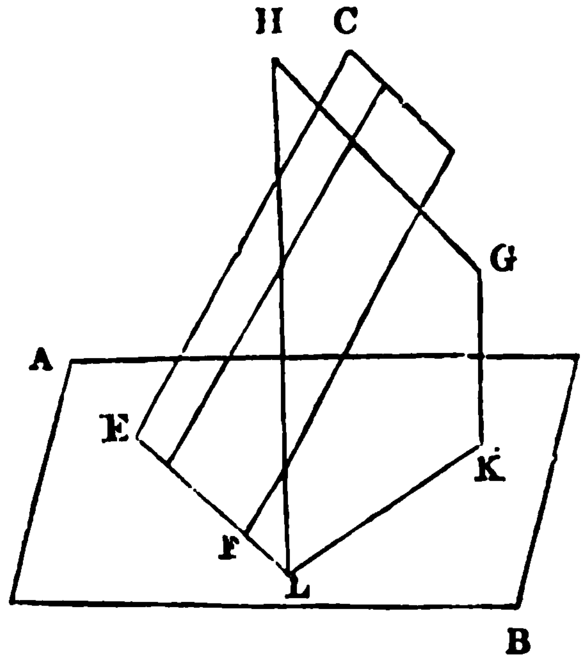
173. If two regular polygons P and Q of the same number of sides be, the one inscribed in a circle, and the other circumscribed about it, and also a polygon of twice the number of sides R be inscribed in the same circle, then R shall be a geometrical mean between P and Q.

GEOMETRICAL EXERCISES ON BOOK XI.

THEOREM I.

Define the projection *of a straight line on a plane; and prove that if a straight line be perpendicular to a plane, its projection on any other plane, produced if necessary, will cut the common intersection of the two planes at right angles.*

Let AB be any plane and CEF another plane intersecting the former at any angle in the line EF; and let the line GH be perpendicular to the plane CEF.

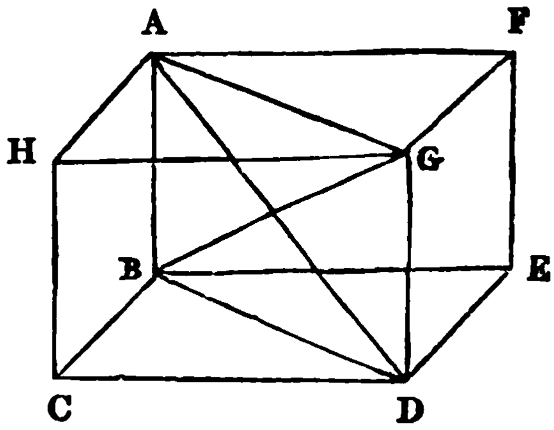


Draw GK, HL perpendicular on the plane AB, and join LK,
then LK is the projection of the line GH on the plane AB;
produce EF, to meet KL in the point L;
then EF, the intersection of the two planes, is perpendicular to LK, the projection of the line GH on the plane AB.
Because the line GH is perpendicular to the plane CEF,
every plane passing through GH, and therefore the projecting plane $GHKL$ is perpendicular to the plane CEF;
but the projecting plane $GHLK$ is perpendicular to the plane AB; (constr.)
hence the planes CEF, and AB are each perpendicular to the third plane $GHLK$;
therefore EF, the intersection of the planes AB, CEF, is perpendicular to that plane;
and consequently, EF is perpendicular to every straight line which meets it in that plane;
but EF meets LK in that plane.
Wherefore, EF is perpendicular to KL, the projection of GH on the plane AB.

THEOREM II.

Prove that four times the square described upon the diagonal of a rectangular parallelopiped, is equal to the sum of the squares described on the diagonals of the parallelograms containing the parallelopiped.

Let AD be any rectangular parallelopiped; and AD, BG two diagonals intersecting one another; also AG, BD, the diagonals of the two opposite faces HF, CE.



Then it may be shewn that the diagonals AD, BG, are equal; as also the diagonals which join CF and HE: and that the four diagonals of the parallelopiped are equal to one another.

The diagonals AG, BD of the two opposite faces HF, CE are equal to one another: also the diagonals of the remaining pairs of the opposite faces are respectively equal.

And since AB is perpendicular to the plane CE, it is perpendicular to every straight line which meets it in that plane,

therefore AB is perpendicular to BD,

and consequently ABD is a right-angled triangle.

Similarly, GDB is a right-angled triangle.

And the square of AD is equal to the squares of AB, BD, (I. 47.)

also the square of BD is equal to the squares of BC, CD,

therefore the square of AD is equal to the squares of AB, BC, CD;

similarly the square of BG or of AD is equal to the squares of AB, BC, CD.

Wherefore the squares of AD and BG, or twice the square of AD, is equal to the squares of AB, BC, CD, AB, BC, CD;

but the squares of BC, CD are equal to the square of BD, the diagonal of the face CE;

similarly, the squares of AB, BC are equal to the square of the diagonal of the face HB;

also the squares of AB, CD, are equal to the square of the diagonal of the face BF; for CD is equal to BE.

Hence, double the square of AD is equal to the sum of the squares of the diagonals of the three faces HF, HB, BC.

In a similar manner, it may be shewn, that double the square of the diagonal is equal to the sums of the squares of the diagonals of the three faces opposite to HF, HB, BC.

Wherefore, four times the square of the diagonal of the parallelopiped is equal to the sum of the squares of the diagonals of the six faces.

PROBLEMS.

1. Two of the three plane angles which form a solid angle, and also the inclination of their planes being given, to find the third plane angle.

2. Having three points given in a plane, find a point above the plane equidistant from them.

3. To describe a circle which shall touch two given planes, and pass through a given point.

4. Two triangles have a common base, and their vertices are in a straight line perpendicular to the plane of the one; there are given the vertical angle of the other, the angles made by each of its sides with the plane of the first and the distance of the vertices of the two triangles, to find the common base.

5. Find the distance of a given point from a given line in space.

6. Find a point in a given straight line such that the sums of its distances from two given points (not in the same plane with the given straight line) may be the least possible.

7. Draw a line perpendicular to two lines which are not in the same plane.

8. Two planes being given perpendicular to each other draw a third perpendicular to both.

9. Required the perpendicular from the vertex upon the base of a triangular pyramid, all the sides of which are equilateral triangles of a given area.

10. Given the lengths and positions of two straight lines which do not meet when produced and are not parallel; form a parallelopiped of which these two lines shall be two of the edges.

11. How many triangular pyramids may be formed whose edges are six given straight lines, of which the sum of any three will form a triangle?

12. Bisect a triangular pyramid by a plane passing through one of its angles, and cutting one of its sides in a given direction.

13. Define a *cube*. When this solid is cut by a plane obliquely to any of its sides, the section will be an *oblong*, always greater than the side, if made by cutting opposite sides. Draw a plane cutting two adjacent sides, so that the section shall be equal and similar to the side.

THEOREMS.

3. If two straight lines are parallel, the common section of any two planes passing through them is parallel to either.

4. If three limited straight lines be parallel, and planes pass through each two of them, and the extremities be joined, a prism will be formed, the ends of which will be parallel if the straight lines be equal.

5. If two straight lines be parallel, and one of them inclined at any angle to a plane, the other also shall be inclined at the same angle to the same plane.

6. Parallel planes are cut by parallel straight lines at the same angle.

7. If from a point A above a plane, straight lines AB, AC be drawn, meeting it in B and C, of which AB is perpendicular to the plane, and AC perpendicular to a straight line DC in that plane, and CB be joined, CB shall be perpendicular to DC.

8. Three straight lines not in the same plane, but parallel to and equidistant from each other, are intersected by a plane, and the points of intersection joined; shew when the triangle thus formed will be equilateral and when isosceles.

9. If two straight lines in space be parallel, their projections on any plane will be parallel.

10. Of all the angles, which a straight line makes with any straight lines drawn in a given plane to meet it, the least is that which measures the inclination of the line to the plane.

11. Three parallel straight lines are cut by parallel planes, and the points of intersection joined, the figures so formed are all similar and equal.

12. If a straight line be at right angles to a plane, the intersection of the perpendiculars let fall from the several points of that line, on another plane, is a straight line which makes right angles with the common section of the two planes.

13. If two straight lines be cut by four parallel planes, the two segments, intercepted by the first and second planes, have the same ratio to each other as the two segments intercepted by the third and fourth planes.

14. If there be two straight lines which are not parallel, but which do not meet, though produced ever so far both ways, shew, that two parallel planes may be determined so as to pass, the one through the one line, the other through the other; and that the perpendicular distance of these planes is the shortest distance of any point that can be taken in the one line from any point taken in the other.

15. If from a point above the plane of a circle straight lines be drawn to the circumference there will be only two of them equal in length, and they will be equidistant from the shortest and longest, and on opposite sides. What is the exception to this proposition?

16. Two planes intersect each other, and from any point in one of them a line is drawn perpendicular to the other, and also another line perpendicular to the line of intersection of both; shew that the plane which passes through these two lines is perpendicular to the line of intersection of the plane.

17. Two points are taken on a wall and joined by a line which passes round a corner of the wall. This line is the shortest when its parts make equal angles with the edge at which the parts of the wall meet.

18. If four straight lines in two parallel planes be drawn, two from one point and two from another, and making equal angles with another plane perpendicular to these two, then if the first and third be parallel, the second and fourth will be likewise.

19. If, round a line which is drawn from a point in the common

section of two planes at right angles to one of them, a third plane be made to revolve, shew that the plane angle made by the three planes is then the greatest, when the revolving plane is perpendicular to each of the two fixed planes.

20. The content of a rectangular parallelopipedon whose length is any multiple of the breadth and breadth the same multiple of the depth is the same as that of the cube whose edge is the breadth.

21. A rectangular parallelopiped is bisected by all the planes drawn through the axis of it.

22. If a straight line be divided into two parts, the cube of the whole line is equal to the cubes of the two parts together with thrice the right parallelopiped contained by their rectangle and the whole line.

23. If a right-angled triangular prism be cut by a plane, the volume of the truncated part is equal to a prism of the same base and of height equal to one third of the sum of the three edges.

24. If a four-sided solid be cut off from a given cube, by a plane passing through the three sides which contain any one of its solid angles, the square of the number of standard units in the base of this solid, shall be equal to the sum of the squares of the numbers of similar units, contained in each of its sides.

25. In an oblique parallelopiped the sum of the squares of the four diagonals equals the sum of the squares of the twelve edges.

26. If any point be taken within a given cube, the square described on its distance from the summit of any of the solid angles of the cube, is equal to the sum of the squares described on its several perpendicular distances from the three sides containing that angle.

27. Shew that a cube may be cut by a plane, so that the section shall be a square greater in area than the face of the cube in the proportion of 9 to 8.

28. Why cannot a sheet of paper be made to represent the vertex of a pyramid, without folding?

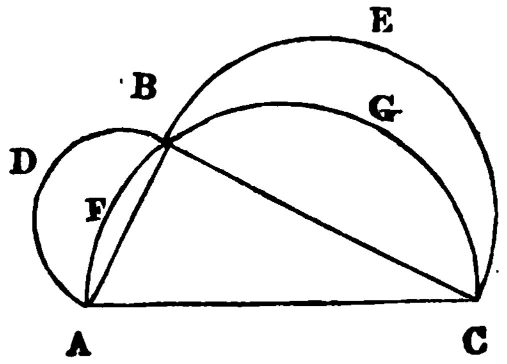
29. A, B are two fixed points in space, and CD a constant length of a given straight line; prove that the pyramid formed by joining the four points A, B, C, D is always of the same magnitude, on whatever part of the given line CD be measured.

30. Define similar rectilineal figures; and prove, that if a pyramid with a polygon for its base be cut by a plane parallel to the base, the section will be a polygon similar to the base.

GEOMETRICAL EXERCISES ON BOOK XII.

THEOREM I.

If semicircles ADB, BEC *be described on the sides* AB, BC *of a right-angled triangle, and on the hypothenuse another semicircle* AFBGC *be described, passing through the vertex* B*; the lunes* AFBD *and* BGCE *are together equal to the triangle* ABC.



It has been demonstrated (XII. 2.) that the areas of circles are to one another as the squares of their diameters; it follows also that semicircles will be to each other in the same proportion.

Therefore the semicircle *ADB* is to the semicircle *ABC*, as the square of *AB* is to the square of *AC*,

and the semicircle *CEB* is to the semicircle *ABC* as the square of *BC* is to the square of *AC*,

hence the semicircles *ADB, CEB,* are to the semicircle *ABC* as the squares of *AB, BC* are to the square of *AC*;

but the squares of *AB, BC* are equal to the square of *AC*: (I. 47.)

therefore the semicircles *ADB, CEB* are equal to the semicircle *ABC*. (V. 14.)

From these equals take the segments *AFB, BGC* of the semicircle on *AC*, and the remainders are equal,

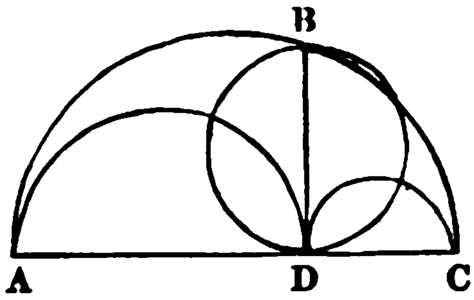
that is, the lunes *AFBD, BGCE* are equal to the triangle *BAC*.

THEOREM II.

If on any two segments of the diameter of a semicircle semicircles be described, all towards the same parts, the area included between the three circumferences (called ἄρβηλος) will be equal to the area of a circle, the diameter of which is a mean proportional between the segments. (Archimedis Lemm. Prop. 4.)

Let *ABC* be a semicircle whose diameter is *AB*,
and let *AB* be divided into any two parts in *D*,
and on *AD, DC* let two semicircles be described on the same side;
also let *DB* be drawn perpendicular to *AC*.

Then the area contained between the three semicircles, is equal to the area of the circle whose diameter is BD.



Since AC is divided into two parts in C,
the square of AC is equal to the squares of AD, DC, and twice the rectangle AD, DC; (II. 4.)
and since BD is a mean proportional between AD, DC;
the rectangle AD, DC is equal to the square of DB, (VI. 17.)
therefore the square of AC is equal to the squares of AD, DC, and twice the square of DB.

But circles are to one another as the squares of their diameters or radii, (XII. 2.)

therefore the circle whose diameter is AC, is equal to the circles whose diameters are AD, DC, and double the circle whose diameter is BD;

wherefore the semicircle whose diameter is AC is equal to the circle whose diameter is BD, together with the two semicircles whose diameters are AD and DC:

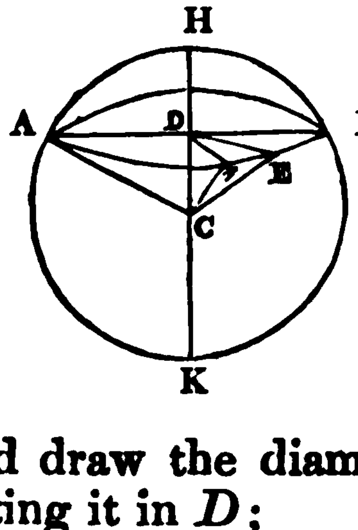
if the two semicircles whose diameters are AD and DC be taken from these equals,
therefore the figure comprised between the three semi-circumferences is equal to the circle whose diameter is DB.

THEOREM III.

Every section of a sphere by a plane is a circle.

If the plane pass through the centre of the sphere, it is manifest that the section is a circle, having the same diameter as the generating semicircle.

But if the cutting plane does not pass through the centre,
let AEB be any other section of the sphere made by a plane not passing through the centre of the sphere.



Take the centre C, and draw the diameter HCK perpendicular to the section AEB, and meeting it in D;
draw AB passing through D, and join AC;
take E, F, any other points in the line AEB, and join CE, DE; CF, DF.

Then since CD is perpendicular to the plane AEB, it is perpendicular to every straight line which meets it in that plane,

therefore the angles CDA, CDE, CDF are right angles,

and CA, CF, CE, being lines drawn from C, the centre of the sphere to points in the surface, are therefore equal to one another.

Hence, in the right-angled triangles, CDA, CDF, CDE;

the square of DA is equal to the difference of the squares of CA and CD;

and the square of DF is equal to the difference of the squares of CF, and CD;

also the square of DE is equal to the difference of the squares of CE and CD:

therefore the squares of DA, DF, DE are equal to one another;

and therefore the lines DA, DF, DE are equal to one another.

But DA, DF, DE are three equal lines drawn from the same point D, in the same plane,

hence the points A, F, E lie in the circumference of a circle of which D is the centre.

THEOREM IV.

There can be only five regular solids.

If the faces be equilateral triangles. The angle of an equilateral triangle is one third of two right angles; and six angles, each equal to the angle of an equilateral triangle, are equal to four right angles; and therefore a number of such angles less than *six*, but not less than *three* are necessary to form a solid angle. Hence there cannot be more than three regular figures whose faces are equal and equilateral triangles.

If the faces be squares. Since four angles, each equal to a right angle, can fill up space round a point in a plane. A solid angle may be formed with three right angles, but not with a number greater or less than three. Hence, there cannot be more than one regular solid figure whose faces are equal squares.

If the faces be equal and regular pentagons. Since each angle of a regular pentagon is a right angle and a fifth of a right angle: the magnitude of three such angles being less than four right angles, may form a solid angle, but four, or more than four, cannot form a solid angle. Hence, there cannot be more than one regular figure whose faces are equal and regular pentagons.

If the faces be equal and regular hexagons, heptagons, octagons, or any other regular figures; it may be shewn that no number of them can form a solid angle.

Wherefore there cannot be more than five regular solid figures, of which, there are three, whose faces are equal and equilateral triangles; one, whose faces are equal squares; and one, whose faces are equal and regular pentagons.

PROBLEMS.

1. In what sense is it said that the circle does not admit of quadrature? Describe generally the process by which Archimedes obtained a first approximation to the ratio of the circumference of a circle to its diameter.

2. Given a circle traced upon a plane, describe another whose area is exactly twice as great as that of the former.

3. ABC is a circle of given radius, describe another concentric circle abc whose area shall be equal to 3 times the area of ABC.

4. Divide a circle into any number of equal parts by means of concentric circles.

5. To divide a circle into any number of equal parts, the perimeters of which shall be equal to the circumference of the circle.

6. Two circles touch each other internally, and the area of the *lune* cut out of the larger is equal to twice the area of the smaller circle. Required the ratio of the diameters of these circles.

7. If on one of the radii of a quadrant a semicircle be described; and on the other, another semicircle so described as to touch the former and the quadrantal arc; compare the area of the quadrant with the area of the circle described in the figure bounded by the three curves.

8. The centres of three circles A, B, and C are in the same right line, B and C touch A internally, and each other externally; P, Q being the points where A is touched by B, C respectively: to find a point R on A such that the portion of the lune PR intercepted between B and A may be equal to the portion of QR between C and A.

9. The diameter of a circle is divided into two parts, upon each of which as diameters circles are described; when the remaining area of the great circle is equal to that of one of these two circles, find the ratio which the parts of the diameter bear to one another.

10. Describe a sphere about a given regular tetrahedron; and the edge of the tetrahedron being 1, find the radius of the sphere.

11. Having given an irregular fragment, which contains a portion of spherical surface: shew how the radius of the sphere, to which the fragment belongs, may be practically determined.

12. Let three given spheres be placed on a horizontal plane in mutual contact with each other; find the sides of the triangle formed by joining the points in which the spheres touch the plane.

13. Construct the five regular solids.

14. Having given six straight lines, of which each is less than the sum of any two, determine how many tetrahedrons can be formed, of which these straight lines are the edges.

15. Inscribe a sphere within a tetrahedron.

16. Find the dihedral angle contained by two adjacent faces of a regular octahedron: and find its solidity.

17. Shew how to find the content of a pyramid, whatever be the figure of its base, the altitude and area of the base being given.

18. A pyramid of triangular base is composed of ten spherical balls of given radius; the base is composed of six, the next layer of three, and the remaining one is placed upon them. Find the distance of the upper ball from the ground.

THEOREMS.

5. The area of a circle is equal to half the rectangle contained by two straight lines which are equal to its circumference and radius.

6. Compare the angle in a segment, which is one fourth part of a circle, with a right angle.

7. In different circles the radii which bound equal sectors contain angles reciprocally proportional to their circles.

8. Prove that the sectors of two different circles are equal, when their angles are inversely as the squares of the radii.

9. If the diagonals of a quadrilateral inscribed in a circle cut each other at right angles, and circles be described on the sides; prove that the sum of two opposite circles will be equal to the sum of the other two.

10. If circles be inscribed in the triangles formed by drawing the altitude of a triangle right-angled at the vertex, the circles and the triangles are proportional.

11. Two straight lines are inclined to each other at a given angle, find the area of all the circles which can be described touching each other and the two given lines, the position of the centre of the last circle being given.

12. If two chords of a circle intersect each other either within or without the circle at right angles; and if on these segments as diameters, circles be described, the areas of these four circles are together equal to that of the original circle.

13. Shew that the semicircles described on the diagonals of a right-angled parallelogram together equal the sum of the semicircles described on the sides.

14. The diameter of a semicircle is divided into any number of parts, and on these parts semicircles are described. Shew that their circumferences are together equal to that of the given semicircle.

15. If two circles be so placed that their planes are parallel, and the straight line which joins their centres perpendicular to the plane of each; the straight lines which join the opposite extremities of any pair of parallel diameters will all intersect the straight line which joins the centres in the same point: if the circles be equal, that point will be the bisection of the aforesaid straight line.

16. If through a point without the plane of a circle straight lines be drawn to its circumference and a plane be drawn parallel to the circle on either side of the point, the points of intersection of the lines with the plane will be in a circle, and the area of this and of the first circle will be as the squares of their distances from the point given.

17. If the arc of a semicircle be trisected, and from the points of section lines be drawn to either extremity of the diameter, the difference of the two segments thus made will be equal to the sector which stands on either of the arcs.

18. The centres of three circles A, B, and C are in the same right line, B and C touch A internally, and each other externally. Shew that the portion of the area of A, which is outside B and C, is equal to the area of the semicircle described on the chord of A which touches B and C at their point of contact.

19. If three equal circles intersect, so that each of the circumferences pass through the centres of the other two, the spaces bounded by the circumferences intercepted will be all equal.

20. Take any three points, A, B, C in the circumference of a circle. Join AB, BC, AC, and draw AD. AE parallel to the tangents at B and C, and meeting BC produced if necessary in D and E; and prove that the segments BD and EC are to each other in the duplicate ratio of AB to AC.

21. If the diameter AB of a circle be divided into an odd number (n) of equal parts, and C and D be the $\left(\frac{n-1}{2}\right)^{\text{th}}$ and $\left(\frac{n+1}{2}\right)^{\text{th}}$ divisions; and AEC, AFD, CGB, DHB be semicircles: shew that the perimeter of the figure $AECGBHDF$ is equal to that of the circle, and its area an n^{th} part of the area of the circle.

22. The circle inscribed in a square is equal to four equal circles touching one another and the sides of that square internally.

23. If AB be a circular arc, centre O, and AD be drawn perpendicular to BO, and the arc AC taken equal to AD, then the sector BOC equals the segment ACB.

24. Let AB and DC be two diameters of a given circle, at right angles to each other; AEB a circular arc described with radius DB or DA; prove that the area of the lune $AEBC$ = area of triangle ADB.

25. If two points B, D, be taken at equal distances from the ends of the arc of a quadrant, and perpendiculars BG, DH be drawn to the extreme radius; the space $BGHD$ shall be equal to the sector BOD.

26. ABC is an isosceles right-angled triangle. On BC is described a semicircle $BDEC$, and BFC is a circle whose radius is AB and centre A. The segment BCF is equal to the segments BAD, ACE.

27. If a semicircle be described on the hypothenuse AB of a right-angled triangle ABC, and from the centre E, the radius ED be drawn at right angles to AB, shew that the difference of the segments on the two sides equals twice the sector CED.

28. If semicircles be described upon the sides of a right-angled triangle on the interior, the difference between the sum of the circular segments thus standing upon the exterior of the sides and segments of the base equals the space intercepted by the circumferences described on the sides.

29. Semicircles are described upon the radii CA, CB of a quadrant, and intersect each other in a point D, shew (*a*) That the points B, D, A are in one straight line. (*b*) That the area *common* to both semicircles is equal to the area without them. (*c*) That the remaining areas of the two semicircles are equal, each is one fourth of the square on AC.

30. If two parallel planes cut a sphere so that the sections are equal, they are equidistant from the centre.

31. Shew that all lines drawn from an external point to touch a sphere are equal to one another; and thence prove that if a tetrahedron can have a sphere inscribed in it touching its six edges, the sum of every two opposite edges is the same.

32. If two equal circles cut one another in the diameter, and a plane cut them perpendicularly to the same diameter, the points of section of this plane with the circumferences, are in a circle.

33. In any polyhedron having different faces, some with an even, some with an odd number of sides: shew that the number of those faces which have an odd number of sides is necessarily an even number.

34. The angles of inclination of the faces of a regular tetrahedron and of a regular octahedron are supplementary to each other.

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TO THE

PROBLEMS AND THEOREMS

IN THE

GEOMETRICAL EXERCISES.

ABBREVIATIONS.

Senate House Examination for Honors. S. H.
Smith's Mathematical Prizes. S. P.
Bell's University Scholarships. B. S.
St Peter's College. Pet.
Claire Hall. Cla.
Pembroke College. Pem.
Gonville and Caius College. Cai.
Trinity Hall. T. H.
Corpus Christi College. C. C.
King's College. Ki.
Queens' College. Qu.
St Catharine's Hall. Cath.
Jesus College. Jes.
Christ's College. Chr.
St John's College. Joh.
Magdalene College. Mag.
Trinity College. Trin.
Emmanuel College. Emm.
Sidney Sussex College. Sid.
Downing College. Down.

In the years the centuries are omitted and the places are supplied by a comma prefixed, thus ,45 means 1845.

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AN APPENDIX

TO THE LARGER EDITION OF

EUCLID'S
ELEMENTS OF GEOMETRY;

CONTAINING

ADDITIONAL NOTES ON THE ELEMENTS,
A SHORT TRACT ON TRANSVERSALS,
AND HINTS FOR THE SOLUTION OF THE PROBLEMS, &c.

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THIS Appendix, presented to the student, is intended to supply some omissions in the notes, and to afford some assistance in the application of the principles of Geometry to the solution of Problems.

The first part supplies some additional notes on the elements, and a more full exposition of the method of the Geometrical Analysis.

The second consists of a short tract on the theory of Transversals, embracing only the most elementary properties.

The third part consists of hints, remarks, &c. on the Problems and Theorems. The constructive part of the analysis or synthesis is generally given, either wholly or partially, and the rest with the demonstration is left to the student. In some instances only brief remarks and references to Euclid are given.

These hints and remarks, it is hoped, will be found serviceable to the student in his geometrical studies, without paralyzing the efforts of his own mind.

R. P.

TRINITY COLLEGE,
17th November, 1847.

ADDITIONAL NOTES TO BOOK I.

It has been maintained by some writers both of ancient and modern times, that Geometry is a perfectly abstract Science, a body of truths completely independent of all human observation and experience. The truths of Geometry may, possibly, be a portion of absolute and universal truths, such as change not in all time, and which maintain the same constant universality under every conceivable state of existence. Still it must be admitted that however abstract and independent of experience such truths may be in themselves, it was not as such that they were originally discovered, or in that form that they are apprehended by the human mind.

The natural process of the human mind in the acquirement of knowledge and in the discovery of truth, is, to proceed from the *particular* to the *general*, from the *sensible* to the *abstract*. It is perhaps not too much to affirm, that the human mind would never have speculated on the abstract properties of circles and triangles unless some *visible forms* of such figures had first been exhibited to the senses. It does seem more probable and analogous to the rise and progress of other branches of human knowledge, that the fundamental truths of Geometry should have been first discovered from suggestions made to the senses; and this opinion, too, is not repugnant to the earliest historical notices existing on the subject. The human mind is so constituted that exact knowledge in any Science can only be acquired progressively. Every successive step in advance must be taken as a sequel to, and dependent upon, the previous acquirements; and some intelligible facts and first principles must form the basis of all human Science.

Now the only possible way of explaining terms denoting simple perceptions is to excite those simple perceptions. The impossibility of defining a word expressive of a simple perception is well known to every one who has paid any attention to his own intellectual progress. The only way of rendering a simple term intelligible is to exhibit the object of which it is the sign, or some sensible representation of it. A straight line therefore must be drawn, and by drawing a curved line and a crooked line, the distinction will be perfectly understood. Again, the definition of a complex term consists merely in the enumeration of the simple ideas for which it stands, and it will be found that all definitions must have some term or terms equally requiring definition or explanation with the one defined. The sensible evidence of things is only to be acquired by the evidence of the senses.

The definitions of Euclid appeal directly to the senses; and the fundamental theorem (Euc. I. 4) which forms the basis of all the succeeding propositions, is demonstrated by one of the simplest appeals to experience. At every step there is a reference made to something exhibited to the senses, the coincidence of the lines, the angles, and lastly, the surfaces of the two triangles, and by shewing a perfect coincidence, their equality is inferred. The instance exhibited, and the proof applied to it, is equally valid for any triangles whatever which have the same specified conditions given in the hypothesis. The same reasoning may be applied to any similar case which can be conceived, and thus from a single instance demonstrated by appeal to the senses, we are led to admit the statement contained in the general enunciation. These considerations appear to support the opinion, that the truths of Geometry, as a portion of human Science, rest ultimately on the evidence of the senses.

It may also be suggested, whether it be not a point of considerable importance to be able to discriminate, where human Science begins, and how certainty is acquired.

Prop. XXIX. With respect to the different proposals made for the amendment of Euclid's method of treating the subject of parallel straight lines, it may be observed, that they all consist in setting out either with a different or a modified result from that of Euclid,—all true, and more or less obvious to the senses.

Euclid has discussed the elementary properties of *triangles*, or of two lines which meet one another and are intersected by a third line, before he has entered upon the discussion of the properties of two lines which do not meet when they are intersected by a third line. The principal objection to Euclid's method of treating the subject of parallel lines, is the assumption of one truth as an *axiom*, which forms the converse of a *theorem* which he has demonstrated as the seventeenth proposition of the first book. Almost every writer on the subject admits the necessity of assuming some positive property of parallel lines as the basis of the reasonings on such lines: and that amendment of Euclid's method would seem to be the best which simply supplies a defect, and leaves the so-called twelfth axiom to assume its rightful position as a theorem, and to fall into its proper place after the seventeenth proposition.

Two straight lines in the same plane which do not meet, when produced, may be *convergent* or *divergent* with respect to each other, according to the directions in which both lines are produced; or, when produced in either direction, they may be neither *divergent* nor *convergent*.

When a third line falls upon two straight lines and makes the two interior angles on one side of it less than two right angles; on that side of the line, the two straight lines are *convergent*, and will, if produced far enough, meet one another, as it is stated in the so-called twelfth axiom. On the other side of the line, the two interior angles are greater than two right angles, and the two straight lines are *divergent*, and will never meet, how far soever they may be produced.

The limiting position of the two straight lines, is, when they are neither *convergent* nor *divergent*, that is, when they do not meet when produced in either direction; and such lines are then said to be *parallel* to one another. If the two parallel lines be intersected by a third line, the following properties exist respecting the angles, whether the intersecting line be perpendicular or be not perpendicular to either of the parallel lines.

(1) The two interior angles on each side of the intersecting line are equal to two right angles. Euc. I. 28.

(2) The alternate angles on each side of the intersecting line are equal to one another. Euc. I. 27.

(3) The exterior angles are equal to their corresponding interior angles on the same side of the intersecting line. Euc. I. 28.

If the intersecting line be perpendicular to one of the parallel lines; it is also perpendicular to the other: and

(4) The perpendicular distance between the two lines is always the same.

If it has been correctly stated, that all *axioms* are in reality *theorems* assumed without proof, and that all demonstrated truths must depend on some truths assumed or admitted to be true, not necessarily truths first discovered, but truths the most simple and which arise directly from the subject of the definitions; the doctrine of parallel lines may be legitimately treated by assuming some one of the positive properties of such lines as the basis for demonstrating their other properties.

Any one of the four positive properties just stated may be assumed as the foundation of the theory of parallel lines, and that theory may be made to depend on the *distance* between the parallel lines, or on some of the angles made by any intersecting line. If the former assumption be adopted: does it not involve that lines which are perpendicular to one of the parallel lines, are also perpendicular to the other, as well as that all such distances are equal? This would require more to be taken for granted, than

would be necessary by assuming the equality of the exterior and interior angles, or either of the two remaining properties respecting the angles which the parallels make with any intersecting line.

In general, with respect to *indirect* demonstrations, it may be questioned whether they ought ever to be admitted as a legitimate mode of proof in a primary and fundamental proposition. *Indirect* demonstrations are properly and most effectually applied in proving the converse of a proposition which has been demonstrated by a *direct* appeal to assumed or demonstrated principles. To make a negative property or an indirect demonstration the basis of a *positive* doctrine, seems to be an inversion of the natural process the mind pursues in the investigation of truth, and to leave the doctrine exposed to all the objections which may be made from the illogical attempt to prove a *positive* from a *negative* truth.

For a more philosophical view of the subject, reference may be made to Professor Powell's able Pamphlet, "On the Theory of Parallel Lines."

ADDITIONAL NOTES TO BOOK V.

THE doctrine of Ratio and Proportion is one of the most important in the whole course of mathematical truths, and it appears probable that if the subject were read simultaneously in the Algebraical and Geometrical form, the investigations of the properties, under both aspects, would mutually assist each other, and both become equally comprehensible; also their distinct characters would be more easily perceived.

In the definition of Ratio as given by Euclid, all reference to a third magnitude of the same geometrical species, as a measure for comparing the two, whose ratio is the subject of conception, has been carefully avoided. It is their relation one to the other, without the intervention of their sum, their difference, or any standard unit whatever. One of the magnitudes is made the standard by which the other is estimated; but even this is not effected by means which require the inquiry, "how many times is the one contained in the other?" Such a procedure would, undoubtedly, have been legitimate, had it been also convenient: but it would at once have led to considerations respecting fractions or irrational functions. Euclid effects his demonstrations by the aid of *multiples* instead of *quotients*: by repetitions of the magnitudes themselves, instead of finding what multiple the one magnitude is of the other, or what multiple each of them is of some third magnitude. Euclid's results too are obtained with greater logical brevity, and he employs fewer principles in their establishment, than any writer has yet been able to do, *in a strictly legitimate form*, by means of the Modern Algebra.

The simple idea of ratio of itself, and absolutely considered, could not, however, lead to any conclusion respecting the properties of figures any more than the mere idea of magnitude. *It is by the comparison of two or more magnitudes subjected to some specific conditions* in the first four books of the Elements, that all the propositions have been demonstrated: and *it is by the comparison of the ratios of two or more pairs of correlative magnitudes, subject to specified conditions*, that the properties of figures depending on ratio are to be established. Euclid does not offer even a solitary property of a single ratio, or of the magnitudes whose ratio it is: except, indeed, that already adverted to, as constituting, in fact, an axiom.

As each of two ratios involves the idea of two magnitudes, the least number of magnitudes between which a comparison of ratios is possible, is four, two for each of the ratios. When these ratios are equal, the sixth definition gives the name of *proportionals* to them collectively, and points out the mode in which they are to be spoken of, and the ordinary, though somewhat inconvenient mode of writing them.

The fifth definition is that of equal ratios. The definition of ratio itself (defs. 3, 4) contains no criterion by which one ratio may be known to be equal to another ratio; analogous to that by which one magnitude is known to be equal to another magnitude (Euc. I. Ax. 8). The preceding definitions (3, 4) only restrict the conception of ratio within certain limits, but lay down no test for comparison, or the deduction of properties. All Euclid's reasonings were to turn upon this comparison of ratios, and hence it was competent to lay down a criterion of equality and inequality of two ratios between two pairs of magnitudes. In short, his *effective* definition is a definition of proportionals.

The precision with which this definition is expressed, considering the number of conditions involved in it, is remarkable. Like all complete definitions, the terms (the subject and predicate) are convertible: that is,

(*a*) If the four magnitudes be proportionals, and any equimultiples be taken as prescribed, they shall have the specified relations with respect to "greater, greater, &c."

(*b*) If of four magnitudes, two and two of the same Geometrical Species, it can be shewn that the prescribed equimultiples being taken, the conditions under which those magnitudes exist, *must be* such as to fulfil the criterion "greater, greater, &c."; then these four magnitudes shall be proportionals.

It may be remarked, that the cases in which the second part of the criterion ("equal, equal") can be fulfilled, are comparatively few: namely, those in which the given magnitudes, whose ratio is under consideration, are both exact multiples of some third magnitude—or those which are called *commensurable*. When this, however, is fulfilled, the other two will be fulfilled *as a consequence of this*. When this is not the case, or the magnitudes are *incommensurable*, the other two criteria determine the proportionality. However, when no hypothesis respecting commensurability is involved, the contemporaneous existence of the three cases ("greater, greater; equal, equal; less, less") must be deduced from the hypothetical conditions under which the magnitudes exist, to render the criterion valid.

The following *axioms*, though not expressed by Euclid, are virtually employed by him, and may be added to the four he has given.

Ax. 5. A part of a greater magnitude is greater than the same part of a less magnitude.

Ax. 6. That magnitude of which any part is greater than the same part of another, is greater than that other magnitude.

The fifth book of the Elements as a portion of Euclid's System of Geometry ought to be retained, as the doctrine contains some of the most important characteristics of an effective instrument of intellectual Education. This opinion is favoured by Dr Barrow in the following expressive terms: "There is nothing in the whole body of the Elements of a more subtile invention, nothing more solidly established, or more accurately handled than the doctrine of proportionals."

ADDITIONAL NOTES TO BOOK VI.

Prop. XXIII. The doctrine of compound ratio, including duplicate and triplicate ratio, in the form in which it was propounded and practised by the ancient Geometers has been almost wholly superseded. However satisfactory for the purposes of exact reasoning the method of expressing the ratio of two surfaces, or of two solids by two straight lines, may be in itself, it has not been found to be the form best suited for the direct application of the results of Geometry. Almost all modern writers on Geometry and its applications to every branch of the Mathematical Sciences, have adopted

the algebraical notation of a quotient AB : BC; or of a fraction $\frac{AB}{BC}$; for expressing the ratio of two magnitudes: as well as the form of a product AB × BC, or AB . BC, for the expression of a rectangle. The want of a concise and expressive method of notation to indicate the proportion of Geometrical Magnitudes in a form suited for the direct application of the results, has doubtless favoured the introduction of Algebraical symbols into the language of Geometry. It must be admitted, however, that such notations in the language of pure Geometry are liable to very serious objections, chiefly on the ground that pure Geometry does not admit the Arithmetical or Algebraical idea of a *product* or a *quotient* into its reasonings. On the other hand, it may be urged, that it is not the employment of symbols which renders a process of reasoning peculiarly Geometrical or Algebraical, but the ideas which are expressed by them. If symbols be employed in Geometrical reasonings, and be understood to express the *magnitudes themselves* and the *conception of their Geometrical ratio*, and not any *measures*, or *numerical values of them*, there would not appear to be any very great objections to their use, provided that the notations employed were such as are not likely to lead to misconception. It is, however, desirable, for the sake of avoiding confusion of ideas in reasoning on the properties of number and of magnitude, that the language and notations employed both in Geometry and Algebra should be rigidly defined and strictly adhered to, in all cases. At the commencement of his Geometrical studies, the student is recommended not to employ the symbols of Algebra in Geometrical demonstrations (see preface). How far it may be necessary or advisable to employ them when he fully understands the nature of the subject, is a question on which some difference of opinion exists.

The following is an example of the method which is generally used:

If (figure Euc. VI. 23) the parallelograms be supposed to be rectangular.

Then the rectangle AC : the rectangle DG :: BC : CG, Euc. VI. 1.

and the rectangle DG : the rectangle CF :: CD : EC,

whence the rectangle AC : the rectangle CF :: BC . CD : EC . CG.

Or, the areas of two rectangles are proportional to the products of the units contained in their adjacent sides respectively.

If, however, we agree that the ratio of BC . CD to EC . CG shall be interpreted to represent the ratio compounded of the ratios of the adjacent sides of the rectangles; we may express the proportion in the following form.

The ratio of the areas of two rectangles is as the ratio which is compounded of the ratios of their bases and altitudes.

Also, in a similar way, it may be shewn that the ratio of the areas of any two parallelograms, whether they be equiangular or not, is as the ratio compounded of the ratios of their bases and altitudes.

In conclusion, the Student must always remember that the introduction of numerical measures or Algebraical operations, must be regarded as a departure from the ancient Geometry, which, as a Science, recognizes no unit of measurement whatever.

ON THE CLASSIFICATION OF PROPOSITIONS.

THERE are only two forms of Propositions in the Elements, the *theorem* and the *problem*. In the *theorem*, it is asserted, and is to be proved, that if a geometrical figure be constructed with certain specified conditions, then some other specified relations must *necessarily* exist between the constituent parts of that figure. Thus:—if squares be described on the sides and hypothenuse of a right-angled triangle, the square on the

hypothenuse must *necessarily* be equal to the other two squares upon the sides (Euc. I. 47). In the *problem*, certain things are given in magnitude, position, or both, and it is required to find certain other things in magnitude, position, or both, that shall *necessarily* have a specified relation to the data, or to each other, or to both. Thus :—a circle being given, it may be required to construct a pentagon, which shall have its angular points in the circumference, and which shall also have both all its sides equal, and all its angles equal. (Euc. IV. 11.)

In Euclid's propositions, it may be remarked, there is in general, an aim at *definiteness*, considered in reference to the *quæsitum* of the problem, and the *predicate* of the *theorem*. The *quæsitum* of the problem is either a single thing, as the perpendicular in Euc. I. 11; or at most two, as the tangents to the circle in the first case of Euc. III. 17; and in the most general problems, even those which transcend the ordinary geometry, the solutions are always restricted to a definite number, which can always be assigned *a priori* for every problem. In certain cases, however, the conditions given in Euclid are not sufficient to fix entirely the quæsitum *in all respects*. For instance, Euc. IV. 10, the magnitude of the triangle is any whatever, and therefore not entirely fixed in all respects; or, again, in Euclid IV. 11, the pentagon may be any whatever, so that its position in the circle is not fixed. To fix the *magnitude* of the triangle or the *position* of the pentagon, some other condition independent of the data, must be added to the conditions of the problem. The length and position of some line connected with the triangle, (as one of the equal sides, the base, the perpendicular, &c.) would have fixed the triangle in magnitude and position; and the position of one angular point of the pentagon, or the condition that one side of the pentagon should pass through a given point (though this point must be subject to a certain restriction as to position, if within the circle), or any other possible conditions, would have confined the pentagon to a single position, or to the alternative of two positions. Such is the only kind of indeterminateness in the *problems* of "the Elements." In the enunciation of the *theorems* too, the same aim at *singleness* in the property asserted to be consequent on the hypothesis, is apparent throughout. There is, however, a remarkable difference in the characters of the hypotheses themselves, in Euclid's theorems: viz.

(1) That in some of them, one thing alone, or a certain definite number, possess the property which is affirmed in the enunciation.

(2) That all the things constituted subject to the hypothetical conditions, possess the affirmed property.

As instances of the first class, the greater number of theorems in the Elements may be referred to, as Euc. I. 4, 5, 6, 8, which are of the simplest class. In these, only one thing is asserted to be equal to another specified thing. In all the theorems of the Second Book, one thing is asserted to be equal to several other things taken together; and the same occurs in Euc. I. 47, as well as frequently in the other Books. They sometimes also take the form of asserting that no certain magnitude is greater or less than another, as in Euc. I. 16, or that two things together are less than or greater than, some one thing or several things, as Euc. I. 17. In all cases, however, this class is distinguished by the circumstance, that the things asserted to have the property are of a given finite number.

As instances of the second class, reference may be made to Euc. I. 35, 36, 37, 38, where all the parallelograms in the two former, and all the triangles in the two latter, are asserted to have the property of being equal to one given parallelogram or one given triangle. Or to Euc. III. 14, 20, 21; the lines in the circle in Prop. 14, or the angles at the circumference in Props. 20, 21, are any whatever, and therefore *all* the lines or angles constituted as in the enunciations, fulfil the conditions. Or again, in Book V. the two pairs of indefinite multiples, which form the basis of Euclid's definition of pro-

portionals; or his propositions "*ex æquo*," and "*ex æquo perturbato*," and the Propositions F, G, H, K; or, lastly, Euc. VI. 2, in which the property is (really, though not formally,) affirmed to be true when any line is drawn parallel to any one of the sides of the triangle.

The very circumstance, indeed, just noticed parenthetically, prevails so much in Euclid's enunciations, as to render it clear that it was his object as much as possible to render the conditions of the hypothesis *formally* definite in number; and if these remarks had no prospective reference, the circumstance would scarcely deserve notice. Still, with such prospective reference, it is necessary to insist upon the fact, that however the form of enunciation may be calculated to remove observation from it, the hypothesis itself is indefinite, or includes an indefinite number of things, which an additional condition would, as in the case of the problem, have restricted either to one thing or to a certain number of things.

Sometimes too, the theorem is enunciated in the form of a negation of possibility, as Euc. I. 7, III. 4, 5, 6, &c. These offer no occasion for remark, except to the ingenious modes of demonstration employed by Euclid. All such demonstrations must necessarily be indirect, assuming as an admitted truth the possibility of the fact denied in the enunciation.

Both among the Theorems and Problems, cases occur in which the hypotheses of the one, and the data or quæsita of the other, are restricted within certain limits as to *magnitude* and *position*. The determination of these limits constitutes the doctrine of *Maxima and Minima*. Thus:—the limit of possible diminution of the sum of the two sides of a triangle described upon a given base, is the magnitude of the base itself, Euc. I. 20, 22:—the limit of the side of a square which shall be equal to the rectangle of the two parts into which a given line may be divided, is half the line, as it appears from Euc. II. 5:—*the greatest line* that can be drawn from a given point within a circle, to the circumference, Euc. III. 7, is the line which passes through the centre of the circle; and *the least line* which can be so drawn from the same point, is the part produced, of the greatest line between the given point and the circumference. Euc. III. 8, also affords another instance of a maximum and a minimum when the given point is outside the given circle.

The theorem Euc. VI. 27 is a case of the *maximum* value which a figure fulfilling the other conditions can have; and the succeeding proposition is a problem involving this fact among the conditions as a part of the data, in truth, perfectly analogous to Euc. I. 20, 22; and finally, there are instances either direct or virtual in Euc. XI. 20, 21, 22, 23.

The doctrine itself was carefully cultivated by the Greek Geometers, and no solution of a Problem or demonstration of a theorem was considered to be complete, in which it was not determined, whether there existed such limitations to the possible magnitudes concerned in it, and how those limitations were to be actually determined.

Such Propositions as directly relate to *Maxima* and *Minima*, may be proposed either as Theorems or Problems. For the most part, however, it is the more general practice to propose them as Problems; but this has most probably arisen from the greater brevity of the enunciations in the form of a Problem. When proposed as a Problem, there is greater difficulty involved in the solution, as it required to find the limits with respect to *increase* and *decrease*; and then to prove the truth of the construction: whereas in the form of a Theorem, the construction itself is given in the hypothesis.

It may be remarked that though the Differential Calculus is always effective for the determination of Maxima and Minima, (in cases where such exist) yet in numerous cases, where it is applied to Problems of the classes which were cultivated by the Ancient Geometers, it is far less direct and elegant in its determinations than the Geometrical

methods. Now if reference be made to what has been stated respecting Theorems, where the hypothesis is *indeterminate*, or wanting in that completeness which reduces the property spoken of to a single example of the figure in question, a consequence of that *peculiarity* in such classes of Propositions may be remarked.

This peculiarity introduces another class of Propositions, which, though in "the Elements" somewhat disguised, formed an important portion of the Ancient Geometry:— the doctrine of *Loci*.

If the converse of Euc. I. 34, 35, 36, 37, and Euc. III. 20, 21, be taken in the form of Problems, they will become,

(1) Given the base and area, to construct the parallelogram.

(2) Given the base and area, to construct the triangle.

(3) Given the base and vertical angle, to construct the triangle.

Now *three conditions* are necessary to fix the magnitude of a triangle or a parallelogram, and in general, three only are sufficient for the purpose; but here it will be observed that only two are given in each case. The precise triangle or parallelogram, viewed as peculiarly solving the Problem, cannot be separated from all the others, except by adding some third condition to the two already given.

The side of the parallelogram in (1), and the vertex of the triangle in (2), opposite to the base, may be in any positions in a certain line parallel to the base; and the vertex of the triangle in (3), may be at any point in the circumference of a segment of a certain circle. The parallel line in which the vertices of all the equal triangles are situated, in one case, and the arc of the circle in which the vertices of all the triangles having equal vertical angles are situated, is each called the *locus of the vertex* of the triangle, since it occupies, in each case, *all the places* in which that vertex may be situated so as to fulfil the required conditions. In the same way, the parallel to the base is also the locus of all the positions in which the other two angular points of the parallelogram may be situated. These Problems are the simplest instances of that class which is called *Local Problems;* and their peculiar character is, that the data are one less than the number of conditions required by the nature of the Problem to restrict the quæsitum to a single or specified number of cases; as in these Problems the data consist of two conditions, while the exactly defining conditions must be three.

Again, viewed as Theorems, they may be thus enunciated:—

(1) If the base and area of a parallelogram be given, the locus of the other angular points will be a straight line parallel to the base.

(2) If the base and area of a triangle be given, the locus of its vertex is a straight line parallel to the base.

(3) If the base and vertical angle of a triangle be given, the locus of the vertex will be an arc of a circle.

In the original form of the propositions, the entire meaning, and that justified by Euclid's own reasoning, is that which would result from saying, "all triangles," "all parallelograms," &c. It will obviously be the case here, as in the *Maxima* and *Minima*, that the proposition may be enunciated either as a *local theorem* or as a *local problem;* and the circumstances will be similar as to the comparative brevity of enunciation and difficulty of the solution, when the proposition is given in the form of a Problem.

The great use made of *loci* by the Ancient Geometers was in the construction of determinate Problems. A certain number of data are required according to the nature of the Problem for rendering the quæsitum determinate; as, for instance, those in the case of the triangle. The subject will be better illustrated by an example, and one may be founded on the second and third propositions already noticed, which will take the following form:—

Given the base, the area, and the vertical angle of a triangle, to construct it.

When the base and the area of a triangle are given, the locus of its vertex is a straight line which can be determined from these data; and when the base and vertical angle are given, the locus of the vertex is a portion of the circumference of a circle which can be determined from these data. Now the point or points of intersection of these loci, will fulfil both conditions, that the triangle shall have the given area, and the given vertical angle. To express the principle generally:—let there be n conditions requisite for the determination of a point which either constitutes the solution, or upon which the solution of the problem depends. Find the locus of this point subject to $(n-1)$ of these conditions; and again, the locus of the point subject to any other $(n-1)$ of these conditions. The intersection of these two loci gives the point required.

It may be observed that $(n-2)$ of the data must be the same in determining the two loci, and no one of the n data must be a consequence of, or depend upon, the remaining $(n-1)$ data, in other words, the n data must separately express n independent conditions.

There are however cases in which one datum is involved in another, and these are of two different kinds—*essential* and *accidental*. To illustrate this distinction, let the following Problem be taken;

Given the base, the area, and the perpendicular drawn from the vertex to the base of the triangle, to construct it. Or,

Given the base, the vertical angle and the sum of the other two angles at the base of the triangle, to construct it.

Now in each of these problems, the third datum is absolutely determined and invariable, in consequence of its essential dependence on the two previous ones.

This dependence is universal and *essential*.

Again, suppose the problem were;—

Given the base of a triangle and a circle in magnitude and position, and likewise the vertical angle, to construct the triangle.

In this case, the given circle will generally be a different one from that which forms the locus of the vertical angle, and in that case, the intersections, or the point of contact, of the two circles will give either two solutions or one solution of the Problem. But on the other hand, the given circle *may* coincide with the locus, and thus again render the Problem indeterminate in this particular case. Generally the construction is possible, and only *accidentally* it becomes indeterminate.

The distinction between these two cases is very important.

As Problems are generally constructed by the intersections of loci, it is easy to imagine cases and conditions that shall give loci which can never meet.

For instance, in the problem just stated, the two circles may never meet; and in the preceding one, the straight line and circle may never meet. In all such cases the problem is *impossible* with the given conditions: these conditions being incompatible with each other in their nature, or more frequently, in their magnitude and position, and with the co-existence of that which constitutes the quæsitum. The limiting cases of possibility belong to the doctrine of Maxima and Minima.

The importance of the distinction alluded to, when one datum is contained in another, arises from its constituting the foundation of another Class of Propositions. These are the *Porisms*.

Whenever the quæsitum is a point, the problem on being rendered indeterminate, becomes a locus, whether the deficient datum be of the essential or of the accidental kind. When the quæsitum is a straight line or a circle, (which were the only two loci admitted into the ancient Elementary Geometry) the problem *may* admit of an *accidentally indeterminate* case; but will not *invariably* or even very frequently do so. This will be the case, when the line or circle shall be so far arbitrary in its position, as

depends upon the deficiency of *a single* condition to fix it perfectly:—that is, (for instance) one point in the line, or two points in the circle, may be determined from the given conditions, but the remaining one is indeterminate from the accidental relations among the data of the problem.

Determinate Problems become indeterminate by the merging of some one datum in the results of the remaining ones. This may arise in three different ways; first, from the coincidence of two points; secondly, from that of two straight lines; and thirdly, from that of two circles. These, further, are the only three ways in which this accidental coincidence of data can produce this indeterminateness; that is, in other words, convert the Problem into a Porism.

GEOMETRICAL ANALYSIS.

The term Analysis is usually understood to signify the separation of any thing into its constituent parts for the purpose of examining them separately; but as employed in Geometry, it expresses a reversal of the order of the parts of a demonstration, or an examination of the conditions attached to the construction of a Problem. The term used in its strict etymological sense is not exactly in accordance with the use made of it in Geometry. Yet this is of little importance, so that its applied meaning be clearly understood:—a meaning which it is difficult to express in the same words when applied to the *Theorem* and the *Problem*. For the purpose of securing perspicuity, it is therefore deemed the better plan, to consider it separately under the aspect it frequently bears in each of these applications. The very descriptive use of it given by Leslie, in his usually forcible antithetic manner, will in each case, be very striking: "Analysis presents the medium of invention; while Synthesis naturally directs the course of instruction."

THE ANALYSIS OF THEOREMS.

It will have been remarked, that in the Elements, Euclid frequently uses the *indirect* method of demonstration:—that is, of proving the truth of a theorem by demonstrating that a contrary conclusion is incompatible with the hypothesis of that theorem. To effect this, he *supposes* the enunciated property to be false; and its contrary to be true. He reasons from the assumed truth of this false property, till he arrives at a conclusion dependent upon that assumption, which is contrary to the original hypothesis; and thence it is inferred that the assumption being incompatible in its consequences with the original conditions of the theorem, those conditions and that assumption cannot co-exist. If, then, all the alternatives of the alleged property be thus examined, and thereby excluded from compatibility with the original hypothesis, it will necessarily follow, that this property itself is true. Thus in Euc. I. 25, where one included angle BAC is alleged in the enunciation to be greater than the other EDF, under the hypothesis of BA, AC, being respectively equal to ED, DF, but BC greater than EF: instead of proving the assertion itself, he admits, that in the first place the angle BAC is equal to the angle EDF, and in the second, that it is less. The consequences of these admissions are both shewn to be incompatible with the hypothesis, and hence it is inferred that the angle BAC can neither be equal to EDF, nor less than it. Wherefore as these are the only alternatives to the truth of the enunciation, and both these are false, it follows that the alleged relation of the angles BAC, EDF is true.

This method of proof occurs very frequently in the first and third books of Euclid, and occasionally in the fifth and sixth. In the eleventh and twelfth, it also occurs frequently; and it may be remarked generally, that it occurs more often in the outset of the developement of a system of truths than in the more advanced parts, or in the more recondite theorems.

It must naturally have occurred to Geometers, who were familiar with the use of this mode of assumption, to inquire: "What would be the effect of supposing the alleged theorem to be true, instead of false?" He who first asked this question made the first step in the Geometrical Analysis. He would see at once that the conclusion ought to be consistent with the hypothesis, and with all previously known properties of the hypothetical figure. He may, indeed, find it of little convenience, often of none, in suggesting a direct proof of a very elementary theorem, but as he would be of course led to try its efficacy in more complex cases, he would be gradually impressed with the facts: —that in many cases his steps were merely the reversal of the steps which he had employed in the hypothetic demonstration of the theorem; and that in all cases, a reversal in the order of the steps of his analysis would constitute a synthetic demonstration, though perhaps different from any one previously known to him. He had then discovered the true principle of the *Geometrical Analysis of Theorems;* and it would require but little additional skill to reduce the whole process to a complete system. It is then probable that his discovery would lead to some such rules as the following:—

(1) Assume that the Theorem is true.

(2) Proceed to examine any consequences that result from this admission, by the aid of other truths respecting the figure which have been already proved.

(3) Examine whether any of these consequences be themselves such as are already known to be *true*, or to be *false*.

(4) If any one of them be false, we have arrived at a *reductio ad absurdum*, which proves that the theorem itself is false, as in Euc. I. 25.

(5) If none of the consequences so deduced be *known* to be either true or false, proceed to deduce other consequences from all or any of these, as in (2).

(6) Examine these results, and proceed as in (3) and (4); and if still without any conclusive indications of the truth or falsehood of the alleged theorem, proceed still further, until such are obtained.

In the case of the theorem being false, we shall ultimately arrive at some result contradictory either to the original hypothesis, or to some truth depending upon it. Euclid's indirect demonstrations always end with a contradiction to the immediate hypothesis; but as the propositions to which he applies the method are so extremely elementary, this could scarcely happen otherwise, as, so far, deductions would be made from the hypothesis by direct steps. Where, however, we find a contradiction in our results to any of the consequences of the hypothesis, our conclusion, that the theorem is false, is as legitimate as though the contradiction had immediately been of the hypothesis itself. Nevertheless, if it should be imposed as a rule, that the contradiction shall be of the hypothesis itself, it only requires that we reverse the hypothesis of the property which is so contradicted, employing the contradiction instead of the conclusion of that property; and we shall thus have carried back that contradictory result into direct contrast with the original hypothesis.

It may sometimes happen that our attempts thus to analyse a theorem may be carried on through a considerable number of successive steps, and yet no conclusive evidence of the truth or falsehood of the alleged theorem present themselves. Nor can we ever judge, *à priori*, whether we should succeed by continuing the process further in any one particular direction. Under one aspect this may be considered an inconvenience; but even were it a real inconvenience, it is inevitable, and must so far be taken as a

drawback upon the value of the method. The inconvenience is, however, more apparent than real; or, at least, the inconvenience is amply compensated by the advantages it otherwise confers, not indeed in reference to the demonstration of the proposed theorem, but in its extension of geometrical discovery. A mistake might occur in the synthetic deduction of a proposed theorem, or the theorem might be a mistaken inference from analogy, or from the contemplation of carefully drawn diagrams; but it does not often happen that a theorem is proposed for solution, of the truth of which the proposer has not satisfied himself. The probabilities then are greatly in favour of such proposition being correct. Now in this case, all the investigations which have been made with a view to the *analysis* of that theorem will become so many *synthetic demonstrations* of the results which have been obtained during those unsuccessful attempts to analyse. It will in general be found, too, that they are of such a character as it would scarcely have occurred to any Geometer to adopt with pure reference to synthetic purposes. There can, in fact, be little doubt that the greater part of the most profound and original theorems that are found in the writings of the greatest Geometers of ancient and of modern times, have originated in attempts to analyse some proposed theorem; and which have failed merely from the direction which was pursued, lying in that of the more recondite instead of the more simple order of truths connected with the proposed one. Such failures should therefore be always carefully preserved, till the proposition itself, from which they were deduced, be proved either to be true or false.

Should the course of analysing pursued in the first instance not promise to succeed, by the conclusions becoming more and more elementary in their character, some other properties of the figure connected with the assumed truth should be tried in the same manner; and if this should also fail to accomplish the immediate object, the investigations should be pursued as before.

It has occasionally, though extremely seldom, happened that several such attempts have failed in succession. Yet some mode of deduction *must necessarily* become a true analysis of the theorem; and this will always result from adequate perseverance in these attempts. All the results obtained in the preceding efforts to analyse the theorem, will then constitute a circle of truths, connected with each other by the medium of that one from which they all, as it were, radiate; and often among truths so related, a general principle may be detected, that shall prove of the utmost value in the treatment of entire classes of propositions that now stand in an uninteresting state of isolation from each other. Moreover, by systematizing the propositions of Geometry, we simplify their didactic developement; and by contemplating such attempted analyses of single theorems, if taken in connexion with each other, very great benefits may be conferred upon Geometrical Science and its practical applications.

There is not the slightest difference between analysis and synthesis, as far as the course of consecutive deduction is concerned. Both are direct applications of the ordinary enthymeme; and both require the same specific habits of mind, and the same resources as regards truths already known. The only difference consists, as far as mere reasoning is concerned, in the difference of the starting points of the investigation. In Synthesis we start from the enunciated property as a truth temporarily admitted; and ultimately arrive at some property which we previously knew to be true of the hypothetical figure. We have only to reverse the order of the Syllogisms, and of the subject and predicate in each of them, to convert the analysis into the synthesis in one case, or the synthesis into the analysis in the other. They are so connected, in fact, that had the hypothesis of the proposed theorem been already proved by one process, the analysis of which we have spoken, would have become the synthesis of the other property.

It must now be obvious that the synthesis of the theorem can be at once formed from the analysis, by the reversal of the steps already described, that the analysis may, if

desirable, be altogether suppressed. On the other hand, for all the purposes of giving full and legitimate conviction of the truth of a theorem, the analysis is always sufficient, without adding the synthesis. It is, however, desirable, in a course of Geometrical study, to complete the formal draft of the investigation both in the analytic and synthetic form.

THE ANALYSIS OF PROBLEMS.

In every Geometrical operation we perform in the construction of a Problem, we have in mind some precedent reason,—a knowledge of some properties of the figure, either axiomatic or not, which would result from that operation, and a preception of its tendency towards accomplishing the object proposed in the Problem. Our processes for construction are founded on our knowledge of the properties of the figure, *supposed to exist already*, subjected to the conditions which are enunciated in the Proposition itself. No Problem could be constructed (except by mere trials, and verified by mere instrumental experiments) antecedently to the admission of our knowledge of some properties of the figure which it is proposed to construct. The simple reason for the operations employed, is, that they collectively and ultimately fulfil the prescribed conditions; and their so fulfilling the conditions, is only known by previously reasoning upon the figure supposed already to be so constructed as to embody those conditions. Let any Problem be selected from Euclid, and at each step of the operation, let the question be asked, "Why that step is taken?" It will in all cases be found that it is *because* of some known property of the figure required, either in its complete or intermediate states, of which the inventor of the construction must have been in possession. This antecedency of Theorems to all Geometrical construction in Scientific Geometry is universal and essential to its nature.

Let the construction of Euc. IV. 10 be taken in illustration of what has been stated. There are five operations specified in the construction:—

(1) Take *any* line AB.
(2) Divide that line in C, so that, &c.
(3) Describe the circle BDE with centre A and radius AB.
(4) Place BD in that circle, equal to AC.
(5) Join the points A, D.

Why should either of these operations be performed rather than any others? And what clue have we to enable us to foresee that the result of them will be such a triangle as was required? The demonstration affixed to it by Euclid, does undoubtedly prove that these operations must, in conjunction, produce such a triangle: but we are furnished in the Elements with no obvious reason for the adoption of these steps, except we suppose them accidental. To suppose that all the constructions, even the simple ones, were the result of accident only, would be supposing more than could be shewn to be admissible. No construction of the problem could have been devised without a previous knowledge of some of the properties of the figure which was to be produced. In fact, in directing the figure to be constructed, we assume the possibility of its existence; and we study the properties of such a figure on the hypothesis of its actual existence. It is this study of the properties of the figure *that constitutes the Analysis of the problem.*

Let then the existence of a triangle BAD be admitted which has each of the angles ABD, ADB double of the angle BAD, in order to ascertain any properties it may possess which would assist in the actual construction of such a triangle.

Then, since the angle ADB is double of BAD, if we draw a line DC to bisect ADB

and meet AB in C, the angle ADC will be equal to CAD; and hence (Euc. I. 6) the sides AC, CD are equal to one another.

Again, since we have three points A, C, D, not in the same straight line, let us examine the effect of describing a circle through them: that is, describe the circle ACD about the triangle ACD (Euc. IV. 5).

Then, since the angle ADB has been bisected by DC, and since ADB is double of DAB, the angle CDB is equal to the angle DAC in the alternate segment of the circle; the line BD therefore coincides with a tangent to the circle at D (converse of Euc. III. 32).

Whence it follows that the rectangle contained by AB, BC, is equal to the square of BD. (Euc. III. 36.)

But the angle BCD is equal to the two interior opposite angles CAD, CDA; or since these are equal to each other; BCD is the double of CAD, that is of BAD. And since ABD is also double of BAD, by the conditions of the triangle, the angles BCD, CBD are equal, and BD is equal to DC, that is, to AC.

It has been proved that the rectangle AB, BC, is equal to the square of BD; and hence the point C in AB, found by the intersection of the bisecting line DC, is such, that the rectangle AB, BC is equal to the square of AC. (Euc. II. 11.)

Finally, since the triangle ABD is isosceles, having each of the angles ABD, ADB double of the same angle, the sides AB, AD are equal, and hence the points B, D, are in the circumference of the circle described about A with the radius AB. And since the magnitude of the triangle is not specified, the line AB may be of any length whatever.

From this "Analysis of the problem," which obviously is nothing more than an examination of the properties of such a figure supposed to exist already, it will be at once apparent, *why* those steps which are prescribed by Euclid for its construction, were adopted.

The line AB is taken of any length, *because* the problem does not prescribe any specific magnitude to any of the sides of the triangle: the circle BDE is described about A with the distance AB, *because* the triangle is to be isosceles, having AB for one side, and therefore the other extremity of the base is in the circumference of that circle: the line AB is divided in C so that the rectangle AB, BC shall be equal to the square of AC, *because* the base of the triangle must be equal to the segment AC: and the line AD is drawn, *because* it completes the triangle, two of whose sides AB, BD are already drawn.

A careful examination of this process will point out the true character of the method by which the construction of all problems (except perhaps a few simple ones which involve but very few and very obvious steps) have been invented: although the actual analysis itself has been suppressed or concealed, as amongst the ancient Geometers, it appears to have been the general practice.

It will be inferred at once, that the use of the Analysis in reference to the construction of problems, is altogether indispensable in its actual form, where the problem requires several steps for its construction; as it has been shewn to be virtually (though the operations may in certain simple problems be carried on mentally and almost unsuspectedly) essential to the construction of all problems whatever.

Whenever we have reduced the construction to depend upon problems which have been already constructed, our analysis may be terminated; as was the case where, in the preceding example, we arrived at the division of the line AB in C; this problem having been already constructed as the eleventh of the second book.

ON THE THEORY OF TRANSVERSALS.

The theory of Transversals can be applied with great brevity and readiness in the demonstration of inverse propositions, in which it is required to prove, that three, or more than three, convergent lines, under certain conditions, when produced, shall pass through the same point; or, when three, or more than three points, formed by the intersections of lines drawn in certain directions, are situated in the same straight line. It has appeared desirable to give in detail a few of the principal properties: at the same time, it may be remarked, that the theory of Transversals appears to be not an unsuitable transition from the Ancient to the Modern Geometry, as it involves only Geometrical and Algebraical considerations, without reference to Trigonometry.

A rectilinear transversal is defined to be a straight line which intersects a system of other straight lines, (figures, Prop. I.)

Thus, if a straight line cba be drawn intersecting two sides AB, AC of a triangle ABC, in the points c, b, and meeting the third side BC produced in a; or intersecting the three sides produced in a, b, c; the line cba so drawn is called a transversal.

The transversal divides each of the sides of the triangle, or the sides produced, into two segments which lie between the vertices of the triangle and the transversal.

Thus Ac, cB are the segments of the side AB between the two vertices A, B and the transversal cba: Ab, bC the segments of AC; and Ba, aC the segments of BC. Also, Ab, Ca, Bc and bC, aB, cA are respectively the alternate segments of the sides made by the transversal cba.

Any figure formed by the meeting of four straight lines at their extremities, is called a *simple quadrilateral:* thus each of the three figures formed by the four lines Bc, cb, bC, CB; AC, Ca, ac, cA; AB, Ba, ab, bA is called a *simple quadrilateral.*

The *complete figure* formed by the production of the opposite sides of the simple quadrilateral to meet each other, or two adjacent sides to meet the other two sides is called a *complete quadrilateral.*

Thus the *same complete quadrilateral* is formed from each of the three simple quadrilaterals;

(1) By the production of the opposite sides Bc, Cb, and BC, cb, of the simple quadrilateral BcbCB to meet in the points A, a.

(2) By the production of the two opposite sides Ac, aC of the simple quadrilateral AcaCA to meet in the point B.

(3) By the production of the two adjacent sides Ab, ab of the simple quadrilateral ABabA to meet the other two adjacent sides in the points C, c.

The six points at which every two of the four lines meet or intersect are called the vertices of the complete quadrilateral, and the points A, B, C, a, b, c are the six vertices of the complete quadrilateral ABabA.

The three straight lines which join every two opposite vertices of a complete quadrilateral are called its diagonals, thus the lines Aa, Bb, Cc are the three diagonals of the complete quadrilateral ABabA.

The line cba was considered as a transversal intersecting the sides or the sides produced of the triangle ABC.

In a similar manner AbC may be considered as a transversal intersecting the sides of the triangle cBa; and the alternate segments of the sides are BA, cb, aC and Ac, ba, CB respectively.

Also AcB is a transversal to the triangle bCa; and aB, CA, bc and BC, Ab, ca are respectively the alternate segments of the sides of the triangle.

And lastly, aCB is a transversal to the triangle Abc; and AB, ca, bC and Bc, ab, CA are respectively the alternate segments of the sides.

If a straight line be drawn as a transversal intersecting the four sides of a complete quadrilateral and two of its diagonals; the three pairs of points in which it intersects the two diagonals, and the alternate sides, produced if necessary, of the complete quadrilateral, are called *conjugate points of the transversal,* thus m, m'; n, n'; p, p'; are the three pairs of conjugate points of the transversal nm, (figure, Prop. IV.)

PROPOSITION I.

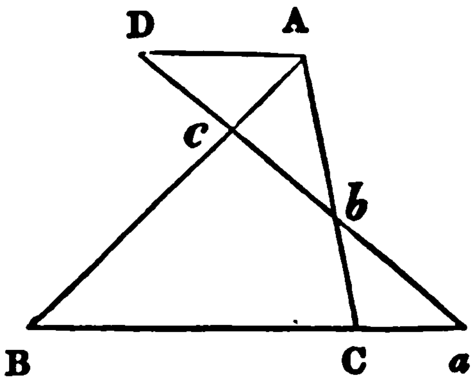
If a straight line intersect two sides AB, AC, *of a triangle* ABC *in the points* c, b, *and the base* BC *produced in* a: *or intersect the three sides* AB, AC, BC *produced in the points* c, b, a: *prove that* Ab.Bc.Ca=aB.bC.cA.

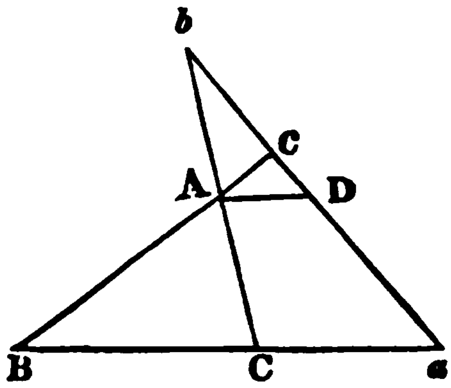
Conversely. If two points c, b *be taken in the sides* AB, AC *of a triangle* ABC, *and a third point* a *in the remaining side produced; or if the three points* c, b, a, *be in the three sides produced of the triangle, such that*

$$Ab.Bc.Ca=aB.bC.cA:$$

the points, a, b, c *shall be in the same straight line.* (Geom. Ex. Theo. 161. p. 364.)

First. Through A the vertex of the triangle ABC, draw AD parallel to BC.





$$\text{Then } \frac{Bc}{Ac}=\frac{aB}{AD}, \text{ by the similar triangles } ADc,\ cBa;$$

$$\text{and } \frac{Ab}{Cb}=\frac{AD}{Ca}, \text{ by the similar triangles } DAb,\ bCa;$$

$$\text{whence } \frac{Ab.Bc}{cA.bC}=\frac{aB}{Ca}.$$

$$\text{And, therefore, } Ab.Bc.Ca=aB.bC.cA;$$

or, the rectangular parallelopipeds contained by the alternate segments intercepted between the vertices of a triangle and a transversal, are equal to one another.

Secondly. Join cb, ba: if ba be not in the same straight line as bc, let ba'* be in the same straight line with it.

Then, since cba' is a transversal to the triangle ABC,

$$\text{therefore, } \mathrm{A}b\,.\,\mathrm{B}c\,.\,\mathrm{C}a' = a'\mathrm{B}\,.\,b\mathrm{C}\,.\,c\mathrm{A};$$

$$\text{but } \mathrm{A}b\,.\,\mathrm{B}c\,.\,\mathrm{C}a = a\mathrm{B}\,.\,b\mathrm{C}\,.\,c\mathrm{A};$$

$$\text{hence } \frac{\mathrm{C}a}{\mathrm{C}a'} = \frac{a\mathrm{B}}{a'\mathrm{B}}; \text{ and } \frac{\mathrm{C}a}{a\mathrm{B}} = \frac{\mathrm{C}a'}{a'\mathrm{B}};$$

$$\text{therefore } \frac{\mathrm{CB}}{a\mathrm{B}} = \frac{\mathrm{CB}}{a'\mathrm{B}}; \text{ and } a\mathrm{B} = a'\mathrm{B};$$

or the point a' coincides with the point a; and therefore the three points c, b, a are in the same straight line.

Cor. 1. The expression $\mathrm{A}b\,.\,\mathrm{B}c\,.\,\mathrm{C}a = a\mathrm{B}\,.\,b\mathrm{C}\,.\,c\mathrm{A}$, may be put under the following form, which perhaps will be found the most convenient in practice, as it connects together the ratio of the segments of each side,

$$\frac{\mathrm{A}b}{b\mathrm{C}}\cdot\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}} = 1;$$

and this form of the expression may be interpreted to mean, in the language of pure Geometry, that the ratio compounded of the ratios of the segments of the three sides of the triangle taken in order, is a ratio of equality.

Or, in the language of Algebraical ratio; that the product of the ratios of the segments of each of the three sides taken in order, is equal to unity.

If, however, this form be objected to, as not being strictly Geometrical; as well as the preceding one, as involving the conception of a solid, and which ought not to be admitted into considerations on plane Geometry: the expression itself may be retained in the form in which it was deduced from the two proportions,

$$\text{thus, } \frac{\mathrm{A}b\,.\,\mathrm{B}c}{c\mathrm{A}\,.\,b\mathrm{C}} = \frac{a\mathrm{B}}{\mathrm{C}a},$$

which may be exhibited under the three following forms:

$$a\mathrm{B} : \mathrm{C}a :: \mathrm{A}b\,.\,\mathrm{B}c : c\mathrm{A}\,.\,b\mathrm{C};$$

$$\text{or } b\mathrm{A} : \mathrm{C}b :: \mathrm{B}a\,.\,\mathrm{A}c : a\mathrm{C}\,.\,c\mathrm{B};$$

$$\text{or } c\mathrm{B} : \mathrm{A}c :: \mathrm{B}a\,.\,\mathrm{C}b : a\mathrm{C}\,.\,b\mathrm{A};$$

and expressed in the following terms:

If a transversal be drawn to any triangle, the segments of any one side between the transversal and two vertices of the triangle, are to each other as the ratio compounded of the ratios of the alternate segments of the other two sides: or, as the rectangles contained by the alternate segments of the other two sides.

Cor. 2. If, in the same way as the line acb was considered as a transversal to the triangle ABC, the lines AC, AB, BC be considered as transversals to the triangles $c\mathrm{B}a$, $b\mathrm{C}a$, $\mathrm{A}cb$, respectively; the four following results are obtained:

$$\frac{\mathrm{A}b}{b\mathrm{C}}\cdot\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}} = 1; \text{ (I.)} \qquad \frac{\mathrm{BC}}{\mathrm{C}a}\cdot\frac{ab}{bc}\cdot\frac{c\mathrm{A}}{\mathrm{AB}} = 1; \text{ (II.)}$$

$$\frac{a\mathrm{B}}{\mathrm{BC}}\cdot\frac{\mathrm{CA}}{\mathrm{A}b}\cdot\frac{bc}{ca} = 1; \text{ (III.)} \qquad \frac{\mathrm{AB}}{\mathrm{B}c}\cdot\frac{ca}{ab}\cdot\frac{b\mathrm{C}}{\mathrm{CA}} = 1. \text{ (IV.)}$$

These four results are not independent of each other, but any one of the four may be deduced from the remaining three.

* ba' is not drawn in the diagram.

Cor. 3. Again, if every two of the three independent results be combined, three other expressions, each consisting of eight segments, are obtained, which will express the relations which subsist between the four lines that form the complete quadrilateral, and their eight segments.

Thus, from I. and II., is deduced $\dfrac{CB.Bc}{cb.bC}=\dfrac{AB.Ba}{Ab.ba}$;

from I. and IV., ... $\dfrac{BA.Ab}{Ba.ab}=\dfrac{CA.Ac}{Ca.ac}$;

from I. and III., ... $\dfrac{Bc.cb}{BC.Cb}=\dfrac{Ac.ca}{AC.Ca}$.

Cor. 4. From the general expression (I) of the product of the ratios of the segments, let the consequences be deduced; first, when the transversal is parallel to any one of the sides of the triangle: and secondly, when the transversal passes through the vertex A of the triangle, and meets the base or the base produced in the point a.

Cor. 5. If BCa be considered as a transversal to the three straight lines AB, AC, Aa, which are drawn through the same point A; then drawing AD perpendicular to aB, the following property may be proved:

$$BC.Ca^2+Ca.BA^2=AC^2.Ba+BC.Ca.aB.$$

(See Geom. Ex. Theo. 41, p. 355).

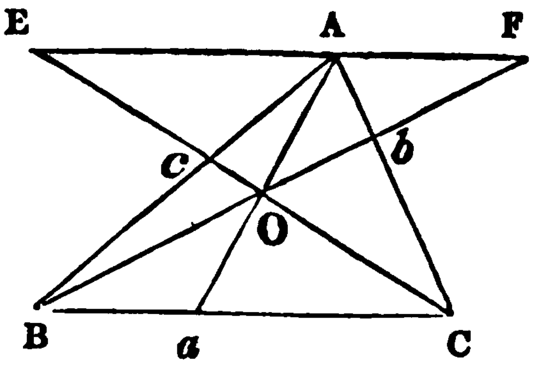
Cor. 6. Ascertain whether this theorem holds good when the point A is supposed to fall any where in the transversal BCa.

PROPOSITION II.

If three straight lines be drawn from the angles of a triangle through any point O *within the triangle, and be produced to meet the opposite sides in* a, b, c: *prove that* $Ac.Ba.Cb=cB.aC.bA$.

Conversely. If three straight lines be drawn from the angles of a triangle ABC *to meet the opposite sides in the points* a, b, c, *so that* $Ac.Ba.Cb=cB.aC.bA$: *then the three straight lines so drawn shall pass through the same point.* (Geom. Ex. Theo. 160, p. 364.)

First. Through the vertex A draw EAF parallel to BC, and meeting Bb, Cc produced, in F, E respectively.



Then $\dfrac{AE}{aC}=\dfrac{AO}{Oa}$, by the similar triangles EAO, OaC;

$=\dfrac{AF}{Ba}$, by the similar triangles AFO, OaB;

Hence $\frac{Ba}{aC}=\frac{AF}{AE}$;

Again, $\frac{Cb}{bA}=\frac{BC}{AF}$, by the similar triangles AbF, CbB;

also, $\frac{Ac}{cB}=\frac{AE}{BC}$, by the similar triangles AcE, BcC;

$$\text{whence } \frac{Ac}{cB}\cdot\frac{Ba}{aC}\cdot\frac{Cb}{bA}=1;$$

$$\text{or } Ac.Ba.Cb=cB.aC.bA.$$

Secondly. Let Aa, Bb intersect each other in the point O; and let c be such a point in AB, that $Ac.Ba.Cb=cB.aC.bA$.

Join CO, then CO produced, passes through the point c.

If CO produced do not pass through the point c, let it pass through c'*, some other point in AB.

Since O is a point within the triangle, and Aa, Bb, Cc', are drawn through it, and meet the sides of the triangle in a, b, c'; therefore

$$Ac'.Ba.Cb=c'B.aC.bA.$$

But $Ac.Ba.Cb=cB.aC.bA$, by hypothesis;

$$\text{therefore } \frac{Ac'}{Ac}=\frac{c'B}{cB}, \text{ and } \frac{Ac'}{c'B}=\frac{Ac}{cB};$$

$$\text{whence } \frac{AB}{c'B}=\frac{AB}{cB}, \text{ and } c'B=cB;$$

or, the point c' coincides with the point c, and therefore COc' coincides with COc, and the three lines Aa, Bb, Cc, pass through the same point O.

This Proposition is also true when the point O is outside the triangle and on either side of the base BC; also the expression $Ac.Ba.Cb=cB.aC.bA$ may be put under the forms:

$$Ac : cB :: aC.bA : Ba.Cb;$$
$$Ca : aB :: cA.bC : Bc.Ab;$$
$$Cb : bA :: cB.aC : Ac.Ba;$$

and expressed in the following terms:

If three straight lines be drawn from the angles of a triangle through any point to meet the opposite sides, or the opposite sides produced; the segments of each side are in the same ratio, as the ratio compounded of the ratios of the alternate segments of the other two sides.

Cor. 1. The following relations may also be shewn to exist between the six lines and their twelve segments.

(1) When AOa is taken as a transversal to the triangles BCc, CBb,

$$\frac{Ba}{aC}\cdot\frac{CA}{Ab}\cdot\frac{bO}{OB}=1; \qquad \frac{Ba}{aC}\cdot\frac{CO}{Oc}\cdot\frac{cA}{AB}=1.$$

(2) When BOb is taken as a transversal to the triangles ACc, CAa,

$$\frac{CO}{Oc}\cdot\frac{cB}{BA}\cdot\frac{Ab}{bC}=1; \qquad \frac{Cb}{bA}\cdot\frac{AO}{Oa}\cdot\frac{aB}{BC}=1.$$

* Oc' is not drawn in the diagram.

(3) When COc is taken as a transversal to the triangles ABb, BAa,

$$\frac{Bc}{cA}\cdot\frac{AC}{Cb}\cdot\frac{bO}{OB}=1; \qquad \frac{Ac}{cB}\cdot\frac{BC}{Ca}\cdot\frac{aO}{OA}=1.$$

(4) When AcB is taken as a transversal to the triangles COa, COb,

$$\frac{Cc}{CO}\cdot\frac{OA}{Aa}\cdot\frac{aB}{BC}=1; \qquad \frac{Cc}{cO}\cdot\frac{OB}{Bb}\cdot\frac{bA}{AC}=1.$$

(5) When AbC is taken as a transversal to the triangles BOa, BOc,

$$\frac{Bb}{bO}\cdot\frac{OA}{Aa}\cdot\frac{aC}{CB}=1; \qquad \frac{Bb}{bO}\cdot\frac{OC}{Cc}\cdot\frac{CA}{AB}=1.$$

(6) When BaC is taken as a transversal to the triangles AOb, AOc,

$$\frac{Aa}{aO}\cdot\frac{OB}{Bb}\cdot\frac{bC}{CA}=1; \qquad \frac{Aa}{aO}\cdot\frac{OC}{Cc}\cdot\frac{cB}{BA}=1.$$

These twelve relations are deduced when the point O is considered to be within the triangle; they are also true when the point O is outside the triangle and on either side of the base.

It is also worth while to ascertain how many of these properties form independent conditions of relation between the lines and their segments.

Cor. 2. In a similar manner, if the three diagonals of any complete quadrilateral be drawn, it will be found that the figure contains thirty-three simple quadrilaterals, and forty-four relations may be deduced from them by means of Prop. I. and II.

Cor. 3. By combining the relation proved in Prop. II. with the second, third, and sixth in Cor. 1: the following relation between the sides of the triangle, the transversals, and their segments is deduced

$$\frac{AO}{Oa}\cdot\frac{BO}{Ob}\cdot\frac{CO}{Oc}=\frac{AB}{Ab}\cdot\frac{BC}{Bc}\cdot\frac{CA}{Ca}.$$

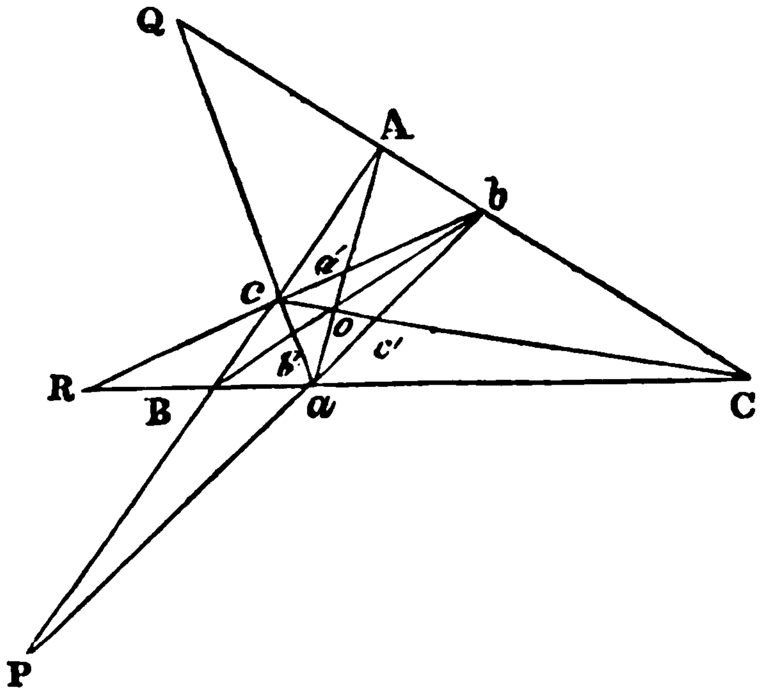
PROPOSITION III.

If through a given point within a triangle lines are drawn from the angles to the opposite sides, and the points of section be joined, the first three lines will be harmonically divided. (Geom. Ex. Theo. 72, p. 357.)

Let Aa, Bb, Cc be drawn through any point O within the triangle ABC meeting the opposite sides in a, b, c.

Draw ab, bc, ca, intersecting Cc, Aa, Bb, in c', a', b', respectively.

Then the lines Aa, Bb, Cc, are each divided harmonically.



From the point b, three straight lines bA, bc, bO, are drawn to the angles of the triangle AcO;

$$\text{therefore } \frac{AB}{Bc}\cdot\frac{cC}{CO}\cdot\frac{Oa'}{a'A}=1, \text{ by Prop. II.,}$$

$$\text{and } \frac{AB}{Bc}\cdot\frac{cC}{CO}\cdot\frac{Oa}{aA}=1,$$

for BaC is a transversal to the triangle AcO;

$$\text{Hence } \frac{Oa}{aA}=\frac{Oa'}{a'A},$$

$$\text{and } \frac{Aa}{Oa}=\frac{Aa'}{Oa'}=\frac{Aa-aa'}{aa'-Oa},$$

$$\text{or } Aa : Oa :: Aa-aa' : aa'-Oa;$$

wherefore Aa, aa', Oa are in harmonical proportion; or the line Aa is divided harmonically in the points a', O.

In a similar way, it may be shewn that Bb, bb', Ob', are in harmonical proportion; as also Cc, cc', Oc.

Cor. 1. If the lines ab, bc, ca be produced to meet the three sides of the triangle produced in the points P, Q, R; the lines bP, aQ, bR, are divided harmonically in the points c', a; b', c; a', c; respectively: as also AP, CQ, CR, the sides produced of the triangle, in the points, c, B; b, A; a, B; respectively.

First. From the point A, the straight lines Ab, AO, Ac are drawn to b, O, c, the angular points of the triangle bcO, and these lines cut the side bc in a' and meet the other sides bO, cO produced in B, C;

$$\text{therefore } \frac{ba'}{a'c}\cdot\frac{cC}{CO}\cdot\frac{OB}{Bb}=1, \text{ by Prop. II.}$$

And the transversal RC intersects the sides produced of the same triangle bcO;

$$\text{therefore } \frac{bR}{Rc}\cdot\frac{cC}{CO}\cdot\frac{OB}{Bb}=1, \text{ by Prop. I.}$$

$$\text{Hence } \frac{bR}{Rc}=\frac{ba'}{a'C}=\frac{bR-Ra'}{Ra'-Rc};$$

or bR, Ra', Rc are in harmonical proportion.

Similarly, it may be shewn that Pb is divided harmonically in a, c'; as also Qa in c, b'.

$$\text{Secondly. Since } \frac{Bc}{cA}\cdot\frac{Ab}{bC}\cdot\frac{Ca}{aB}=1, \text{ by Prop. II.,}$$

$$\text{and } \frac{Ab}{bC}\cdot\frac{Ca}{aB}\cdot\frac{BP}{PA}=1, \text{ by Prop. I.,}$$

For the transversal Pab intersects the triangle ABC.

$$\text{Hence } \frac{BP}{PA}=\frac{Bc}{cA},$$

$$\text{and } \frac{AP}{PB}=\frac{cA}{Bc}=\frac{PA-Pc}{Pc-PB};$$

or AP, Pc, PB are in harmonical proportion, and the line PA is divided harmonically in c, B.

In a similar way, it may be shewn, that CQ, and CR, are harmonically divided in b, A; a, B, respectively.

Cor. 2. Since the lines $\mathrm{A}a'\mathrm{O}a$, $ba'c\mathrm{R}$, $\mathrm{C}a\mathrm{BR}$; $\mathrm{B}b'\mathrm{O}b$, $ab'c\mathrm{Q}$, $\mathrm{C}b\mathrm{AQ}$; $\mathrm{C}c'\mathrm{O}c$, $bc'a\mathrm{P}$, $\mathrm{A}c\mathrm{BP}$; are the diagonals produced of the three complete quadrilaterals BACOB, ABCOA, ACBOA, respectively: the results of Prop. III. and Cor. 1. may be generally expressed in the following terms:

If the three diagonals of a complete quadrilateral be drawn and be produced to meet one another; each of the diagonals is divided harmonically by the other two.

Cor. 3. The three points P, Q, R, are in the same straight line.

For considering $\mathrm{P}ab$, $\mathrm{Q}ca$, $\mathrm{R}cb$, as transversals to the triangle ABC, by Prop. I. we obtain;

$$\frac{\mathrm{AP}}{\mathrm{PB}}\cdot\frac{\mathrm{B}a}{a\mathrm{C}}\cdot\frac{\mathrm{C}b}{b\mathrm{A}}=1, \text{ or } \frac{\mathrm{AP}}{\mathrm{PB}}=\frac{\mathrm{A}b}{b\mathrm{C}}\cdot\frac{\mathrm{C}a}{a\mathrm{B}};$$

$$\frac{\mathrm{CQ}}{\mathrm{QA}}\cdot\frac{\mathrm{A}c}{c\mathrm{B}}\cdot\frac{\mathrm{B}a}{a\mathrm{C}}=1, \text{ or } \frac{\mathrm{CQ}}{\mathrm{QA}}=\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}};$$

$$\frac{\mathrm{BR}}{\mathrm{RC}}\cdot\frac{\mathrm{C}b}{b\mathrm{A}}\cdot\frac{\mathrm{A}c}{c\mathrm{B}}=1, \text{ or } \frac{\mathrm{BR}}{\mathrm{RC}}=\frac{\mathrm{B}c}{c\mathrm{A}}\cdot\frac{\mathrm{A}b}{b\mathrm{C}}.$$

$$\text{Hence } \frac{\mathrm{AP}}{\mathrm{PB}}\cdot\frac{\mathrm{BR}}{\mathrm{RC}}\cdot\frac{\mathrm{CQ}}{\mathrm{QA}}=\left(\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}}\cdot\frac{\mathrm{A}b}{b\mathrm{C}}\right)^2.$$

$$\text{But } \frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}}\cdot\frac{\mathrm{A}b}{b\mathrm{C}}=1, \text{ by Prop. II.;}$$

$$\text{therefore } \frac{\mathrm{AP}}{\mathrm{PB}}\cdot\frac{\mathrm{BR}}{\mathrm{RC}}\cdot\frac{\mathrm{CQ}}{\mathrm{QA}}=1;$$

which is the condition fulfilled when a straight line is drawn intersecting the three sides of a triangle, AB, CA, CB produced, in the points P, Q, R.

Hence the three points P, Q, R, are in the same straight line.

Cor. 4. The lines $\mathrm{A}a$, $\mathrm{B}b$, $\mathrm{C}c$ may be considered as transversals to the triangle abc, and a similar series of relations may be deduced, as in Prop. II.

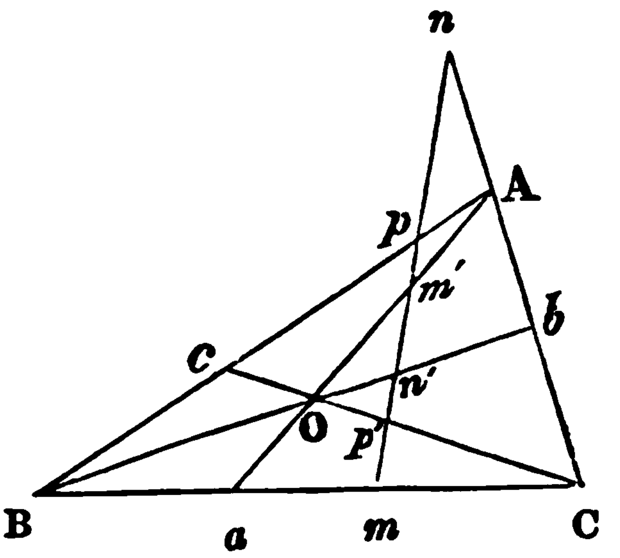
PROPOSITION IV.

If three lines be drawn from the angles A, B, C *of a triangle through any point* O *to meet the opposite sides in the points* a, b, c*; and if a transversal be drawn intersecting these lines in* m′, n′, p′, *and the sides of the triangle, produced if necessary, in the points* m, n, p*: then the three following relations exist between the parts into which the transversal is divided.*

$$\frac{mp}{pm'}\cdot\frac{m'n}{nm}\cdot\frac{mp'}{p'm'}\cdot\frac{m'n'}{n'm}=1; \quad (1) \qquad \frac{nm'}{m'n}\cdot\frac{n'p}{pn}\cdot\frac{nm}{mn'}\cdot\frac{n'p'}{p'n}=1; \quad (2)$$

$$\frac{mp}{pn'}\cdot\frac{n'p'}{p'm'}\cdot\frac{m'p}{pn}\cdot\frac{np'}{p'm}=1; \quad (3).$$

First, the triangle amm' is intersected by the transversals AB, Bb,



$$\text{therefore } \frac{m'A}{Aa}\cdot\frac{aB}{Bm}\cdot\frac{mp}{pm'}=1; \qquad \frac{mB}{Ba}\cdot\frac{aO}{Om'}\cdot\frac{m'n'}{n'm}=1.$$

Again, the same triangle is intersected by the transversals AC, Cc,

$$\text{therefore } \frac{m'n}{nm}\cdot\frac{mC}{Ca}\cdot\frac{aA}{Am'}=1; \qquad \frac{m'O}{Oa}\cdot\frac{aC}{Cm}\cdot\frac{mp'}{p'm'}=1,$$

$$\text{whence is deduced } \frac{mp}{pm'}\cdot\frac{m'n}{nm}\cdot\frac{mp'}{p'm'}\cdot\frac{m'n'}{n'm}=1.$$

Secondly. In a similar way since the triangle $nn'b$ is intersected by the transversals BA, Aa; and by BC, Cc; four relations arise, from which may be deduced

$$\frac{nm'}{m'n'}\cdot\frac{n'p}{pn}\cdot\frac{nm}{mn'}\cdot\frac{n'p'}{p'n}=1.$$

Thirdly. The triangle $pp'c$ is intersected by the transversal CA, Aa; and by CB, Bb; other four relations are found from which there results

$$\frac{mp}{pn'}\cdot\frac{n'p'}{p'm'}\cdot\frac{m'p}{pn}\cdot\frac{np'}{p'm}=1.$$

Cor. From these three relations, each involving eight of the segments of the transversal, a relation may be found which involves only six segments; multiplying these results together, is deduced

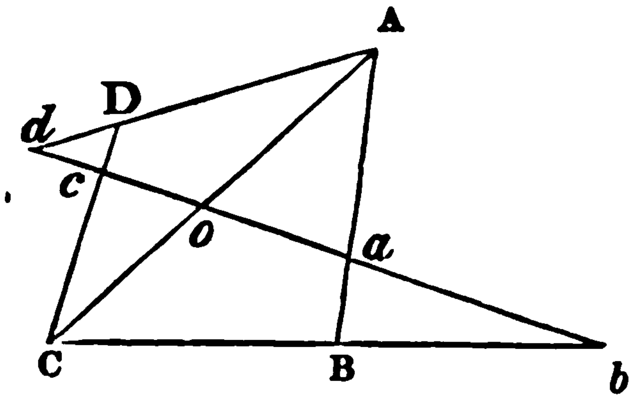
$$\frac{mp'}{p'n}\cdot\frac{nm'}{m'p}\cdot\frac{pn'}{n'm}=1.$$

Three other forms may be deduced from the same three expressions, (I.) by multiplying (1) and (2) together and dividing by (3); (II.) by multiplying (1) and (3) together and dividing by (2); (III.) by multiplying (2) and (3) together and dividing by (1).

PROPOSITION V.

If any polygon be intersected by a transversal, the segments of the sides have to each other a relation similar to that of the segments of the sides of a triangle.

Let ABCD be a polygon of four sides, and let its opposite sides AB, CD be intersected in b, c by a transversal which meets AD, CB produced in d, b.



Join AC intersecting the transversal db in O.

Then the transversal dcO intersects the triangle ACD,

$$\text{therefore } \frac{Ad}{dD} \cdot \frac{Dc}{cC} \cdot \frac{CO}{OA} = 1, \quad \text{Prop. I.}$$

And the transversal baO intersects the triangle ABC,

$$\text{therefore } \frac{Aa}{aB} \cdot \frac{Bb}{bC} \cdot \frac{CO}{OA} = 1. \quad \text{Prop. I.}$$

$$\text{Whence } \frac{Aa}{aB} \cdot \frac{Bb}{bC} \cdot \frac{Cc}{cD} \cdot \frac{Dd}{dA} = 1,$$

or, the ratio compounded of the ratios of the segments of the four sides of the polygon taken in order, is a ratio of equality.

This result may also be expressed thus

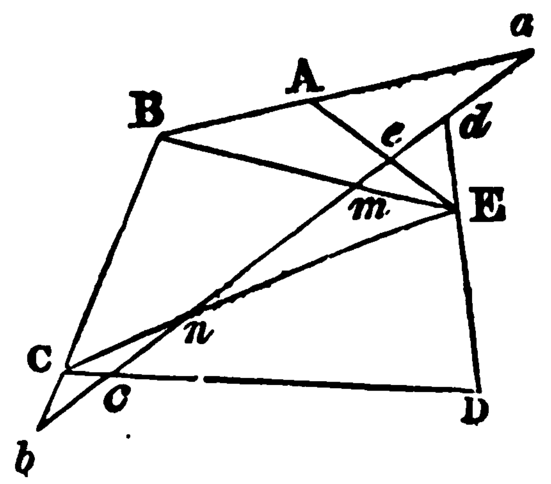
$$Aa.Bb.Cc.Dd = aB.bC.cD.dA,$$

or it may be exhibited in the form of a proportion, for instance

$$\frac{Aa.Cc}{aB.cD} = \frac{bC.dA}{bB.dD}.$$

The transversal bOacd was supposed in the preceding result to intersect AB, CD, *two opposite* sides of the polygon; the same property is also true when the transversal intersects *two adjacent* sides, or the *four sides produced.*

Let ABCDE be a polygon of five sides, and let AE, CD, any two sides of it, be intersected by a transversal in the points e, c: let BA, BC, DE produced, meet the transversal in a, b, d, respectively. Draw the diagonals EB, EC intersecting the transversal in m, n,



then $\dfrac{Em}{mB} \cdot \dfrac{Ba}{aA} \cdot \dfrac{Ae}{eE} = 1$, for mea intersects the triangle ABE,

$$\frac{Bm}{mE} \cdot \frac{En}{nC} \cdot \frac{Cb}{bB} = 1, \text{ } ncb \text{ intersects BCE,}$$

$$\frac{Cn}{nE} \cdot \frac{Ed}{dD} \cdot \frac{Dc}{cC} = 1, \text{ } cnd \text{ intersects CDE.}$$

$$\text{Whence } \frac{Ba}{aA} \cdot \frac{Ae}{eE} \cdot \frac{Ed}{dD} \cdot \frac{Dc}{cC} \cdot \frac{Cb}{bB} = 1;$$

or, as before, the ratio compounded of the ratios of the segments of the sides of the figure successively taken in order is a ratio of equality.

In a similar manner, for any polygon ABCDEF of six sides, when intersected by a transversal, adopting the same notation as for the polygon of five sides, it may be proved, that

$$\frac{Ba}{aA} \cdot \frac{Af}{fF} \cdot \frac{Fe}{eE} \cdot \frac{Ed}{dD} \cdot \frac{Dc}{cC} \cdot \frac{Cb}{bB} = 1;$$

or the ratio compounded of the ratios of the segments of the six sides taken in order is a ratio of equality.

And similarly for any polygon whatever, whether the transversal intersect two of the sides of the polygon or only the sides produced.

Cor. If the hexagon be inscriptible in a circle, then the three points of intersections of the alternate sides produced, are in the same straight line.

Carnot was the first who systematically pointed out the importance of the *relations of position* in Geometrical figures. This he exhibited in a tract entitled "*De la Correlation des figures de Géométrie,*" published in 1801. His greater work on the subject appeared in 1803, under the title of "*Géométrie de Position,*" and three years afterwards, in 1806, his tract "*Sur la Théorie des Transversales.*" The subject has been both extended and simplified by subsequent writers; among whom may be named C. J. Brianchon, who published, in 1818, his "*Application de la Théorie des Transversales.*" Many interesting properties of Transversals will be found in the notes of M. Chasles's "*Aperçu Historique des Methodes en Géométrie.*" The only English work in which the subject is systematically treated, is the last edition of Hutton's Course of Mathematics, in which the theory is applied both to Elementary Geometry, and to properties of the Conic Sections.

HINTS, &c. TO THE PROBLEMS. BOOK I.

2. There is another method whereby a line may be divided into three equal parts:—by drawing from one extremity of the given line, another making any acute angle with it, and taking three equal distances from the extremity, then joining the extremities, and through the other two points of division, drawing lines parallel to this line through the other two points of division, and to the given line; the three triangles thus formed are equal in all respects. This may be extended for any number of parts.

5. This is a particular case of Euc. I. 22.

6. The same remark applies.

7. Let A be the given point without the line, and B the given point in the line. Suppose D the point required in the line. If AB and AD be joined, DAB is an isosceles triangle: draw DE perpendicular to AB, and the construction is obvious.

8. This may be effected in two ways, (1) by Euc. I. 9, 10, when the two lines meet; (2) when the lines are produced beyond the point of intersection, by Euc. I. 9, 31. It may be remarked also, that the line drawn through the given point makes equal angles with the two given lines, Euc. I. 5.

9. Suppose the two lines CB, DB to meet in B, and that EAF drawn through the given point A, is bisected in that point. Through A, draw AG parallel to BC, and through G, GH parallel to EF. Then EAGH is a parallelogram, and BG is equal to GF. Hence the synthesis.

10. Apply Euc. I. 1, 9.

11. The angle to be trisected is one-fourth of a right angle. If an equilateral triangle be described on one of the sides of the triangle which contains the given angle, and a line be drawn to bisect that angle of the equilateral triangle which is at the given angle, the angle contained between this line and the other side of the triangle, will be one twelfth of a right angle, or equal to one third of the given angle.

It may be remarked, generally, that any angle which is the half, fourth, eighth, &c. part of a right angle, may be trisected by Plane Geometry.

12. This may be done by means of Euc. I. 47, taking three lines of 3, 4, 5 units. Or, by means of *equal* straight lines, figure Euc. I. 1, if an equilateral triangle CBF be described on CB, and another CFG on CF, the line drawn from G to B is at right angles to AB. Or, take a point D in AB, draw DE equal to DC and inclined to AB, C being the given point. Join EC, and produce it till CG be equal to CD; make CF equal to CE, join FG and produce it till GH be equal to GC. Then CH will be perpendicular to AB.

13. Take any point in the given line, and apply Euc. I. 23, 31.

14. The only distinction between this and Prob. 8, above, is in the quæsitum. The quæsita in both are *essentially* connected.

15. The two given points may be both on the same side, or one point may be on each side of the line. If the point required in the line be supposed to be found, and lines be drawn joining this point and the given points, an isosceles triangle is formed, and if a perpendicular be drawn on the base from the point in the line: the construction is obvious.

16. The line to which the given line is to be parallel, may pass through an angle of the triangle, or it may not; if it do not, draw through an angle of the triangle a line parallel to this line, Euc. I. 31. If a line equal to the given line be supposed to be drawn parallel to the line through one of the angles, a parallelogram may be formed and the construction determined.

17. The construction of this problem may be effected from Prob. 1, p. 293.

18. If the point in the base be supposed to be determined, and lines drawn from it parallel to the sides, it will be found to be in the line which bisects the vertical angle of the triangle.

19. This may be effected by Prob. 3, p. 295.

20. The line required is obviously the diagonal joining the two obtuse angles.

21. If AC be drawn from A one extremity of the given line AB making *any*

angle less than the given angle, and if from A, a line AD be drawn making with AC an angle equal to the given angle, and BC, BD be drawn from B parallel to AD, AC respectively:—the parallelogram is formed with the given conditions. The Problem is indeterminate.

22. If in the figure Euc. I. 1, the circles intersect each other in C, C′, and CA, AC′, C′B, BC be joined, ACBC′ is the parallelogram required.

23. See the figure, Euc. I. 35.

24. Construct a right-angled parallelogram by Euc. I. 44, equal to the given quadrilateral figure, and from one of the angles draw a line to meet the opposite side and equal to the base of the rectangle, and a line from the adjacent angle parallel to this line will complete the rhombus.

25. Bisect BC in D, and through the vertex A draw AE parallel to BC, with centre D and radius equal to half the sum of AB, AC, describe a circle cutting AE in E.

26. Make a triangle ABC equal to the given figure, by Prob. 27. Produce the base BC, if necessary, making BD equal to the given base. On BD make the triangle BDE equal to ABC; through E draw EG parallel to BD, and upon BD describe a segment of a circle containing an angle equal to the given angle, Euc. III. 33, and cutting EG in H; join HB, HD. HBD is the required triangle.

27. If the figure ABCD be one of four sides; join the opposite angles A, C of the figure, through D draw DE parallel to AC meeting BC produced in E, join AE:—the triangle ABE may be proved equal to the foursided figure ABCD.

If the figure ABCDE be one of five sides, produce the base both ways, and the figure may be transformed into a triangle, by two constructions similar to that employed for a figure of four sides. If the figure consist of six, seven, or any number of sides, the same process must be repeated.

28. From the given point A, let fall AB on the given straight line, and upon AB describe an equilateral triangle PAB. Produce AP to meet the given line in D, and bisect the angle BAD by AC meeting the given line in C. On AC, BD, describe the equilateral triangles QAC, RAD; if P, Q, R can be proved to be in a straight line the locus of the vertices of the triangles on the same side of AB will be a straight line.

29. Let ABC, be the given triangle whose base is BC. In BC, produced if necessary, take BD equal to the given base of the required isosceles triangle: on BD let BD be the base of the required isosceles triangle. On BD let a triangle BDE be formed equal to ABC. If this triangle be not isosceles, an equal triangle which shall be isosceles, may be formed on the same base BD, Euc. I. 37.

30. See Theorem 69, p. 304.

31. Let ABC be the required triangle having the right angle ABC, and such that the sum of AC and AB is double of BC.

Now since $AC + AB = 2BC$, therefore $AC - BC = BC - AB$. On AC take AD equal to this difference. Then $AC = BC + AD$, $AB = BC - AD$, and since $AC^2 = AB^2 + BC^2$, it follows that $BC^2 = 4BC.AD$, and $BC = 4AD$. Hence AD the difference between the hypothenuse and one side BC is known, and therefore the hypothenuse $AC = 5AD$. Hence the construction of the triangle depends on the division of the hypothenuse into five equal parts. See the remark on Prob. 2.

32. Let the two given lines meet in A, and let B be the given point.

If BC, BD be supposed to be drawn making equal angles with AC, AD and DC be joined, BCD is the triangle required, and the figure ACBD may be shewn to be a parallelogram. Whence the construction.

33. This is a particular case of Prob. 76, p. 351.

34. This problem cannot be solved without proportion.

35. This is a case of the more general problem:—To divide a triangle into two parts, having a given ratio to one another by a line parallel to any line given in position. See note on Prob. 76, p. 351.

36. Let D be the required point within the triangle ABC, such that the lines AD, BD, CD trisect the triangle. If these lines be produced to meet the sides of the triangle in E, F, G, it may be shewn that the sides are bisected in these points.

37. It is proved, Euclid I. 34, that each of the diagonals of a parallelogram bisects the figure, and it may easily be shewn that they also bisect each other. It is hence manifest that any straight line which bisects a parallelogram *must* pass

through the intersection of the diagonals. The different positions of a line through the intersection of the diagonals will suggest the constructions in the different cases.

38. (1) Reduce the trapezium ABCD to a triangle BAE by Prob. 27, and bisect the triangle BAE by a line AF from the vertex. If F falls without BC, through F draw FG parallel to AC or DE, and join AG.

Or thus. Draw the diagonals AC, BD; bisect BD in E, and join AE, EC. Draw FEG parallel to AC the other diagonal, meeting AD in F, and DC in G. AG being joined bisects the trapezium.

(2) Let E be the given point in the side AD. Join EB. Bisect the quadrilateral EBCD by EF. Make the triangle EFG equal to the triangle EBA, on that side of EF on which the greater part of ABCE lies. Bisect the triangle EFG by EH. EH bisects the figure.

39. (1) DF bisects the triangle ABC (fig. Prob. 3, p. 295). On each side of the point F in the line BC, take FG, FH, each equal to one third of BF, the lines DG, DH shall trisect the triangle. Or,

Let ABC be any triangle, D the given point in BC. Trisect BC in E, F. Join AD, and draw EG, FH parallel to AD. Join DG, DH: these lines trisect the triangle. Draw AE, AF and the proof is manifest.

(2) Let ABC be any triangle; trisect the base BC in D, E, and join AD, AE. From D, E, draw DP, EP parallel to AB, AC and meeting in P. Join AP, BP, CP; these three lines trisect the triangle.

(3) Let P be the given point within the triangle ABC. Trisect the base BC in D, E. From the vertex A draw AD, AE, AP. Join PD, draw AG parallel to PD and join PG. Then BGPA is one third of the triangle. The problem may be solved by trisecting either of the other two sides and making a similar construction.

40. Trisect the side AB in E, F, and draw EG, FH parallel to AD or BC, meeting DC in G and H. If the given point P be in EF, the two lines drawn from P through the bisections of EG and FH will trisect the parallelogram. If P be in FB, a line from P through the bisection of FH will cut off one third of the parallelogram, and the remaining trapezium is to be bisected by a line from P, one of its angles. If P coincide with E or F, the solution is obvious.

41. If a line be drawn from the given point in the side of the parallelogram through the intersection of the diagonals, the parallelogram is bisected; and the problem is reduced to the bisection of a trapezium two of whose sides are parallel, by a line drawn from the extremity of one of the parallel sides.

42. If a straight line be drawn from the given point through the intersection of the diagonals and meeting the opposite side of the square; the problem is then reduced to the bisection of a trapezium by a line drawn from one of its angles.

43. If the angles at the base of the isosceles triangle be bisected, the line joining the points where the bisecting lines meet the opposite sides of the triangle, will cut off the trapezium required.

44. The base may be divided into nine equal parts, and lines may be drawn from the vertex to the points of division. Or, the sides of the triangle may be trisected, and the points of trisection joined.

45. (1) By supposing the point P found in the side AB of the parallelogram ABCD, such that the angle contained by AP, PC may be bisected by the line PD; CP may be proved equal to CD; hence the solution is obvious.

(2) By supposing the point P found in the side AB produced, so that PD may bisect the angle contained by ABP and PC; it may be shewn that the side AB must be produced, so that BP is equal to BD.

46. Produce one side of the square till it become equal to the diagonal, the line drawn from the extremity of this produced side and parallel to the adjacent side of the square, and meeting the diagonal produced, determines the point required.

47. It is sufficient to suggest, that triangles on equal bases, and of equal altitudes, are equal.

48. Let ABC be the required triangle having the angle ACB a right angle. In BC produced, take CE equal to AC, and with centre B and radius BA describe a circular arc cutting CE in D, and join AD. Then DE is the difference between the sum of the two sides AC, CB and the hypothenuse AB; also one side AC the perpendicular is given. Hence the construction. On any line EB take EC equal to

the given side, ED equal to the given difference. At C, draw CA perpendicular to CB, and equal to EC, join AD, at A in AD make the angle DAB equal to ADB, and let AB meet EB in B. Then ABC is the triangle required.

49. Let AD be the sum of the base and hypothenuse, AB the sum of the base and perpendicular (figure, Euc. II. 10). At D draw a line perpendicular to AD, and through B draw EBG making with AB an angle ABE equal to half a right angle, and meeting the perpendicular to AD in G. Join AG and draw BH equal to BD and meeting AG in H. From A draw AE parallel to HB meeting EBG in E; draw EF parallel to AD meeting the perpendicular at D in F; and lastly, draw EC perpendicular to AD. Then the triangle AEC is the triangle required.

The proof of this problem involves the consideration of similar triangles; it should have been placed among the Problems to Book VI.

50. Draw two straight lines making an angle A equal to the given angle, and on one of them take a part AD equal to half the sum of the sides; bisect the angle A; then the problem is reduced to constructing a right-angled triangle, having given the sum of the base and the hypothenuse, and the angle contained between the perpendicular and the hypothenuse. Suppose the thing done, and that EF is the position of the base, join DF, and draw DG parallel to EF, the angle EDG is bisected by DF.

51. On the line which is equal to the perimeter of the required triangle describe a triangle having its angles equal to the given angles. Then the rest of the process is suggested by Prob. 2, p. 221.

52. Let ABC (fig. to Euc. I. 20) be the required triangle, having the base CB equal to the given base, the angle ABC equal to the given angle, and the two sides BA, AC together equal to the given line BD. Join DC, then since AD is equal to AC, the triangle ACD is isosceles, and therefore the angle ADC is equal to the angle ACD. Hence the construction of the triangle.

53. Let ABC be the required triangle (fig. to Euc. I. 18), having the angle ACB equal to the given angle, and the base BC equal to the given line, also CD equal to the difference of the two sides AB, AC. If BD be joined, then ABD is an isosceles triangle. Hence the synthesis. Does this construction hold good in all cases?

54. Let ABC be the required triangle, (fig. Euc. I. 18,) of which the side BC is given and the angle BAC, also CD the difference between the sides AB, AC. Join BD; then AB is equal to AD, because CD is their difference, and the triangle ABD is isosceles, whence the angle ABD is equal to the angle ADB; and since twice the angle ABD and BAD are equal to two right angles, it follows that ABD is half the supplement of the given angle BAC. Hence the construction of the triangle.

55. Let AB be the given base, at B draw BE at right angles to AB, with centre A and radius equal to the sum of the two remaining sides describe a circle cutting BE in E, and join AE. On BE take BF equal to the given altitude, and through F draw FC parallel to AB and meeting AE in C: join CB, then ACB is the triangle required.

56. Let CD be the given difference of the sides AB, AC (figure, Euc. I. 18), and BC the given base. Let ABC be the triangle required; at B draw BF perpendicular to BC, through A draw AE parallel to BC meeting BF in E, and produce CA to meet BF in F. Join BD. Then AD, AB, AF are equal to one another, and the triangle may be constructed on the base BC with altitude BE, and having the difference of the sides equal to CD. For the point D is determined by two circles which touch one another, one described with centre C and radius CD, and the other described passing through the points F, B, and touching the circle whose radius is CD in D. This problem requires the principles of Euc. III. for its construction.

57. Let ABC be the triangle, at C draw CD perpendicular to CB and equal to the sum of the required lines, through D draw DE parallel to CB meeting AC in E, and draw EF parallel to DC, meeting BC in F. Then EF is equal to DC. Next produce CB, making CG equal to CE, and join EG cutting AB in H. From H draw HK perpendicular to EAC, and HL perpendicular to BC. Then HK and HL together are equal to DC. The proof depends on Theorem 66, p. 303.

58. Let ABC, EBC, DBC (DB being joined) be three equal triangles on the same base BC and on the same side of it (fig. Euc. I. 41). Join AD, DE. Then AD is parallel to BC, and DE is parallel to BC.

59. The diameters of a square bisect one another at right angles.

60. This may be exhibited in different ways: one of the most simple, however, is the following. In the figure, Euc. II. 4. On DE, EB, take DE, EM, each equal to BC. Join CM and LM cutting GK in R and GF in Q; also join CH, HL, and draw MP parallel to BA, meeting GF in P. Then the square on CH is equal to the squares on HG, GC. The square on CG is divided into two parts by the line CR; and the square on HG into three parts by HL and LQ. The parts of the two squares HF, CK may be so arranged as to cover exactly the square CL.

61. Draw two indefinite lines AM, AN at right angles to each other. On AM take AB equal to the side of one of the given squares, and on AN take AC equal to the side of the second square, join BC, then the square on BC is equal to the squares on AB and AC. Again, on AM take AD equal to BC, and on AN take AE equal to the side of the third square; join DE, and the square on DE is equal to the three squares on AB, AC, AE.

62. The square described upon the diagonal of a square being equal to double the given square; a square may be described 8 times or n times any given square, where n is any power of the number 2.

63. This is an Algebraical Problem;—to find two rational numbers, the difference of whose squares is a given rational number.

Let the given base of the triangle contain (a) units, and if x, y, denote the hypothenuse and perpendicular; then $a^2 = x^2 - y^2 = (x+y)(x-y)$.

$$\text{Assume } x+y = na, \text{ and } x-y = \frac{1}{n}a:$$

$$\text{whence } x = \frac{(n^2+1).a}{2n}, \text{ and } y = \frac{(n^2-1).a}{2n}.$$

If $a = 2n$, then $2n$, n^2+1, n^2-1, are the three sides of the triangle, where n is any integer greater than unity.

64. This problem is the same as to construct a right-angled triangle, having given the hypothenuse and one side.

65. If x, y denote the base and perpendicular of the triangle, these values will be found to be 4 and 3 from the equations $x-y=1$, and $x^2+y^2=25$.

If a Geometrical Analysis be required. Let ABC be (fig. Euc. I. 18) the triangle required having the right angle at A, BC the given hypothenuse, and CD the given difference between the base AC and the perpendicular AB. Join BD, then BAD is an isosceles right-angled triangle. Since the angle ADB is half a right angle, and the two sides DC, CB are given, the point B can be found. Hence the synthesis.

66. See Theorem 70, p. 304.

67. First, let a parallelogram be formed on one side of the square, and having two of its sides of the required length; next, let a rhombus be formed on one of the sides of the required length. Euc. I. 35.

HINTS, &c. TO THE THEOREMS. BOOK I.

2. Apply Euc. I. 6, 8.

3. This is proved by Euc. I. 32, 13, 5.

4. Let CAB be the triangle (fig. Euc. I. 10) CD the line bisecting the angle ACD and the base AB. Produce CD, and make DE equal to CD, and join AE. Then CB may be proved equal to AE, also AE to AC.

5. Construct the figure and apply Euc. I. 5, 32, 15.

If the isosceles triangle have its vertical angle less than two thirds of a right angle, the line ED produced meets AB produced towards the base and then 3.AEF = 4 right angles + AFE. If the vertical angle be greater than two thirds of a right angle, ED produced meets AB produced towards the vertex, then 3.AEF = 2 right angles + AFE.

6. For in the figure Euc. I. 18, the two sides CB and BA are greater than CA, but AB is equal to AD, therefore the remainder BC is greater than DC or the difference between the two sides BA, AC of the triangle, is less than BC,

It may also be proved that the sum of the three sides of the triangle are *greater than* double any one of the sides, but *less than* the double of any two of the sides.

7. In the Theorem, for AC read BC. At C make the angle BCD equal to the angle ACB, and produce AB to meet CD in D.

8. If the given triangle have both of the angles at the base, acute angles; the difference of the angles at the base is at once obvious from Euc. I. 32. If one of the angles at the base be obtuse, does the property hold good?

9. Let ABC be a triangle, having the right angle at A, and the angle at C greater than the angle at B, also let AD be perpendicular to the base and AE be the line drawn to E the bisection of the base.

Then AE may be proved equal to BE or EC independently of Euc. III. 31.

10. Let ABC be a triangle having the angle ACB double of the angle ABC, and let the perpendicular AD be drawn to the base BC. Take DE equal to DC and join AE. Then AE may be proved to be equal to EB.

If ACB be an obtuse angle, then AC is equal to the sum of the segments of the base made by the perpendicular from the vertex A.

11. By bisecting the hypothenuse, and drawing a line from the vertex to the point of bisection, it may be shewn that this line forms with the shorter side and half the hypothenuse an equilateral triangle.

12. Let the sides AB, AC of any triangle ABC be produced, the exterior angles bisected by two lines which meet in D, and let AD be joined, then AD bisects the angle BAC. For draw DE perpendicular on BC, also DF, DG perpendiculars on AB, AC produced, if necessary. Then DF may be proved equal to DG, and the squares of DF, DA are equal to the squares of FG, GA of which the square of FD is equal to the square of DG; hence AF is equal to AG, and Euc. I. 8, the angle BAC is bisected by AD.

13. Let ABC be the obtuse-angled triangle having the obtuse angle at A. Let the perpendiculars from D, E the bisections of AB, AC meet in G, join G and F the bisection of BC. If GF be proved perpendicular to BC, the theorem is proved.

NOTE. It may be more readily proved by transversals.

14. See Theorem 29, p. 321.

15. Constructing the figure, then by the Method of Transversals, D, E, G may be shewn to be in a straight line. See Cor. 3. Prop. III. Appendix, p. 22.

16. Let the two sides be produced and the exterior angles of the triangle be bisected: join the point in which the bisecting lines meet with the third interior angle of the triangle. If this line be proved to bisect the third interior angle of the triangle, the truth of the theorem is proved. (Conv. of Theorem 12, supra.)

17. This theorem is the converse of Theorem 10, supra.

18. The triangle ABE is proved equal to the triangle DCF, (fig. 2.) In FD if FG be taken equal to ED, and GH be drawn parallel to DC, the triangle FGH is equal to the triangle whose base is DE.

19. Let AD bisect the base, and AE the vertical angle A, and meeting the base in the points D, E. The angle AED may be shewn to be greater than the angle ADE.

20. Let ABC be the triangle; AD perpendicular to BC, AE drawn to the bisection of BC, and AF bisecting the angle BAC. Produce AD and make DA′ equal to AD: join FA′, EA′.

21. In the figure, Euc. I. 47, FC is always equal to AD, and AE to BK. If AB be equal to AC, the truth of the proposition is manifest.

22. This theorem is misplaced, as it cannot be proved by the first book.

First. Prove that the perpendiculars Aa, Bb, Cc pass through the same point O, as Theo. 29, p. 321; or by the theory of Transversals, Prop. II. Appendix, p. 22. Secondly. That the triangles Acb, Bac, Cab are equiangular to ABC. Euc. III. 21. Thirdly. That the angles of the triangle abc are bisected by the perpendiculars; and lastly, by means of Prob. I. p. 293, that $ab+bc+ca$ is a minimum.

23. Let FC be perpendicular to AB and FE be drawn to any other point E in AB: then FC may be proved less than FE, Euc. I. 32, 19. Let CD be taken equal to CE and FD be joined, then FD is equal to FE, and no other line can be drawn from F equal to FE; if possible let FA be equal to FE, which may be shewn to be impossible.

24. Let a circle be described through the points A, B, C; bisect AB in D and

draw the diameter EDF; then the line which bisects the angle C may be proved to pass through the point F on the other side of AB. Euc. III. 21.

25. This proposition requires for its proof the case of equal triangles omitted in Euclid:—namely, when two sides and one angle are given, but not the angle included by the given sides.

26. Let A be the given point, and BC, BD the two straight lines intersecting each other in B. Suppose AEF the line required, such that AF is terminated by BD, and bisected in E by BC. Join AB, draw AG parallel to BD, and join GF. Then ABFG is a parallelogram.

27. This is Prop. 33, of Euclid's Data. There are obviously two points in the given line to which lines may be drawn from the given point.

28. This may very easily be shewn. A restriction however is necessary—namely, that the angles of the interior figure are turned *from* the base.

29. This is possible in certain cases. As an instance, a right-angled triangle ABC may be taken having the right angle at B. From A draw any line AD to meet the base BC in D. Take DE equal to AB; bisect AE in F, and join FC; then the sum of the lines CF, FD shall be greater than the sum of CA, AB, the sides of the triangle. Pappi Coll. Math. Lib. III. Prop. 28.

30. See the notes on Euc. I. 29, p. 50; also, Appendix, p. 2.

31. See the notes on Euc. I. 29, p. 50; and Appendix, p. 2.

32. This is manifest from Euc. I. 29.

33. This will appear from Euc. I. 29, 15, 26.

34. Draw the diagonal and apply Euc. I. 8, 28. The figure is either a square or a rhombus.

35. This is only a more general case of the last Proposition.

36. Apply Euc. I. 29, 26.

37. If the square and parallelogram be upon the same base and between the same parallels, the truth is obvious from Euc. I. 37.

38. The former assertion is proved from Euc. I. 29, 26. The latter may be shewn indirectly.

39. This is proved by applying Euc. I. 8, 4.

40. Let fall upon the diagonal perpendiculars from the opposite angles of the parallelogram. These perpendiculars may be proved to be equal, and each pair of triangles is situated on different sides of the same base and has equal altitudes.

41. Let the line drawn from A fall without the parallelogram, and let CC′, BB′, DD′ be the perpendiculars from C, B, D, on the line drawn from A; from B draw BE parallel to AC′, and the truth is manifest. Next, let the line from A be drawn so as to fall within the parallelogram.

42. Let FH, GE (figure, Euc. I. 43) be joined and produced, they will meet the diagonal CA produced in the same point L. The lines CA, GE, FH may be proved by similar triangles *to converge* to one point, and when produced, to meet in that point. This theorem properly falls under the theorems on Euc. VI.

43. The perpendiculars drawn from B and C are to be perpendicular to the sides AB, AC respectively. Let ABDC be the parallelogram, DE perpendicular on BC the diagonal. At B let BF be drawn meeting ED, produced if necessary in F. Join F, C. If FC can be shewn to be perpendicular to AC, the theorem is proved.

44. One case of this theorem is included in Theorem 40, supra. The other case, when the point is in the diagonal produced, is obvious from the same principle.

45. Let ABCD be a parallelogram and P any point without it, and AC the diagonal. Let AP, PD, PB, PC be joined. Then the triangles APD and APB together are equivalent to the triangle APC. Draw PGE parallel to AD meeting AB and DC in G, E, and join DG, CG. Then by Euc. I. 37.

46. If the four sides of the figure be of different lengths, the truth of the theorem may be shewn. If, however, two adjacent sides of the figure be equal to one another, as also the other two: the lines drawn from the angles to the bisection of the longer diagonal, will be found to divide the trapezium into four triangles which are equal in area to one another.

47. Let BCED be a trapezium (fig. Euc. VI. 2.) of which DC, BE are the diagonals intersecting each other in G. If the triangle DGB be equal to the triangle EGC, the side DE may be proved parallel to the side BC, by Euc. I. 39.

48. Through the point of bisection of one of the opposite sides which are not parallel, draw a line parallel to the opposite side, and meeting the parallel sides, produced, if necessary.

49. This may be shewn by Euc. I. 20.

50. This may be shewn by Euc. I. 35.

51. Draw the two diagonals, then four triangles are formed, two on one side of each diagonal. Then two of the lines drawn through the points of bisection of two sides may be proved parallel to one diagonal, and two parallel to the other diagonal, in the same way as Theorem 45, (which ought to have been placed earlier). The other property is manifest from the relation of the areas of the triangles made by the lines drawn through the bisections of the sides.

52. Join A′B′, B′C′, C′D′, D′A′; then the triangles D′AA′, B′CC′, may be proved to be equal in all respects; as also the triangles A′BB′, C′DD′: whence the figure A′B′C′D′ may be proved to be a parallelogram.

53. See the fig. Euc. VI. 2. The triangle ABC is double of the triangle ABE, and the triangle ABE is double of the triangle ADE. Hence the triangle ADE is one fourth of the triangle ABC. The line DE which bisects the sides may easily be shewn to be parallel to BC.

54. The lines bisecting the opposite sides of a trapezium may be shewn to be the diagonals of the parallelogram formed by joining the four points of bisection of the sides of the trapezium.

55. Let the figure be constructed, and let AC, BD, intersect each other in E. Then by Euc. I. 6, 15, 26, 4, 5, 32, 29.

56. If the *isosceles triangle* be obtuse-angled, by Euc. I. 5, 32, the truth will be made evident. If the triangle be acute-angled the enunciation of the proposition requires some modification.

57. Let ED be bisected in F, and join AF, then AF is equal to EF, and by Euc. I. 5, 29, 32, the angle ABD is proved to be double the angle DBC.

58. The points A, C are two points in the adjacent sides BF, BD produced of the parallelogram. It may be shewn that so long as the figure BFED is a parallelogram, the angles made by FE, DE at the point E, with AE and CE, are together equal to two right angles, and therefore by Euc. I. 14, the line AE is in the same straight line with EC.

In the enunciation for "points A, G and C," read "points A, E and C."

59. The most direct method of shewing that the three other diagonals (which bisect the sides of the triangle) pass through the same point, is by means of transversals.

60. Let ABC be an isosceles triangle (fig. Euc. I. 42), AE perpendicular to the base BC, and AECG the equivalent rectangle. Then AC is greater than AE, &c.

61. This is a case of the preceding theorem.

Of all the triangles which are equal to one another, the isosceles has the least perimeter, and it is easily shewn that the perimeter of the triangle or twice the side and diagonal of the square is greater than the perimeter of the square.

62. Let the angles at the base BC be acute angles. Join DE, CD, CE. Then C is a point within the parallelogram DABE, and the triangles ACD, BCE may be shewn to be together equal to half the parallelogram or double of the triangle ABC.

63. If ABC, DBC be two equal triangles on the same base, of which ABC is isosceles, fig. Euc. I. 37. By producing AB and making AG equal to AB or AC and joining GD, the perimeter of the triangle ABC may be shewn to be less than the perimeter of the triangle DBC.

64. Let two triangles be constructed on the same base with equal perimeters, of which one of them is isosceles. Through the vertex of that which is not isosceles draw a line parallel to the base, and intersecting the perpendicular drawn from the vertex of the isosceles triangle upon the common base. Join this point of intersection and the extremities of the base.

65. Let ABC be a triangle whose vertical angle is A, and whose base BC is bisected in D; let any line EDG be drawn through D, meeting AC in G and AB produced in E, and forming a triangle AEG having the same vertical angle A. Draw BH parallel to AC, and the triangles BDH, GDC are equal. Euc. I. 26.

66. Let ABC be an isosceles triangle, and from any point D in the base BC,

and the extremity B, let three lines DE, DF, BG be drawn to the sides and making equal angles with the base. Produce ED and make DH equal to DF and join BH.

67. Let fall also a perpendicular from the vertex on the base.

68. In the fig. Euc. I. 1, produce AB both ways to meet the circles in D and E, join CD, CE, then CDE is an isosceles triangle, having each of the angles at the base one fourth of the angle at the vertex. At E draw EG perpendicular to DB and meeting DC produced in G. Then CEG is an equilateral triangle.

69. On the same base AB, and on the same side of it, let two triangles ABC, ABD be constructed, having the side BD equal to BC, the angle ABC a right angle, but the angle ABD not a right angle; then the triangle ABC may be shewn to be greater than the triangle ABD whether the angle ABD be acute or obtuse.

70. Let any parallelograms be described on any two sides AB, AC of a triangle ABC, and the sides parallel to AB, AC be produced to meet in a point P. Join PA. Then on either side of the base BC, let a parallelogram be described having two sides equal and parallel to AP. Produce AP and it will divide the parallelogram on BC into two parts respectively equal to the parallelograms on the sides. Euc. I. 35, 36.

71. (*a*) Apply Euc. I. 5, 29.

(*b*) The question is imperfectly expressed.

If FC intersect AE in p and AB in Q; also BK intersect AD in q and AC in P: then Ap is equal to Pq and Aq to Qp. In the case only of AB being equal to AC will the parts Ap, Aq cut off from AE, AD be equal to one another.

(*c*) Let AL meet the base BC in P, and let the perpendiculars from F, K meet BC produced in M and N respectively: then the triangles APB, FMB may be proved to be equal in all respects, as also the triangles APC, CKN.

(*d*) If FH, FG be produced and meet in O, and OB, OC be joined, the triangle OBC is equal in all respects to the triangle ADE, and joining the points A, O, the line AO is in the same straight line with AL which meets the base BC in R. Then OR is a perpendicular from an angle on the opposite side BC of the triangle OBC. If BK, CF can be proved to be perpendicular to the other two sides OC, OB respectively: then BK and CF intersect AL in the same point.

(*e*) Let fall DQ perpendicular on FB produced. Then the triangle DQB may be proved equal to each of the triangles ABC, DBF; whence the triangle DBF is equal to the triangle ABC.

Perhaps however the better method is to prove at once that the triangles ABC, FBD are equal, by shewing that they have two sides equal in each triangle, and the included angles, one the supplement of the other.

(*f*) If DQ be drawn perpendicular on FB produced, FQ may be proved to be bisected in the point B, and DQ equal to AC. Then the square of FD is found by the right-angled triangle FQD. Similarly, the square of KE is found, and the sum of the squares of FD, EK, GH will be found to be six (not eight) times the square of the hypothenuse; but the sum of the squares of FD, DE, EK, KH, HG, GF are together equal to eight times the square of the hypothenuse.

(*g*) The three triangles formed by joining GH, KE, DF are each equal to the triangle ABC. Whence the sum of the four triangles and the three squares may be shewn to be equal to the sum of two squares, namely of BC and of AB + AC.

72. The former part is at once manifest by Euc. I. 47. Let the diagonals of the square be drawn, and the given point be supposed to coincide with the intersection of the diagonals, the minimum is obvious. Find its value in terms of the side.

73. Apply Euc. I. 47.

74. Let the base BC be bisected in D, and DE be drawn perpendicular to the hypothenuse AC. Join AD: then Euc. I. 47.

75. This is at once obvious from Euc. I. 47.

76. Let the equilateral triangles ABD, BCE, CAF be described on AB, BC, CA the sides respectively of the triangle ABC having the right angle at A.

Join DC, AK: then the triangles DBC, ABE are equal. Next draw DG perpendicular to AB and join CG: then the triangles BDG, DAG, DGC are equal to one another. Also draw AH, EK perpendicular to BC; the triangles EKH, EKA are equal. Whence may be shewn that the triangle ABD is equal to the triangle BHE, and in a similar way may be shewn that CAF is equal to CHE.

The restriction is unnecessary: it only brings AD, AE into the same line.

HINTS, &c. TO THE PROBLEMS. BOOK II.

2. PROB. I. p. 306, suggests the method to be employed.

3. Let AB be the given straight line (fig. Euc. II. 4). On AB describe a square, draw the diagonal BD, and take AC equal to half the diagonal BD.

4. Let BF be the given line, (fig. Euc. II. 14) and suppose E the point of division, such that the rectangle BE, EF is equal to the square of the difference of BE and EF. On FB describe the semicircle BHF and draw EH perpendicular to BF to meet the circumference in H. Join G, H, and produce GH to meet in K, the line FK drawn perpendicular to BF. Then FK may be shewn to be equal to FB. Hence the construction.

5. A square may be found equal to the given rectangle; and then the Prob. is reduced to Prob. I. p. 306.

6. This is a repetition of Prob. 2, by mistake.

7. Let AB be the given line. Find a line AE of which the square shall be three times the square of AB; from AB cut off AC equal to the difference between AE and AB, and C is the point of section such that the squares of AB and BC are double the square of AC.

If the square of one part were required to be three times the square of the other, the problem, by aid of Euc. II. 7, is at once reduced to depend on Euc. II. 11.

8. From the preceding Problem $AB^2 + BC^2 = 2AC^2$;

$$\text{therefore } AB^2 - AC^2 = AC^2 - BC^2;$$
$$\text{or } (AB + AC).(AB - AC) = (AC + BC).(AC - BC);$$
$$\text{or } (AB + AC).BC = AB.(AC - BC);$$
$$\text{or } AB.BC + AC.BC = AB.AC - AB.BC,$$
$$\text{therefore } 2.AB.BC = AC.(AB - BC) = AC^2.$$

Whence the problem depends upon the preceding.

9. See note on Euc. II. 11, p. 72.

10. By assuming the points of division, it will be found that the line must be so divided that the square of the middle part is equal to twice the rectangle contained by the extreme parts. Let AB be the given line. Describe on AB any right-angled triangle not isosceles. With centre B and radius BC describe a circle cutting AB in D; and with centre A and radius AC describe a circle cutting AB in E. Then D, E, are the points required.

11. Let the point C be supposed to be determined; then since the rectangle of the sum and difference of AB and BC is equal to the difference of the squares of AB and BC, of which AB is known and BC unknown, which difference is equal to the square of a given line AC. Hence BC is known by Euc. I. 47.

12. Let AB the given straight line be divided in C, it is required to produce AB so that the rectangle contained by the whole line produced and the part produced, may be equal to the rectangle contained by AB, AC. Find by Euc. II. 14, the side of a square which is equal to the rectangle contained by AB, AC: and then the problem is reduced to that of producing the line AB to some point D, so that the rectangle contained by AD, DB is equal to a given square.

13. This follows more simply from Euc. III. 36. If BD be the side of the given square, and AC the difference of the adjacent sides of the rectangle.

14. Let EH be the side of the given square (fig. Euc. II. 14), and BF the sum of the adjacent sides—the construction is obvious.

15. Euc. I. 45, 47; II. 14; will suggest the necessary constructions.

16. Let ABCD be the given rectangle. From A draw AB′ equal to the given length to meet DC, or DC produced in B′; draw BE perpendicular to AB′, and upon the other side of AB′ describe a rectangle AB′C′D′ having AD′ equal to BE. The line C′D′ may cut DC, or DC produced, either way. If D′C′ cut AD in F so that AF is less than FD, then FG must be taken equal to AF and a line be drawn through G parallel to D′C′ to meet DC. The two rectangles may be shewn to consist of parts common or mutually equal to each other.

17. There seems to be some inaccuracy in the enunciation of this Problem. In its present form it appears to be impossible, except when C is the middle of AB, and D coincides with it.

18. In the question for $4c$ read $2c$; otherwise the problem is impossible, Euc. I. 19. To construct the problem generally:—

Assume any line AB to represent c; on AB describe a square ABCD; produce AB to E. On AE describe a semicircle intersecting CD in F. Draw FG perpendicular to AE. Then AG, AB, GE are the sides of the triangle. If BE be equal to AB, F coincides with C, and the triangle is equilateral.

HINTS, &c. TO THE THEOREMS. BOOK II.

3. The area of the rhombus is equal to the areas of the four *right-angled triangles* formed by the diagonals and sides of the figure.

4. Let ABCD be a trapezium having the side AB parallel to CD. Draw the diagonal AC; then the area of the trapezium is equal to the two triangles. See note on Euc. II. def. p. 68.

5. Apply Euc. II. 5, Cor. and note p. 68.

6. In the figure, Euc. II. 7. Join BF, and draw FL perpendicular on GD. Half the rectangle DB, BG, may be proved equal to the rectangle AB, BC. Or,

Join KA, CD, KD, CK. Then CK is perpendicular to BD. And the triangles CBD, KBD are each equal to the triangle ABK. Hence, twice the triangle ABK is equal to the figure CBKD; but twice the triangle ABK is equal to the rectangle contained by AB, BC; and the figure CBKD is equal to half the rectangle contained by DB and CK, the diagonals of the squares on AB, BC. Wherefore, &c.

7. This follows from Euc. II. 7.

8. The difference between the two unequal parts may be shewn to be equal to twice the line between the points of section.

9. This proposition is only another form of stating Euc. II. 7.

10. This may be shewn from Euc. II. 5, Cor.

11. See the notes on Euc. II. 5, 6, 10, 11, p. 69, &c.

12. The Problem is, in other words, given the sum of two lines and the sum of their squares, to find the lines.

Take AB equal to the side of the given square, and on AB describe a semicircle ADB, D being the middle point. Join DA, DB. With centre D and distance DA or DB describe a circle AEB; and with centre A and radius equal to the sum of the given lines, describe another circle cutting AEB in E. Draw AE cutting ADB in C, and join CB. AC, CB are the lines required.

13. Through E draw EG parallel to AB, and through F, draw FHK parallel to BC and cutting EG in H. Then the area of the rectangle is made up of the areas of four triangles; whence it may be readily shewn that *twice the area* (not the area) of the triangle AFE, and the figure AGHK is equal to the area of the rectangle.

14. The lines must be reckoned *positive* or *negative*, according to the direction in which they are measured; and the theorem is rather algebraical: From

$$AF^2=(AE+EF)^2;\ BF^2=(BE+EF)^2;\ CF^2=(EF-CE)^2;\ DF^2=(EF-DE)^2,$$

the enuncicated property may be found.

15. Draw lines from the angles of the equilateral triangle to the point from which the perpendiculars are drawn to the sides. The equilateral triangle is divided into three triangles, the sum of whose areas are equal to the area of the whole triangle. Euc. I. 41; II. def. 1. The point may next be supposed to fall on one of the sides, and the consequence remarked; and lastly, outside the triangle.

16. In the question, for C read H, as in the figure, Euc. II. 11. If D be the point in AH, so that $HD=BH$, then $AB=AH+BH$; and since $AB.BH=AH^2$,

$$\therefore (AH+BH).BH=AH^2, \text{ and } \therefore BH^2=AH^2-AH.BH=AH.(AH-BH);$$

$$\text{or, } HD^2=AH.AD; \text{ that is, AH is divided in D,}$$

so that the rectangle contained by the whole line and one part, is equal to the square

of the other part. By a similar process, HD may be so divided; and so on, by always taking from the greater part of the divided line, a part equal to the less.

Also, if BA be produced, and AK be taken equal to AB, KM to KH, ML to MA, and so on; KH, MA, LK, &c. may be proved to be similarly divided to AB. The proof that KH is so divided appears from the figure: for CF.FA = CA^2; and AK = AC, also AF = AH.

17. The succession of steps may be traced through the first and second books, the final step being Euc. II. 14.

18. From C let fall CF perpendicular on AB. Then ACE is an obtuse-angled triangle, and BEC is an acute-angled triangle. Apply Euc. II. 12, 13; and by Euc. I. 47, the squares of AC and CB are equal to the square of AB.

19. Let ABDE be the square on AB; from C draw CF, CG perpendiculars on DB, EA produced. Then by Euc. II. 12.

20. In a right-angled triangle (Euc. I. 47), the square of the side subtending the right angle is *equal to* the squares of the sides containing the right angle; but in an obtuse-angled triangle (Euc. II. 12), the square of the side subtending the obtuse angle *is greater than* the square of the side containing the obtuse angle; and in an acute-angled triangle (Euc. II. 13), the square of the side subtending an acute angle *is less than* the squares of the sides which contain that angle.

21. This will be found to be that particular case of Euc. II. 12, in which the distance of the obtuse angle from the foot of the perpendicular is half of the side subtended by the right angle made by the perpendicular and the base produced.

22. In every scalene triangle, the line drawn from the vertex to the bisection of the base, divides the triangle into two triangles, one obtuse-angled and the other acute-angled. Apply Euc. II. 12, 13.

23. (1) Let the triangle be acute-angled (Euc. II. 13, fig. 1.)

Let AC be bisected in E, and BE be joined; also EF be drawn perpendicular to BC. DF is equal to FC. Then the square of BE may be proved to be equal to the square of EC together with the rectangle BD, BC.

(2) If the triangle be obtuse-angled, the perpendicular EF falls *within* or without the base according as the bisecting line is drawn from the *obtuse* or the *acute* angle at the base.

24. See Theorem 29, p. 354.

25. The truth of this theorem follows at once from Euc. I. 47.

26. The common intersection of the three lines divides each into two parts, one of which is double of the other, and this point is the vertex of three triangles which have lines drawn from it to the bisection of the bases. Apply Euc. II. 12, 13.

27. Draw a perpendicular from the vertex to the base, and apply Euc. I. 47; II. 5. Cor. Enunciate and prove the proposition when the straight line drawn from the vertex meets the base produced.

28. This follows directly from Euc. II. 13, Case 1.

29. The truth of this proposition may be shewn from Euc. I. 47; and Euc. II. 4.

30. Let the square on the base of the isosceles triangle be described. Draw the diagonals of the square, and the proof is obvious.

31. Apply Euc. I. 47, to express the squares of the three sides in terms of the squares of the perpendicular and of the segments of AB.

32. Draw EF parallel to AB and meeting the base in F: draw also EG perpendicular to the base. Then DF is a parallelogram, and by Euc. I. 47; II. 5. Cor.

33. Bisect the angle B by BD meeting the opposite side in D, and draw BE perpendicular to AC. Then by Euc. I. 47; II. 5. Cor.

34. This follows directly from Theorem 22, p. 309.

35. Let the point O be within the rectangle. Draw the diagonals intersecting each other in P and join OP. Euc. II. 12, 13. Let O be without the triangle.

36. Draw from any two opposite angles, straight lines to meet in the bisection of the diagonal joining the other angles. Then by Euc. II. 12, 13.

37. Draw two lines from the point of bisection of either of the bisected sides to the extremities of the opposite side; and three triangles will be formed, two on one of the bisected sides and one on the other, in each of which is a line drawn from the vertex to the bisection of the base. Then by Theorem 22, p. 309.

38. If the extremities of the two lines which bisect the opposite sides of the

trapezium be joined, the figure formed is a parallelogram which has its sides respectively parallel to, and equal to, half the diagonals of the trapezium. The sum of the squares of the two diagonals of the trapezium may be easily shewn to be equal to the sum of the squares of the four sides of the parallelogram.

39. Draw perpendiculars from the extremities of one of the parallel sides, meeting the other side produced, if necessary. Then from the four right-angled triangles thus formed, may be shewn the truth of the proposition.

40. Let ABC be any triangle; AHKB, AGFC, BDEC, the squares upon their sides; EF, GH, KL the lines joining the angles of the squares. Produce GA, KB, EC, and draw HN, DQ, FR perpendiculars upon them respectively: also draw AP, BM, CS perpendiculars on the sides of the triangle. Then AN may be proved to be equal to AM; CR to CP; and BQ to BS; and by Euc. II. 12, 13.

HINTS, &c. TO THE PROBLEMS. BOOK III.

3. LET A be the centre, the radius of the circle being unknown. Take any point B in the circumference and find by means of the compasses only, a third point C in the circumference, such that B, A, C, shall be in the same straight line.

Many problems of this class are solved in the "*Géométrie du Compas,*" *par L. Mascheroni.*

4. Euc. III. 3, suggests the construction.

5. The least chord drawn through a given point, is the line perpendicular to that diameter which passes through the given point.

6. The given point may be either within or without the circle. Find the centre of the circle, and join the given point and the centre, and upon this line describe a semicircle, a line equal to the given distance may be drawn from the given point to meet the arc of the semicircle. When the point is without the circle, the given distance may meet the diameter produced.

7. Let two unequal circles cut one another, and let the line ABC drawn through B, one of the points of intersection, be the line required, such that AB is equal to BC. Join OO′ the centres of the circles, and draw OP, O′P′ perpendiculars on ABC, then PB is equal to BP′; through O′ draw O′D parallel to PP′; then ODO′ is a right-angled triangle, and a semicircle described on OO′ as a diameter will pass through the point D. Hence the synthesis. If the line ABC be supposed to move round the point B and its extremities A, C to be in the extremities of the two circles, it is manifest that ABC admits of a maximum.

8. It is sufficient to suggest that the angle between a chord and a tangent is equal to the angle in the alternate segment of the circle. Euc. III. 32.

9. There is some inaccuracy in the enunciation:—for if the two points be given in the circumference of the circle, the angle which the tangents make with the given line is dependent on the position of the two points in the circumference.

10. Let D be the point required in the diameter BA produced, such that the tangent DP is half of DB. Join CP, C being the centre. Then CPD is a right-angled triangle, having the sum of the base PC and hypothenuse CD double of the perpendicular PD. See Prob. 31, p. 298.

11. Let P be the given point, and PBA the given line cutting the circle ABC in the points B, A. Let PCD be the line required: join OC, OD, O being the centre. Then the arc AB being given, and the sum of the arcs BC, AD; the arc CD is also given in magnitude, and the angle COD which it subtends at the centre.

Whence the construction. Take the arc RS equal to the defect of the sum of the three arcs AB, DA, BC from the whole circumference: join RS, and with centre O describe a circle touching RS, and from P draw PCD to touch this circle.

12. At any point P in the circumference of the given circle, draw a line APB touching the circle at P, the parts AP, BP being equal to the given line. With centre C and radius CA or CB describe a circle, produce the given chord DE to meet the circumference of this circle in F. From F draw FG to touch the given circle in G, then FG is the line required.

14. Describe a circle through the three given points; from A any one of them, draw any chord, and from the centre D draw the perpendicular DE upon it. With the same centre and radius DE describe a circle. Then from B, C draw lines BF, CG touching this circle; then AE, BF, CG are equal to one another. The circle, which is the superior limit, is obviously the circle passing through the points A, B, C.

15. Find the centre, and the construction of Euc. III. 15 will suggest how the radius may be found, supposing FG and BC on the same side of the diameter.

16. The lengths of the lines may be found by Euc. III. 15.

17. Let ABC be the required triangle on the given base AB, having the sum of the squares of its sides AC, CB equal to the given square. If the base AB be bisected in D and CD be joined, then by Theo. 22, p. 309, the difference between the sum of the squares of the sides and of the square of half the base may be shewn to be equal to double the square of CD. Hence CD is constant, and therefore the locus is a circle whose centre is D and radius DC.

18. Let O be the centre of the given circle. Draw OA perpendicular to the given straight line: at O in OA make the angle AOP equal to the given angle, produce PO to meet the circumference again in Q. Then P, Q are two points from which tangents may be drawn fulfilling the required condition.

19. (1) When the tangent is on the same side of the two circles. Join C, C′ their centres, and on CC′ describe a semicircle. With centre C′ and radius equal to the *difference* of the radii of the two circles, describe another circle cutting the semicircle in D; join DC′ and produce it to meet the circumference of the given circle in B. Through C draw CA parallel to DB and join BA; this line touches the two circles.

(2) When the tangent is on the alternate sides. Having joined C, C′; on CC′ describe a semicircle; with centre C, and radius equal to the *sum* of the radii of the two circles describe another circle cutting the semicircle in D, join CD cutting the circumference in A, through C draw CB parallel to CA and join AB.

20. Let A be the given point, PQ any quadrantal arc of the given circle. From C the centre draw CD perpendicular to PQ and with centre C and radius CD, describe a circle, which will touch PQ in D. Then two tangents may be drawn to this circle from A, either of which is a solution. The same construction serves for any given chord *less than* the diameter of the given circle.

21. Let AB, BC, CD be the three given straight lines in the same straight line. On AC as a base describe the locus of the vertex of the triangle whose sides are as AB to BC. On the base BD describe the locus of the vertex of the triangle whose sides shall be as BC to CD. Let P be the intersection of these loci. Join PA, PB, PC, PD; the angles ABP, BPC, CPD are equal angles. See Prob. 8, p. 348.

22. Let A be the given point in the diameter BC; through A draw DAE perpendicular to BC, and join DB, BE. Through A draw any other chord FAG; and join BF, BG; draw FH, GK parallel to BC meeting DE in H, K respectively; join BH, BK, and draw FM, GN perpendicular to BC. Then HK may be proved to be less than DE, and the triangle FBG less than the triangle DBA.

23. From A suppose ACD drawn, so that when BD, BC are joined, AD and DB shall together be double of AC and CB together. Then the angles ACD, ADB are supplementary, and hence the angles BCD, BDC are equal, and the triangle BCD is isosceles. Also the angles BCD, BDC are given, hence the triangle BDC is given in species.

Again $AD + DB = 2 . AC + 2 . BC$, or $CD = AC + BC$.

Whence, make the triangle *bdc* having its angles at *d*, *c* equal to that in the segment BDA; and make $ca = cd - cb$, and join *ab*. At A make the angle BAD equal to *bad*, and AD is the line required.

24. This is the same as Problem 12, p. 315.

25. Let the given straight line AB be divided at any point C. On AB as a diameter describe a circle. Let the point D in the arc be the point required, such that when DA, DC, DB are joined, the angles ADC, CDB are each equal to half a right angle. Produce DC to meet the circumference in E, then Euc. III. 21.

26. The line which divides the circle into two segments is equal to the line which joins the two points of intersection of the circles in the figure Euc. I. 1.

27. Let ABC be a triangle of which the base or longest side is BC, and let a segment of a circle be described on BC. Produce BA, CA to meet the arc of the

segment in D, E, and join BD, CE. If circles be described about the triangles ABD, ACE, the sides AB, AC shall cut off segments similar to the segment described upon the base BC.

28. This is the same as Euc. III. 34, with the condition, that the line must pass through a given point.

29. By supposing the thing done, the line joining the points of intersection of the two circles, will pass through the centre of the given circle. From one of the given points draw a line through the centre of the given circle and cutting the two circles in four points; then by Euc. III. 35.

30. As a particular case, if the segment be a semicircle: on AB the diameter, the point K will be found to be distant from B, one half of the radius.

31. If a segment of a circle containing an angle equal to the given angle be described upon the line which joins the two given points, the circumference of the segment will meet the circumference of the given circle in two points, which will give two solutions of the Problem; or in one point at which the angle will be a maximum.

32. Let ABCD be the required trapezium inscribed in the given circle (fig. Euc. III. 22) of which AB is given, also the sum of the remaining three sides and the angle ADC. Since the angle ADC is given, the opposite angle ABC is known, and therefore the point C and the side BC. Produce AD and make DE equal to DC and join EC. Since the sum of AD, DC, CB is given, and DC is known, therefore the sum of AD, DC is given, and likewise AC, and the angle ADC. Also the angle DEC being half of the angle ADC is given. Whence the segment of the circle which contains AEC is given, also AE is given, and hence the point E, and consequently the point D. Whence the construction.

33. Join the centres A, B; at C the point of contact draw a tangent, and at A draw AF cutting the tangent in F, and making with CF an angle equal to one fourth of the given angle. From F draw tangents to the circles.

34. At any points P, R in the circumferences of the circles whose centres are A, B, draw PQ, RS, tangents equal to the given lines, and join AQ, BS. These being made the sides of a triangle of which AB is the base, the vertex of the triangle is the point required.

35. Let C, C′ be the centres of the given circles, join CC′, and let DD′ be the line required, making the given angle with CC′. Through C draw CE making with CC′ an angle equal to the given angle, and equal to the given line, join CD, ED′.

36. Let ABCD be the required line, such that the chords AB, CD are each equal to the given line. Take O, O′ the centres of the circles, and draw OE perpendicular to AB, and O′F to CD. With centres O, O′ and radii OE, O′F respectively, describe two circles. Then the line ABCD is a common tangent to these circles, and all chords which touch the interior and are terminated by the exterior circles are equal to one another.

37. Suppose the thing done, then it will appear that the line joining the points of intersection of the two circles is bisected at right angles by the line joining the centres of the circles. Since the radii are known, the centres of the two circles may be determined.

38. Let the two circles cut one another in A and B. Join AB, and suppose ACD to be the line required meeting the circumferences in C, D, such that the part DE is equal to the given line. Join also BC, BD. Then the angles at C, D may be shewn to be given, and the point D. The proof of this problem involves proportion.

39. Let A, B, C be the three given points. Join A, B, and on AB describe a segment of a circle ADB containing an angle equal to that which the lines drawn to the points A, B is to contain, and complete the circle. On the other side of AB at the point B, make the angle ABE equal to the angle which the lines drawn to the points A, C, is to contain, and let the line BE meet the circumference in E. Join EC and produce it to meet the circumference again in D. D is the point required.

40. Let the two circles touch each other at A, AD, AD′, their diameters; and let PQM be the semichord perpendicular to the diameter. If PQ be supposed equal to QM, the line MD may be shewn to be equal to four times MD′.

41. Let the straight line joining the centres of the two circles be produced both ways to meet the circumference of the exterior circle.

42. Let A be the common centre of the two circles, and BCDE the chord such that BE is double of CD. From A, B draw AF, BG perpendicular to BE. Join AC, and produce it to meet BG in G. Then AC may be shewn to be equal to CG, and the angle CBG being a right angle, is the angle in the semicircle described on CG as its diameter.

43. Let C be the common centre. Draw any diameter BCD of the interior circle, and produce it, making DE equal to CD. On DE describe a semicircle cutting the circumference of the exterior circle in F, join FD and produce it to meet the interior circumference in G and the exterior in H.

44. It may be proved that of all triangles having equal vertical angles, and their distances from their vertices to the bisections of their bases, equal to one another, the greatest is that which is isosceles.

45. The locus of the point C when the string ACB is kept stretched is an ellipse. This Problem appears here by mistake.

46. Let A be the given point (fig. Euc. III. 36, Cor.) and suppose AFC meeting the circle in F, C, to be bisected in F, and let AD be a tangent drawn from A. Then $2.AF^2 = AF.AC = AD^2$, but AD is given, hence also AF is given.

To construct. Draw the tangent AD. On AD describe a semicircle AGD, bisect it in G; with centre A and radius AG, describe a circle cutting the given circle in F. Join AF and produce it to meet the circumference again in C.

47. Let C be the centre of the given circle, AC the given radius; produce AC to meet the circumference in B. With centre B and radius equal to the given line, describe a circle cutting the given circle in D. Through D draw DE parallel to BA, meeting the circumference in E, at B make the angle CBG equal to the given angle, and from E, draw EF parallel to DB, meeting the given radius in F.

48. On any two sides of the triangle, describe segments of circles each containing an angle equal to two thirds of a right angle, the point of intersection of the arcs within the triangle will be the point required, such that three lines drawn from it to the angles of the triangle shall contain equal angles. Euc. III. 22.

49. Let AB, AC be the lines containing the given angle BAC, and let PQ be the line of given length. Bisect PQ in R and on PR describe a rectangle equal to the given area; also on PQ describe a segment of a circle containing an angle equal to the angle BAC, and let the arc of this segment cut the side of the rectangle opposite to PR in S. Join SP, SQ. On AB take AD equal to SP, and on AC take AE equal to SQ; join DE, then DAE is the triangle required. It may be shewn that the triangle cut off has the greatest possible area when it is isosceles.

50. Let A, B be the extremities of the diameter of the circle, CD the given chord not parallel to AB; and let KL be the length to be cut from CD by lines drawn from A, B, and meeting at a point in the circumference.

Draw BE equal to KL and parallel to CD, join AE and on AE describe a semicircle cutting CD in F, join AF and produce it to meet the circumference in H, and join HB cutting CD in G. Then FG is equal to KL, as may be shewn from BEFG being a parallelogram. If the semicircle touch CD there is only one solution, if it cut CD there are two, if it do not meet CD, there is no solution to the problem.

51. This problem is a slightly varied form of Problem 39, p. 317.

52. On any two sides AC, CB describe internally two segments of circles each containing an angle equal to one third of four right angles; and let the segments intersect each other in D. Join AD, BD, CD. Let a circle be described with centre A and radius AD and a tangent be drawn to it at D. Then the sum of BD, DC is a minimum when they make equal angles with the tangent, and therefore when they make equal angles with the radius AD. Similarly it may be shewn that the sum of AD, DC is a minimum. But the lines AD, BD, CD make equal angles with each other, D therefore is the point required.

53. Let Aa, Bb, Cc be drawn from the angles to the bisections of the opposite sides of the triangle ABC. At B, b draw BD, bD parallel to aA, Cc respectively, and meeting each other in D. Join Dc, bc. Then Db is proved equal to Cc from the triangles DbA, cCb: also in a similar way DB is proved equal to aA.

54. Let ABC be the given isosceles triangle having the vertical angle at C, and let FG be any given line. Required to find a point P in FG such that the distance PA shall be double of PC. Divide AC in D so that AD is double of DC, produce AC

to E and make AE double of AC. On DE describe a circle cutting FG in P, then PA is double of PC. This is found by shewing that $AP^2 = 4 . PC^2$.

55. The equal sides of the equivalent isosceles triangle are each a mean proportional between the two sides of the scalene triangle. Euc. VI. 15.

56. This is made to depend upon constructing a right-angled triangle of which the perpendicular and the opposite angle are given.

57. Let ABC be the triangle required; BC the given base, BD the given difference of the sides, and BAC the given vertical angle. Join CD and draw AM perpendicular to CD. Then MAD is half the vertical angle and AMD a right angle: the angle BDC is therefore given, and hence D is a point in the arc of a given segment on BC. Also since BD is given, the point D is given, and therefore the sides BA, AC are given. Hence the synthesis.

58. On any base BC describe a segment of a circle BAC containing an angle equal to the given angle. From D the middle point of BC draw DA to make the given angle ADC with the base. Produce AD to E so that AE is equal to the given bisecting line, and through E draw FG parallel to BC. Join AB, AC and produce them to meet FG in F and G. The demonstration depends on Book VI. ;

59. Analysis. Let ABC be the triangle, and let the circle ABC be described about it; draw AF to bisect the vertical angle BAC and meet the circle in F, make AV equal to AC, and draw CV to meet the circle in T; join TB and TF, cutting AB in D; draw the diameter FS cutting BC in R, DR cutting AF in E; join AS, and draw AK, AH perpendicular to FS and BC. Then shew that AD is half the sum, and DB half the difference of the sides AB, AC. Next, that the point F in which AF meets the circumscribing circle is given, also the point E where DE meets AF is given. The points A, K, R, E are in a circle, Euc. III. 22. Hence KF.FR = AF.FE, a given rectangle; and the segment KR, which is equal to the perpendicular AH, being given, RF itself is given. Whence the construction.

60. On AB the given base describe a circle such that the segment AEB shall contain an angle equal to the given vertical angle of the triangle. Draw the diameter EMD cutting AB in M at right angles. At D in ED, make the angle EDC equal to half the given difference of the angles at the base, and let DC meet the circumference of the circle in C. Join CA, CB; ABC is the triangle required.

For, make CF equal to CB, and join FB cutting CD in G.

61. Let ABC be the triangle, AD the perpendicular on BC. With centre A, and AC the less side as radius, describe a circle cutting the base BC in E, and the longer side AB in G, and BA produced in F, and join AE, EG, FC. Then the angle GFC being half the given angle, BAC is given, and the angle BEG equal to GFC is also given. Likewise BE the difference of the segments of the base, and BG the difference of the sides, are given by the problem. Wherefore the triangle BEG is given (with two solutions). Again, the angle EGB being given, the angle AGE, and hence its equal AEG is given; and hence the vertex A is given, and likewise the line AE equal to AC the shorter side is given. Hence the construction.

62. On the given base AB describe a segment of a circle containing an angle equal to the given angle, and through C the bisection of the base draw the diameter DE perpendicular to AB. Join AE, and on AE describe a semicircle ACE. From A, place in this semicircle AF equal to half the given difference of the two sides of the triangle. Produce AF to meet the circle in G and join GB; ABG is the triangle required.

63. Let ABC be the triangle, D, E the bisections of the sides AC, AB. Join CE, BD intersecting in F. Bisect BD in G and join EG. Then EF, one third of EC is given, and BG one half of BD is also given. Now EG is parallel to AC; and the angle BAC being given, its equal opposite angle BEG is also given. Whence the segment of the circle containing the angle BEG is also given.

Hence F is a given point, and FE a given line, whence E is in the circumference of the given circle about F whose radius is FE. Wherefore E being in two given circles, it is itself their given intersection.

64. Of all triangles on the same base and having equal vertical angles, that triangle will be the greatest whose perpendicular from the vertex on the base is a maximum, and the greatest perpendicular is that which bisects the base. Whence the triangle is isosceles.

65. When the vertical angle and area are given, the rectangle under the sides is also given. Likewise the sum of the squares and the rectangle of the two lines which constitute the sides are given. These lines may hence be found, and the triangle constructed.

66. Let AB be the base, and AHB the segment containing the vertical angle. At any point H draw the tangent HP, making the square of HP equal to the given rectangle under the sum of the sides and one of them. Find O the centre of the circle AHB and join PO, with which as radius describe a circle PDQ. Also from R the middle of the arc AHB, with radius RA or RB describe the circle ADB cutting PDQ in D. Draw DA cutting AHB in C, and join CB. ACB is the triangle required. For, draw the tangent DE, and join HO, EO, DO. Then $HP^2 = DE^2 = AD \, . \, DC = (AC + CB) \, . \, CB$; for the triangle DCB is isosceles.

67. Let C be the centre of the given circle, B the given point in the circumference, and A the other given point through which the required circle is to be made to pass. Join CB, the centre of the circle is a point in CB produced. The centre itself may be found in three ways.

68. Let A be the given point, and B the given point in the given line CD. At B draw BE at right angles to CD, join AB and bisect it in F, and from F draw FE perpendicular to AB and meeting BE in E. E is the centre of the required circle.

69. Let AB, AC be the given lines and P the given point. Then if O be the centre of the required circle touching AB, AC in R, S, the line AO will bisect the given angle BAC. Let the tangent from P meet the circle in Q, and draw OQ, OS, OP, AP. Then there are given AP and the angle OAP. Also since OQP is a right angle, we have $OP^2 - QO^2 = OP^2 - OS^2 = PQ^2$ a given magnitude. Moreover the right-angled triangle AOS is given in species, or OS to OA is a given ratio. Whence in the triangle AOP there is given, the angle AOP, the side AP, and the excess of OP^2 above the square of a line having a given ratio to OA, to determine OA. Whence the construction is obvious.

70. Bisect the angle BAC by AF, at A draw AG perpendicular to AB and equal to half of DE. Through G draw GH parallel to AB, meeting AF in H. Then H is the centre of the required circle.

71. Draw any line AB so that AC is equal to BC; on this describe a segment to contain the given angle; through the given point P draw CQ, meeting the segment in Q, and join QA, QB; draw PD, PE parallel to QA, QB respectively; and DO perpendicular to AC, and EO to BC. Then O is the centre.

72. Let D be the given point and EF the given straight line. (fig. Euc. III. 32.) Draw DB to make the angle DBF equal to that contained in the alternate segment. Draw BA at right angles to EF, and DA at right angles to DB and meeting BA in A. Then AB is the diameter of the circle.

73. Let A, B be the given points, and CD the given line. From E the middle of the line AB, draw EM perpendicular to AB, meeting CD in M, and draw MA. In EM take any point F; draw FH to make the given angle with CD; and draw FG equal to FH, and meeting MA produced in G. Through A draw AP parallel to FG; and CPK parallel to FH. Then P is the centre, and C the third defining point of the circle required: and AP may be proved equal to CP by means of the triangles GMF, AMP; and HMF, CMP, Euc. VI. 2. Also CPK the diameter makes with CD the angle KCD equal to FHD, that is, to the given angle.

74. See Euc. III. 37, and Theo. 3, p. 313.

75. Let A be the base of the tower, AB its altitude, BC the height of the flag-staff, AD a horizontal line drawn from A. If a circle be described passing through the points, B, C, and touching the line AD in the point E: E will be the point required. Give the analysis.

76. Let A, B, C be the centres of the three given circles. On the same side draw the tangent DE to the circles whose centres are A, B; and FG to the circles whose centres are B, C; bisect DE in H, and FG in K; and draw HL, KM perpendicular to AB, AC, (the lines joining the centres of the circles) to meet in O. O is the point required. For, join DA, HA, FA, KA, EB, HB, KC, CG, AO, BO, CO; draw the tangents OP, OQ, OR to the circles whose centres are A, B, C respectively, and join AP, BQ, CR. Then the difference of the squares of OP and OQ may be shewn to be equal to the difference of the squares of DH and HE. But

DH is equal to HE by construction, it follows that OP is equal to OQ. In the same way it may be shewn, that OP is equal to OR.

NOTE. The line HO is called *the radical axis* of the circles whose centres are A and B; and similarly, KO of the circles whose centres are B and C.

The point O is *the radical centre* of the circles whose centres are A, B, C.

77. Let POQ be the common diameter, O being the point of contact of the circles B, C. Let DEOFG be any line drawn through O and meeting the circumferences of the circles. Join PE, QF, then DE is equal to FG by Theorem I. p. 344.

Can DE be shewn to be equal to FG without proportion?

78. Let the point E in AO be supposed to be found subject to the conditions of the Problem. Produce PO to meet the circumference in Q. Then by Euc. II. 5, 6, III. 36, Cor. combined with the given conditions, the squares of OE and OP may be shewn to be equal to five times the square of the radius AO. And the triangle PEO is right-angled. Hence the square of PE is five times the square of AO, and AO is known, therefore the line PE is known, also the point P is given. Whence the point E is determined subject to the given conditions.

79. This is the same as the following problem with only a slight alteration.

80. From A draw the chord AC equal to the side of the given square, join BC and produce it to meet AP in P, the point P thus found is the point required, as may be shewn by Euc. III. 36, I. 47, II. 3. The limits of the possibility may be readily determined.

81. On PC describe a semicircle cutting the given one in E, and draw EF perpendicular to AD: then F is the point required.

82. Let AB, AC be the two given lines meeting at the point A, and D the given point. From D draw DE perpendicular to AB and produce ED to F, so that the rectangle contained by ED, DF may be equal to the given rectangle. On DF as a diameter describe a circle cutting AC in G, join GE and produce it to meet AB in H. Then a circle may be described through the points E, G, F, H.

83. If the ladder be supposed to be raised in a vertical plane, the locus of the middle point may be shewn to be a quadrantal arc of which the radius is half the length of the ladder.

84. The point E will be found to be that point in BC from which two tangents to the circles described on AB and CD as diameters, are equal. Euc. III. 36.

85. Draw from the given point A a straight line AB, and divide AB on C, so that the rectangle AB, BC is equal to the given rectangle. If the line given by position (which is a circle) be described so as to pass through the points A, B, the locus of the extremities of all such lines drawn from A will be in the convex circumference of the circle.

86. Suppose AD, DB the tangents to the circle AEB and which contain the given angle. Draw DC to the centre C and join CA, CB. Then the triangles ACD, BCD are always equal: DC bisects the given angle at D and the angle ACB. The angles CAB, CBD, being right angles, are constant, and the angles ADC, BDC are constant, as also the angles ACD, BCD; also AC, CB the radii of the given circle. Hence the locus of D is a circle whose centre is C and radius CD.

87. (1) This locus may be shewn to be a circle, as the sum of the angles at the base is constant, the vertical angle is also constant. Euc. III. 21.

(2) This locus is a hyperbola, and was inserted here by an oversight.

88. The locus is a hyperbola. Also the equation to the locus may be put under the form, $4xy \tan\phi = (a^2 - b^2)\sec^2\phi$.

89. Join AB, and upon it describe a segment of a circle which shall contain an angle equal to the given angle. If the circle cut the given line, there will be two points; if it only touch the line, there will be one; and if it neither cut nor touch the line, the problem is impossible.

90. Describe a circle having the given line AB as its diameter and centre C. Draw any radius CD, and at D draw a tangent DE equal to AB and join CE. In AB produced, make CF equal to CE. F is the point required.

Let EC meet the circle in B′, and when produced, in A′.

91. This is only a case of the preceding, instead of taking the tangent DE equal to AB, it must be taken equal to the side of the given square.

92. If the rectangle of the sides of the right-angled triangle be equal to the

square of their difference, it may be shewn that the rectangle is also equal to one third of the square of the hypothenuse. Hence, if a semicircle be described on AB the given hypothenuse, and a line AC be drawn at right angles to AB and equal to one third of AB, and from C a line CD be drawn parallel to AB meeting the semicircle in D, and DA, DB be joined: DAB is the required triangle.

93. Let ABC be the required triangle in which are given AD, BE, CF, lines drawn from the angles to the bisections of the opposite sides of the triangle, and intersecting one another in G. Then AG is double of GD. Produce GD to H, making DH equal to DG and join BH, CH. Then BGCH is a parallelogram, and the sides of the triangle BHG are respectively each two thirds of the lines drawn from the angles to the bisections of the sides.

94. Let AB, BC, CA be the three given lines, and D the given point. Draw any line DE through D to meet AC in E; and any line FG parallel to DE to meet AB in F, and BC in G; and join EF, EG; and lastly, through F and G, draw parallels to EG and EF meeting DE in R and P, and each other in Q. Then the triangle PQR fulfils the conditions.

It is obvious that there is no limit to the number of triangles which can be constructed to fulfil the given conditions. For the *direction* of DE from D is arbitrary; and likewise the distance of the parallels DE, FG.

95. Let ABCD be the given square, and let the diagonals intersect each other in E. Let F be any point in the locus, and join FA, FB, FD, FC, FE. Then by Theorem 22, p. 309, the sum of the squares of the lines from F to the angles of the square are shewn to be equal to twice the area of the square and four times the square of EF. And E is a fixed point; if EF be constant, the locus of F is a circle whose centre is E and radius EF.

HINTS, &c. TO THE THEOREMS. BOOK III.

5. First. Join the extremities of the chords, then Euc. I. 27; III. 28.

Secondly. Draw any straight line intersecting the two parallel chords and meeting the circumference.

6. This is the converse of the former part of the preceding Theorem 5.

7. Let the circles intersect in A, B; and let CAD, EBF be any parallels passing through A, B and intercepted by the circles. Join CE, AB, DF. Then the figure CEFD may be proved to be a parallelogram. Whence CAD is equal to EBF.

8. Construct the figure and the arc BC may be proved equal to the arc B′C′.

9. See Theorem 67, p. 351.

10. Let AB be drawn from the given point A, touching the given circle whose centre is D. Join AD, DB, then the position of the point B may be shewn to be given, and therefore the line AB both in position and magnitude. (Euclid's Data, Prop. 94.)

11. Let C be the centre of the circle, and E the point of contact of DF with the circle. Join DC, CE, CF.

12. Let AB, AC be the sides of a triangle ABC. From A draw the perpendicular AD on the opposite side, or opposite side produced. The semicircles described on AB, BC both pass through D. Euc. III. 31.

13. Let A be the right angle of the triangle ABC, the first property follows from the preceding Theorem 12. Let DE, DF be drawn to E, F the centres of the circles on AB, AC, and join EF. Then ED may be proved to be perpendicular to the radius DF of the circle on AC at the point D.

14. Let ABC be a triangle, and let the arcs be described on the sides externally containing angles, whose sum is equal to two right angles. It is obvious that the sum of the angles in the remaining segments is equal to four right angles. These arcs may be shewn to intersect each other in one point D. Let a, b, c be the centres of the circles on BC, AC, AB. Join ab, bc, ca; Ab, bC, Ca, aB; Bc, cA; bD, cD, aD. Then the angle cba may be proved equal to one half of the angle AbC. Similarly, the other two angles of abc.

15. The angle OAQ may be proved equal to the angle OCQ, and the truth of the theorem is manifest from Euc. I. 32.

16. Bisect the lines and join the points of bisection with the centre of the circle, the two triangles thus formed may be shewn to be equal, and by Euc. III. 14, the equality of the lines is inferred.

17. Let the diagram be drawn, and from the centre of the circle draw a perpendicular on the chord which passes through the middle points of the two equal chords. Then Euc. III. 3.

18. Let AB a chord in a circle be bisected in C, and DE, FG two chords drawn through C; also let their extremities DG, FE be joined intersecting CB in H, and AC in K; then AK is equal to HB. Through H draw MHL parallel to EF meeting FG in M, and DE produced in L. Then by means of the equiangular triangles, HC may be proved to be equal to CK, and hence AK is equal to HB.

19. Let AB, AC be the bounding radii, and D any point in the arc BC. The circle described on AD will always be of the same magnitude, and the angle EAF in it, is constant:—whence the arc EDF is constant, and therefore its chord EF.

20. This is manifest from Euc. III. 23.

21. Join BC. Since the angles at B, C are right angles, a circle may be described about the figure BECF. Euc. III. 22. Let the circle be described. Then the angle BEF is equal to BCF or BCA; and BCA is equal to BDA or FDG; also BFE is equal to DFG: whence two of the angles of the triangle FGD being equal respectively to two angles of the triangle BFE; the third angle FGD is equal to the third angle EBF, which is a right angle, Euc. III. 31.

22. The chord PQ is proved greater than any other chord TR passing through the same point N, by Euc. I. 19; III. 15. If a circle be described about the triangle PSQ, it will touch the circumference of the given circle in P, and the angle SPQ may be shewn to be greater than the angle STR.

23. The perpendicular from the vertex bisects the base of the isosceles triangle, and the circle described upon one of the sides will pass through the bisection of the base, Euc. III. 31.

24. Let AB, CD be any two diameters of a circle, and let two other circles through B, D cut the diameters in E, F; and in G, H. Join BD. Then DE, BF may be proved to be equal, as also BD, DG.

25. Constructing the figure and producing the tangent QP, the triangle CPQ may be shewn to be an isosceles triangle, as also the triangle C′QP: also CQ (not CQ′) and C′Q may be each shewn to be equal to QP.

26. The chord AB is common to the two equal circles. The angles ADB, ACB may be shewn to be equal. Hence the triangle BCD is isosceles.

27. This is only the extreme case of Theorem 5, p. 320. Also the angle contained by the tangents may be shewn to be equal to the difference between the angles in the two segments formed by joining the points of contact.

28. By constructing the figure and joining BC, the truth will appear from Euc. I. 32; and III. 20.

29. It may be remarked, that generally, the mode by which, in pure geometry, three lines must, under specified conditions, pass through the same point, is that by reductio ad absurdum. This will for the most part require the converse theorem to be first proved or taken for granted.

The converse theorem in this instance is, "If two perpendiculars drawn from two angles of a triangle upon the opposite sides, intersect in a point, the line drawn from the third angle through this point will be perpendicular to the third side." Theorem 21, supra, is the same under a modified form of expression.

The proof will be formally thus, taking the same figure as in Theorem 21 supra. Let EHD be the triangle, AC, BD two perpendiculars intersecting in F. If the third perpendicular EG do not pass through F, let it take some other position as EH; and through F draw EFG to meet AD in G. Then it has been proved that EG is perpendicular to AD: whence the two angles EHG, EGH of the triangle EGH are equal to two right angles:—which is absurd.

30. The truth of this appears at once from Euc. III. 21.

31. Since all the triangles are on the same base and have equal vertical angles, these angles are in the same segment of a given circle. The lines bisecting the vertical angles may be shewn to pass through the extremity of that diameter which bisects the base.

32. This is the converse of the preceding Theorem.

33. It can be shewn that of all triangles on the same base and between the same parallels, the isosceles triangle has the least perimeter. The equilateral triangle, being also isosceles, may be shewn to be greater than any other isosceles triangle of the same perimeter, and hence of all triangles with equal perimeters, the equilateral has the greatest area.

34. Apply Euc. III. 31.

35. Let AC be the common base of the triangles, ABC the isosceles triangle, and ADC any other triangle on the same base AC and between the same parallels AC, BD. Describe a circle about ABC, and let it cut AD in E and join EC. Then, Euc. I. 17, III. 21.

36. Let two lines AP, BP be drawn from the given points A, B, making equal angles with the tangent to the circle at the point of contact P, take any other point Q in the convex circumference, and join QA, QB: then by Prob. 1, p. 293, and Euc. I. 21.

37. Let DKE, DBO (fig. Euc. III. 8) be two lines equally inclined to DA, then KE may be proved to be equal to BO, and the segments cut off by equal straight lines in the same circle, as well as in equal circles are equal to one another.

38. Produce the radii to meet the circumference. See Theorem 27, p. 321.

39. Let F be any point in the diameter AD of a circle whose centre is E (fig. Euc. III. 7) and let HFK, KFL, LFM, &c. be equal angles at F, then the arc HK is less than KL, KL less than LM, &c. Join HK, KL, LM, &c. and prove HK less than KL. Take FN equal to FH, and join KN; KN is equal to KH. Produce KF to meet the circumference in P, and join LP, HP, LH. Then the angle KNL may be proved to be greater than KLN.

40. Join the point of intersection with the centre of the circle and let fall from the centre perpendiculars upon the chords.

41. The diagram of Euc. III. 7, suggests the method of proof.

42. See Theorem 16, p. 321.

43. If BE intersect DF in K (fig. Euc. III. 37). Join FB, FE, then by means of the triangles, BE is shewn to be bisected in K at right angles.

44. The angle between the chord BE and the diameter BFM may be shewn equal to the angle between the tangent BD and the line DF drawn from D through the centre of the circle. (fig. Euc. III. 37.)

45. Let AB, CD be any two diameters of a circle, O the centre, and let the tangents at their extremities form the quadrilateral figure EFGH. Join EO, OF, then EO and OF may be proved to be in the same straight line, and similarly HO, OK.

NOTE.—This Proposition is equally true if AB, CD be any two chords whatever. It then becomes equivalent to the following proposition:—The diagonals of the circumscribed and inscribed quadrilaterals, intersect in the same point, the points of contact of the former being the angles of the latter figure.

46. Let the chord AB, of which P is its middle point, be produced both ways to C, D, so that AC is equal to BD. From C, D, draw the tangents to the circle forming the tangential quadrilateral CKDR, the points of contact of the sides, being E, H, F, G. Let O be the centre of the circle. Join EH, GF, CO, GO, FO, DO.

Then EH and GF may be proved each parallel to CD, they are therefore parallel to one another. Whence is proved that both EF and DG bisect AB.

47. Let ABC be the *isosceles* triangle which fulfils the conditions; take any point F in the base BC, from which draw FH and FG parallel to AB and AC; make GD equal to GA and draw DFE. Then since the side AG of the triangle ADE is bisected in G, and GF is parallel to AE, the base DE is bisected in F. Again, since GF is equal to AH, and FH to AG; and since BGF, HFC, are isosceles triangles; it follows that $AG + AH = AG + GB = AB$. Wherefore $AD + AE = 2 . AG + 2 . AH = 2 . AB = BA + AC$. The variable triangle ADE therefore has its vertical angle and the sum of its sides constant, and the middle of its base is in the line BC.

48. Let the tangent AB touch the circle in C, and let CD be drawn from C perpendicular on any diameter EF, and let the perpendiculars from E and F meet the tangent in B and A respectively. Join CE, CF. Then the angles BCE, DCE, may be shewn each equal to the angle CFE; and by means of the triangles BCE,

DCE, BE may be shewn equal to ED, and in a similar way FA may be shewn equal to FD.

49. The line drawn from the point of intersection of the two lines to the centre of the given circle may be shewn to be constant, and the centre of the given circle is a fixed point.

50. Let AD, DF be two lines at right angles to each other; O, the centre of the circle BFQ; A, any point in AD from which tangents AB, AC are drawn; then the chord BC shall always cut FD in the same point P, wherever the point A is taken in AD. Join AP. Then BAC is an isosceles triangle: and

$$FD.DE + AD^2 = AB^2 = BP.PC + AP^2 = BP.PC + AD^2 + DP^2.$$

$$\text{Or again, } BP.PC = FD.DE - DP^2.$$

The point P, therefore, is independent of the position of the point A; and is consequently the same for all positions of A in the line AD.

51. Let C be the point without the circle from which the tangents CA, CB are drawn, and let DE be any diameter, also let AE, BD be joined, intersecting in P, then if CP be joined and produced to meet DE in G: CG is perpendicular to DE. Join DA, EB and produce them to meet in F.

Then the angles DAE, EBD being angles in a semicircle, are right angles; or DB, EA are drawn perpendicular to the sides of the triangle DEF: whence GPCF is perpendicular to the third side DE. See Theorem 29, p. 321.

52. Let AB, AC be drawn from A and touch the circle in B, C; let AB be perpendicular to the diameter BD, and CE perpendicular to BD, also let AD intersect CE in F, then CE is bisected in F. Join DC and produce it to meet BA produced in G. DG may be shewn to be equal to AD, and EF to FB by means of similar triangles. Euc. VI. 4.

53. This is the same as Theorem 44.

54. Let the radius BC produced meet the circumference of the quadrantal arc when continued in F, and join FE, CD, BE. Then FE is parallel to CD, and the angles DEB, EBD may be each shewn to be equal to half a right angle.

Each of the tangents to the larger circle at A and B makes with AB an angle equal to half a right angle, it follows that if AD make with AB an angle *less* than half a right angle, AD must cut the arc of the quadrant.

55. By constructing the diagram in accordance with the directions given, it will be found that two circles described from the centres B, H, and with radius BH, *do not intersect each other* in the centre of the circle ABF: it would hence appear that there is some inaccuracy in the terms or letters of the enunciation.

56. Let AB be the diameter of the given circle of which the centre is C, and E the bisection of any chord AD. Join EC, then the angle AEC may be proved to be always a right angle in whatever position the chord AD may be situated.

57. Join AD, and the first equality follows directly from Euc. III. 20, I. 32. Also by joining AC, the second equality may be proved in a similar way. If however the line AD do not fall on the same side of the centre O as E, it will be found that the *difference*, not the *sum* of the two angles, is equal to 2.AED. See note to Euc. III. 20, p. 108.

58. This problem cannot be constructed by the line and circle, as the Algebraical equation which arises for finding D is of the third degree.

59. See Theorem 85, p. 325.

60. Complete the circle whose segment is ADB; AHB being the other part. Then since the angle ACB is constant, being in a given segment, the sum of the arcs DE and AHB is constant. But AHB is given, hence ED is also given and therefore constant.

61. Let A, B, be the centres, EF the tangent intersecting the line AB joining the centres, and CD the other tangent. Draw the radii AC, AE, BD, BF to the points of contact; and draw BG parallel to DC meeting AC in G; and BH parallel to FE meeting AE produced in H. Then $BG = CD$, $BH = EF$, $AG = AC - BD$, and $AH = AE + BF = AC + BD$: and $CD^2 - EF^2$ may be proved to be equal to $4.AC.BD$, or the rectangle of the diameters of the circles.

62. Let the segments AHB, AKC be externally described on the given lines AB, AC, to contain angles equal to BAC. Then by the converse to Euc. III. 32, AB touches the circle AKC, and AC the circle AHB.

63. See Theorem 82, p. 325.

64. Let A, B, be the centres, and C the point of contact of the two circles; D, E the points of contact of the circles with the common tangent DE, and CF a tangent common to the two circles at C, meeting DF in E. Join DC, CE. Then DF, FC, FE may be shewn to be equal, and FC to be at right angles to AB.

65. The possibility is obvious, and the centre of the required circle will be found to be the point of intersection of two circles described from the centres of the given circles with their radii increased by the radius of the required circle.

66. This may be directly shewn from Euc. III. 36, 37.

67. See Theorem 67, p. 351.

68. This is the same as Theorem 26, p. 321: repeated by mistake.

69. This follows directly from Euc. III. 36.

70. The line drawn through the point of contact of the two circles parallel to the line which joins their centres, may be shewn to be double of the line which joins the centres, and greater than any other straight line drawn through the same point and terminated by the circumferences. The greatest line is therefore dependent on the distances between the centres of the two circles.

71. A repetition of Theorem 7, p. 320.

72. This is the same as Theorem 60, p. 324, under another form of expression.

73. This is at once obvious from Euc. III. 36.

74. Each of the lines CE, DF may be proved parallel to the common chord AB.

75. This may be proved by shewing that the line joining the centres, bisects the angles at the centres which are contained by radii drawn to the points of intersection of the circles.

76. By constructing the figure and applying Euc. I. 8, 4, the truth is manifest.

If two diameters from one of the points of intersection be drawn in both circles, and the other extremities of them be joined with the other intersection of the circles: then these two lines are in the same straight line.

77. The third circle must be defined as that whose radius is equal to the diameter of either of the equal circles; or else that *touch* must be used instead of *cut*, in the enunciation. For in that case only will the angles at the other point of intersection be right angles.

78. Let E be that point in the circumference of one circle which is the centre of the other. Join CE and produce it to meet the circumference in F. Join also FA, EA, OA. The triangles FEA, OHA may be proved to be equiangular.

79. Let the two circles touch one another in the point C, and let A, B be any two points in the circumference of the interior circle, and D any point in the circumference of the exterior circle; the angle ACB is greater than the angle ADB. Let AD intersect the interior circumference in E and join EB.

80. Let perpendiculars from the centre of the larger circle be drawn on those straight lines, then Euc. III. 15.

81. Let the line which joins the centres of the two circles be produced to meet the circumferences, and let the extremities of this line and any other line from the point of contact be joined. From the centre of the larger circle draw perpendiculars on the sides of the right-angled triangle inscribed within it. In the enunciation of the Theorem, for *externally* read *internally*.

82. This is only a slight variation of Theorem 63, p. 324.

83. The sum of the distances of the centre of the third circle from the centres of the two given circles, is equal to the sum of the radii of the given circles, which is constant.

84. This may be shewn to follow directly from Euc. III. 36, and I. 47.

85. Let the circles touch at C either externally or internally, and their diameters AC, BC through the point of contact will either coincide or be in the same straight line. DCE any line through C will cut off similar segments from the two circles. For joining AD, BE, the angles in the segments DAC, EBC are proved to be equal. The remaining segments are also similar, since they contain angles which are supplementary to the angles DAC, EBC.

86. In the enunciation, read "are *not* similar" instead of "are similar." The lines joining the common centre and the extremities of the chords of the circles, may be shewn to contain unequal angles, and the angles at the centres of the circles are dou-

ble the angles at the circumferences, it follows that the segments containing these unequal angles are not similar.

87. Let AB, AC be the straight lines drawn from A, a point in the outer circle to touch the inner circle in the points D, E, and meet the outer circle again at B, C. Join BC, DE. Prove BC double of DE.

Let O be the centre, and draw the common diameter AOG intersecting BC in F, and join EF. Then the figure DBFE may be proved to be a parallelogram.

88. Let A, B, C, be the centres of the three equal circles, and let them intersect each other in the point D: and let the circles whose centres are A, B intersect each other again in E; the circles whose centres are B, C in F; and the circles whose centres are C, A in G. Then FG is perpendicular to DE; DG to FC; and DF to GE. Since the circles are equal, and all pass through the same point D, the centres A, B, C are in a circle about D whose radius is the same as the radius of the given circles. Join AB, BC, CA; then these will be perpendicular to the chords DE, DF, DG. Again, the figures DAGC, DBFC, are equilateral, and hence FG is parallel to AB; that is, perpendicular to DE. Similarly for the other two cases.

89. This is true not only for three circles, but for all circles.

Let A be the point of contact, C, the centre of any circle in the line AP, and B the given point from which the tangents are to be drawn. Join BA and make the angle ABF equal to BAF, and produce BF till FG be equal to FB. The chord of contact DE will always pass through G. For join BC cutting DE in H: then BHG is a right angle. Also BF, FA, FG are equal to one another, and A is in a semicircle on BG, and AG being joined, BAG is a right angle. The angles BHG, BAG, therefore are right angles in the same semicircle; and hence HE always passes through G, the extremity of the hypothenuse of the triangle BAG.

90. Let E be the centre of the circle which touches the two equal circles whose centres are A, B. Join AE, BE which pass through the points of contact F, G. Whence AE is equal to EB. Also CD the common chord bisects AB at right angles, and therefore the perpendicular from E on AB coincides with CD.

91. Let the three chords be AB, AC, AD, the middle points of which are b, c, d; let E, F, G be the intersections of the circles on AB and AC, on AB and AD, and on AC and AD respectively; join BC, CD, DB, bc, cd, db, AE, AF, AG; and let m be the intersection of bc, AE. The proof may be made to depend on the following theorem;—If a circle be inscribed in a triangle and perpendiculars be drawn upon the sides from any point in the circumference; the three points of intersection are in the same straight line.

92. By joining the points of intersection of the circles, the foursided figure so formed may be shewn to have its opposite sides equal and its angles right angles. The diagonals may easily be shewn to cut one another at the centre of the middle circle.

93. Let three circles touch each other at the point A, and from A, let a line ABCD be drawn cutting the circumferences in B, C, D. Let O, O′, O″ be the centres of the circles, join BO, CO′, DO″, these lines are parallel to one another. Euc. I. 5, 28.

94. Let the fixed circle CDE be cut in C, D by any circle whatever passing through the fixed points A, B: draw CD to meet BA produced in F. Then BF.FA = DF.FC; and hence F is independent of the magnitude of the circle ACDB; and is consequently the same for all, that is, all the chords pass through the same point F.

95. This is similar to the last, except that the rectangle AF.FB is exchanged for the square of the tangent.

96. Let any number of circles touch each other internally at the point A, and let a common tangent be drawn at A. With any point B in the tangent and any radius, describe a circle cutting the given circles in C, D, E, &c. Join BC, BD, BE, &c., and produce them to meet the circles again in C′, D′, E′, &c. Then CC′, DD′, EE′, &c., may be shewn to be equal to one another by Euc. III. 36.

97. The construction of the figure suggests a reference to Theorem 22, p. 309.

98. Let AB be a chord parallel to the diameter FG of the circle, fig. Theo. 1, p. 235, and H any point in the diameter. Let HA and HB be joined. Bisect FG in O, draw OL perpendicular to FG cutting AB in K, and join HK, HL, OA. Then the square of HA and HF may be proved equal to the squares of FH, HG by Theo. 20, p. 233; Euc. I. 47; Euc. II. 9.

99. Let the chords AB, CD intersect each other in E at right angles. Find F the centre, and draw the diameters HEFG, AFK and join AC, CK, BD. Then by Euc. II. 4, 5; III. 35.

100. Let ABCD be any quadrilateral figure, AC, BD the diagonals, F, G, their points of bisection, E the point of bisection of FG. Let P be any point in the circumference of a circle described from centre E. Join PF, PE, PG, PA, PB, PC, PD, EA, EB, EC, ED. Then by Theorem 22, p. 309.

101. Let the figure be constructed and join DB, DG. Then DABG is a rectangle, and FG is equal to CD, both being diameters of the circle, and by Euc. I. 47; II. 12, 3; III. 36, the property may be proved.

In the enunciation read, "or their lines produced in H, G."

102. This is only another form of stating Theorem, 103.

103. This is manifest from Euc. III. 35.

104. Let ACB be the given acute angle at C the centre of the circle, which is subtended by the arc AB, and suppose ACD to be one third of the angle ACB. Through B draw BEF parallel to CD and meeting the circumference in E and the radius AC produced in F. Join CE, then FE may be shewn equal to the radius CE. If therefore from B the line BEF, &c.

105. Let Q be the centre of the circles. Join QO, QB. Then QOB is a right-angled triangle. Also ON may be proved equal to OM by Theorem 18, p. 321. And $4 . CN . NF + MO^2$ may be shewn to be equal to the difference of the squares of the radii of the circles by Euc. III. 35; II. 6; I. 47.

106. Let the figure be constructed, and the truth is obvious from Euc. I. 41.

107. Let E, F be the points in the diameter AB equidistant from the centre O; CED any chord; draw OG perpendicular to CED, and join FG, OC. The sum of the squares of DF and FC may be shewn to be equal to twice the square of FE and the rectangle contained by AE, EB, by Euc. I. 47; II. 5; III. 35.

108. Let the chords AB, AC be drawn from the point A, and let a chord FG parallel to the tangent at A be drawn intersecting the chords AB, AC in D and E, and join BC. Then the opposite angles of the quadrilateral BDEC are equal to two right angles, and a circle would circumscribe the figure. Hence by Euc. I. 36.

109. Let the line drawn from A touch the circumference in P, and from B, C, let lines be drawn to the points where the circle intersects the two sides of the triangle. Then by Euc. III. 31; II. 13; III. 36.

110. Let QOP cut the diameter AB in O. From C the centre draw CH perpendicular to QP. Then CH is equal to OH, and by Euc. II. 9, the squares of PO, OQ are readily shewn to be equal to twice the square of CP.

111. From P draw PQ perpendicular on AB meeting it in Q. Join AC, CD, DB. Then circles would circumscribe the quadrilaterals ACPQ and BDPQ, and then by Euc. III. 36.

112. In the enunciation for "centre" read "circumference;" and for KG^2 read AH^2. Draw AGDK, AE, AC, AH, EK, KC, CG, CE, and let M be the intersection of AK, EC. Then AEKC is a rhombus, and $MK = MA$: also $GM = MD$. Whence $GK = AD = 2 . AG$, and $AK = 3 . AG$; wherefore $AK . AG = 3 . AG^2$. Also $AH^2 = AC^2 = AG^2 + GC^2 + 2 . AG . GM = 3 . GM^2$. Hence $AK . AG = AH^2$.

113. Here A is the extremity of the diameter and C the centre of the larger circle, the perpendicular BDE meets AC in E, the smaller circle in D and the larger in B. From the right-angled triangles the truth of the property may be shewn.

114. Describe the figure according to the enunciation; draw AE the diameter of the circle, and let P be the intersection of the diagonals of the parallelogram. Draw EB, EP, EC, EF, EG, EH. Since AE is a diameter of the circle, the angles at F, G, H are right angles, and EF, EG, EH are perpendiculars from the vertex upon the bases of the triangles EAB, EAC, EAP. Whence by Euc. II. 13, and Theorem 22, page 309, the truth of the property may be shewn.

115. Let AC, BC be produced to meet the circumference in A′, B′: produce also QRr to meet the circumference in q. Through r draw a line perpendicular to Qq and meeting the circumference in S, s, and join Sq, then Srq may be shewn to be a right-angled triangle having the sides Sq, Sr, qr respectively equal to AB, Qr, QR.

116. If AQ, A′P′ be produced to meet, these lines with AA′ form a right-angled triangle.

117. This Theorem is the same as 119, p. 361.

118. Let the tangents TP, TQ be drawn from any point T in the perpendicular CT to meet the circles in P, Q respectively. Join AP, PQ, then by Euc. I. 47. The square of PT may be shewn to be equal to the square of QT.

119. This theorem requires the aid of proportion. The locus of the point D which fulfils the condition is a circle (Theorem 17, p. 353.) whose diameter is found by drawing the common tangent PQ to meet ACB produced in K. Let AB meet the circles in L and M, and DC meet them in E and F; and join FL, EO, DK. Then since EO, FL are parallel to DK the base of the triangle CDK, and by compounding the two proportions deduced from Euc. VI. 2: we have

$$ED.DF : LK.KM :: CD^2 : CK^2,$$

and it may be shewn that $LK.KM = CK^2$;

$$\text{hence, } ED.DF = DC^2, \text{ or, } ED : DC :: DC : DF;$$

$$\text{whence, } ED.DC : DC^2 :: DC^2 : DC.DF;$$

$$\text{or, } DH^2 : DC^2 :: DC^2 : DG^2;$$

$$\text{and } DH : DC :: DC : DG; \text{ or } DH.DG = DC^2.$$

120. Let the figure be constructed, and let the perpendicular from AG on the diameter be greater than the perpendicular BH. Take O the centre of the circle, join CO, and draw BK perpendicular to AG. Then the triangles ABK, OCF being equiangular; AB is to BK or GH as OC is to CF. But DE is equal to twice OC, and CT is twice CF. Hence AB is to GH as DE is to CF; and therefore the rectangle contained by AB, CF is equal to that contained by GH, DE. This Theorem is misplaced, as it involves proportion.

121. Let AD meet the circle in G, H, and join BG, GC. Then BGC is a right-angled triangle and GD is perpendicular to the hypothenuse, and the rectangles may be each shewn to be equal to the square of BG. Euc. III. 35; II. 5; I. 47. Or, if EC be joined, the quadrilateral figure ADCE may be circumscribed by a circle. Euc. III. 31, 22, 36, Cor.

122. Join EC, ED, FG, FH, then by means of the similar triangles, two proportions may be found from which it is shewn, that the first rectangle is equal to the second, and the second to the third.

123. Let ADBC be the inscribed quadrilateral; let AC, BD produced meet in O, and AB, CD produced meet in P, also let the tangents from O, P meet the circles in K, H respectively. Join OP, and about the triangle PAC describe a circle cutting PO in G and join AG. Then A, B, G, O may be shewn to be points in the circumference of a circle. Whence the sum of the squares of OH and PK may be found by Euc. III. 36, and shewn to be equal to the square of OP.

124. This Theorem is only a different form of stating Theorem 119, p. 361. The consideration of it may be deferred, as it properly involves the idea of Harmonic proportionals.

125. This is an extension of Theorem 4, p. 314, where the chords intersect each other outside the circle. The truth of it may be readily shewn by drawing perpendiculars on the chords from the centre; and by Euc. II. 9; I. 47.

126. This Theorem involves the property:—that the circumferences of circles are proportional to their radii or diameters. In general, the locus of a point in the circumference of a circle which rolls within the circumference of another, is a curve called the *Hypocycloid;* but to this there is one exception, in which the radius of one of the circles is double that of the other: in this case, the locus is a straight line, as may be easily shewn from the figure.

127. Let the diagram be described as in the enunciation, and let CS meet the circle at A (between C and S); draw the tangent at A, and in it take $AE = AE' = AS$; on CD describe a circle cutting the given one in B, B'; join BB' meeting CS in D; draw DE, DE'. Then these lines will be the loci of the point Q.

Analysis. Suppose it true that $QM = Sy$, Q being taken in the line DE. Then draw CP and produce it to meet the circle on SC in N, and join NS.

Since Sy and NP are perpendicular to Py, they are parallel, and since CNS is in a semicircle, it is a right angle. Whence Ny is a rectangle, and $NP = Sy$.

Again, by the similar triangles EAD, QMD, we have

$$QM.AD = EA.MD, \text{ or } QM(AC - CD) = (CS - CA)(CM - CD).$$

Also by the similar triangles CMP, CNS, we have

$CM.CS = CP.CN = CP^2 + CP.PN = CA^2 + CA.Sy$, or $CA.Sy = CM.CS - CA^2$.

Now since the proposition is assumed to be true, namely, that $QM = Sy$, a comparison of their values gives

$$CA : CA - CD :: CM.CS - CA^2 : (CS - CA)(CM - CD),$$

or $CD : CA :: CM.CS - CA^2 - CS.CM + CS.CD + CA.CM - CA.CD : CM.CS - CA^2$;

and since $CD : CA :: CA : CS$, and $CS.CD = CA^2$;

this becomes $CA : CS :: CA.DM : CM.CS - CA^2$;

Whence $CS.DM = CM.CS - CA^2$, or $CA^2 = CS(CM - DM) = CS.SD$, a known truth. Whence the Theorem is true.

128. Let PAB, PDC be the straight lines from P cutting the circle in A, B, C, D. Join PO, and about the triangle PAC describe a circle cutting PO in G, and join AG. Draw the diameter through P, and the centre Q, and through O a perpendicular to PQ cutting it in R, and the circle in H, K; draw HP, KP. Then it may be shewn that the points A, B, G, O, are in the circumference of a circle, also HK is bisected in R. Next by Euc. III. 36, 34, 37; II. 5; I. 47, HP^2 may be proved to be equal to CP.PD; or HP is a tangent drawn from P to the circle. In the same way, PK is a tangent. Whence O is situated in the chord which joins the points of contact of tangents drawn from P to the circle.

129. First. Since the angle PCQ and base PQ of the triangle PCQ are constant, the circle about PCQ is constant in magnitude, and consequently in diameter. Also since the angles PCQ, PRQ are supplementary, R is in the circumference of the circle PCQ. But RQC, RPC being right angles, CR is a diameter; and it has been proved to be of constant magnitude. Wherefore R is always at the same distance from C. Secondly. Draw RS, SC; then PQ is bisected in L, since PSQR is a parallelogram. Also bisect RC in K and join KL. Then since RC is the diameter of a circle given in magnitude, and PQ a given chord in it, the line KL is of constant magnitude. Moreover, since RS, RC are bisected in L and K, CS is equal to twice LK a given line; and the locus of S is a circle.

130. The locus may be shewn to be the circumference of the circle described on the base of the triangle as a diameter.

HINTS, &c. TO THE PROBLEMS. BOOK IV.

4. Draw through the centre a diameter parallel to the given line, and from its extremities, draw two lines perpendicular on the diameter.

5. This is a more general form of Problem 50, p. 217.

6. Place in the circle a straight line AB equal to the given line. Through the centre O draw a line OC perpendicular to AB, and with centre O and radius OC, describe a circle, and through P the given point, draw a straight line touching this circle. Hence, chords in one circle, which are also tangents to another concentric circle, are bisected at the points of contact.

7. Trisect the circumference and join the centre with the points of trisection.

8. Let AB, CD be two diameters intersecting each other in O at right angles, and suppose EFGH the straight line required which is trisected in the points F, G. At F in AF make the angle AFK equal to the angle AFE and join KH; then KF is equal to EF, KFH is a right-angled triangle of which the base KH is the chord of a quadrant. The semicircle described on KH as a diameter touches the diameter AB of the given circle in F, also the point O is given. Hence the construction.

9. If three lines be drawn within the triangle from the angular points, making equal angles with the respective sides of the triangle, an equilateral triangle is formed by the intersection of these lines, except when each line makes equal angles with *two* sides of the triangle: and the smaller the angles are which are contained between the lines and the sides of the triangle, the greater will be the area of the new triangle, which cannot exceed the area of the given triangle. If however the lines are not

required to be drawn within the triangle, the greatest triangle will be that whose sides are parallel to the sides of the given triangle.

10. This may be effected by Euc. IV. 10, 3.

11. From the vertex of the isosceles triangle let fall a perpendicular on the base. Then, in each of the triangles so formed, inscribe a circle, Euc. IV. 4; next inscribe a circle so as to touch the two circles and the two equal sides of the triangle. This gives one solution: the problem is indeterminate.

12. The meaning of this problem does not appear very obvious.

13. Let AB be the base of the given segment, C its middle point. Let DCE be the required triangle. From C draw CF perpendicular to the base DE, and make CH equal to the given line. Join HD and produce it to meet AB produced in K. Then FK is double of DF, and CH double of CF. Draw DL perpendicular to CK.

14. The first part is another form of stating Euc. IV. 5. In order that a circle may pass through four points, the condition may be deduced from Euc. III. 22, 35 or 36, Cor.

15. Apply Euc. IV. 5.

16. Let P, Q, R, be the three given points. Draw PR and take PR.PD = PA.PB; draw QD, and take QD.QE = QB.QC; through E, apply the chord CF to subtend the angle PDQ: C is an angular point in the circumference.

If the three points are in the same straight line, take PQ.PG = PA.PB, and RG.GH = GA.GV: draw the tangent HV, then VC, parallel to PQ, determines C an angular point. Swale's *Apollonius*, p. 48.

17. From the given angle draw a line through the centre of the circle, and at the point where the line intersects the circumference, draw a tangent to the circle, meeting two sides of the triangle. The circle inscribed within this triangle will be the circle required.

18. If four points successively be taken in the sides at equal distances from the angles, the lines joining these points will form a square. When the four points coincide with the bisections of the sides of the given square, the area of the inscribed square is a minimum.

19. Let the diagonals of the rhombus be drawn; the centre of the inscribed circle may be shewn to be the point of their intersection.

20. On the diameter AB describe a rectangle equal to the given rectilineal figure, and let the side parallel to AB meet the circumference in E. Join AE, EB, through A draw AF parallel to BE and join BF.

21. The greatest quadrilateral in a circle is a square. By reference to Euc. III. 22, and Theo. 21, p. 252, it will appear when a circle can be inscribed in, and circumscribed about a quadrilateral figure.

22. Bisect the angle contained by the two lines at the point where the bisecting line meets the circumference, draw a tangent to the circle and produce the two straight lines to meet it. In this triangle inscribe a circle.

23. Join the given point and the centre of the circle, and at the point where the circumference is cut, draw a tangent meeting the two other tangents; the circle inscribed in the triangle thus formed, will be the circle required.

24. If ABCD be the required square. Join O, O′ the centres of the circles and draw the diagonal AEC cutting OO′ in E. Then E is the middle point of OO′ and the angle AEO is half a right angle.

25. The line AB joining the points of contact is bisected in D by the line CC′ joining the centres of the circles. If DE, DF be taken each equal to half the given side, the contruction is obvious.

26. Let A, B, C be the three points, join AB, AC, BC.

The point required, will be found to be that point in which three circles circumscribing the equilateral triangles on AB, AC, BC, meet within the triangle.

27. First shew the possibility of a circle circumscribing such a figure, and then determine the centre of the circle.

28. The centre of the circumscribing circle is determined by the intersection of the two lines which bisect the angles adjacent to any side of the quadrilateral figure.

29. Let ABC be a triangle, having C a right angle, and upon AC, BC let semicircles be described: bisect the hypothenuse in D, and let fall DE, DF perpendiculars on AC, BC respectively, and produce them to meet the circumferences of the semicircles in P, Q; then DP may be proved to be equal to DQ.

30. Let O, O′ be the centres of the semicircles on AC and CB.

At C draw CD perpendicular to AB meeting the circumference in D. Produce CD and make DE equal to the radius of either of the smaller semicircles. Join EO, EO′ and let F be the centre of the circle described about the triangle EOO′, join FO, FO′ meeting the circumferences in G, H. Then FD, FG, FH may be proved to be equal to one another.

By Euc. III. 36, twice OG may be shewn to be equal to three times GF.

31. Suppose the centre of the required circle to be found, let fall two perpendiculars from this point upon the radii of the quadrant, and join the centre of the circle with the centre of the quadrant and produce the line to meet the arc of the quadrant. If three tangents be drawn at the three points thus determined in the two semicircles and the arc of the quadrant, they form a right-angled triangle which circumscribes the required circle.

32. Suppose the parallelogram to be rectangular and inscribed in the given triangle and to be equal in area to half the triangle: it may be shewn that the parallelogram is equal to half the altitude of the triangle, and that there is a restriction to the magnitude of the angle which two adjacent sides of the parallelogram make with one another.

33. Let ABC be the given triangle, and A′B′C′ the other triangle, to the sides of which the inscribed triangle is required to be parallel. Through any point *a* in AB draw *ab* parallel to A′B′ one side of the given triangle and through *a*, *b* draw *ac*, *bc* respectively parallel to AC, BC. Join A*c* and produce it to meet BC in D; through D draw DE, DF, parallel to *ca*, *cb*, respectively, and join EF. Then DEF is the triangle required.

34. (1) Let ABCD be the given square: join AC, at A in AC, make the angles CAE, CAF, each equal to one third of a right angle, and join EF.

(2) Bisect AB any side in P, and draw PQ parallel to AD or BC, then at P make the angles as in the former case.

35. Let ABCD be the given square. With centre A and radius AB describe a circle, which will pass through D: take DE equal to DB. With centres B, E and radius BE describe two arcs cutting each other in F. With centre D and radius DF describe a circle cutting the sides AD, DC of the square in G, H. Then B, G, H, are the angular points of the inscribed equilateral triangle.

36. Join AB, and from A draw AT to touch the circle in T. Divide AB in C so that the rectangle contained by AB, AC shall be equal to the square of AT. Through C draw CP to touch the circle in P, join AP and produce it to meet the circumference in X. Draw XB intersecting the circumference in Q and join PQ. PQ is parallel to AB. The proof requires Euc. VI. 6. How is the construction to be effected if the two points A, B be given within the circle?

37. Let AB be the two given points and C the centre of the given circle. Join AB and describe a circle passing through A, B and touching the circle whose centre is C in the point D, join AD, BD: the angle ADB is greater than the angle AEB, subtended by AB at any other point E in the circumference of the circle whose centre is C.

38. The point P will be found to be that point where a circle passing through the extremities of the given line touches the given circle.

39. The point required will be found to be that point at which the line drawn bisecting the radius is perpendicular to it.

40. The point D may be shewn to be that point in the circumference of the circle which passes through the points A, B and touches the line CD in the point D.

41. The point required is the centre of the circle which circumscribes the triangle. See the notes on Euc. III. 20, p. 108.

42. If the perpendiculars meet the three sides of the triangle, the point is within the triangle, Euc. IV. 4. If the perpendiculars meet the base and the two sides produced, the point is the centre of the *escribed* circle.

43. In general three straight lines when produced will meet and form a triangle, except when all three are parallel or two parallel are intersected by the third. This Problem includes Euc. IV. 5 and all the cases which arise from producing the sides of the triangle. The circles described touching a side of a triangle and the other two sides produced, are called the *escribed* circles.

44. Let the two given lines AB, CD when produced meet in E, and let F be the given point. Bisect the angle AEC by EG and through F draw FG perpendicular to EG and produce it both ways to meet AB in B, and CD in D. Take GH equal to GF, and on BA make BK such that the square of BK is equal to the rectangle contained by BH, BF. The circle described through the points K, F, H shall touch the lines AB, CD. If the lines be parallel there is no difficulty in the construction.

45. Let the circle required touch the given circle in P and the given line in Q. Let C be the centre of the given circle and C′ that of the required circle. Join CC′, C′Q, QP; and let QP produced meet the given circle in R, join RC and produce it to meet the given line in V. Then RCV is perpendicular to TQ. Hence the construction.

46. Let A be the centre of the given circle, B the given point in the circumference, and C the other given point. Join AB, BC, and make the angle BCD equal to ABC: then CD meets AB in D, the centre of the required circle.

The given point C may be *inside* or *outside* the given circle, and the contact appears to be always possible.

47. Let A, B be the centres of the given circles and CD the given straight line. On the side of CD opposite to that on which the circles are situated, draw a line EF parallel to CD at a distance equal to the radius of the smaller circle. From A the centre of the larger circle describe a concentric circle GH with radius equal to the difference of the radii of the two circles. Then the centre of the circle touching the circle GH, the line EF, and passing through the centre of the smaller circle B, may be shewn to be the centre of the circle which touches the circles whose centres are A, B, and the line CD.

48. Let AB, CD be the two lines given in position and E the centre of the given circle. Draw two lines FG, HI parallel to AB, CD respectively and external to them. Describe a circle passing through E and touching FG, HI. Join the centres E, O, and with centre O and radius equal to the difference of the radii of these circles describe a circle; this will be the circle required.

49. Let the circle ACF having the centre G, be the required circle touching the given circle whose centre is B, in the point A, and cutting the other given circle in the point C. Join BG, and through A, draw a line perpendicular to BG; then this line is a common tangent to the circles whose centres are B, G. Join AC, GC. Hence the construction.

50. Let A be the given point, BC the given straight line which the circle is to touch, and DE the line in which the centre is to be situated. Let DE be produced to meet BC in C. Join AC and through A draw AB perpendicular to BC, and produce it to meet DEC in D. With centre D and radius DB describe a circle cutting CA, produced if necessary, in two points F, G, or touching it in one. Join FD, and draw AH parallel to FD meeting DE in H. The circle described with centre H and radius HA is the circle required. Draw HK perpendicular to BC, then by similar triangles, HG is proved to be equal to HA.

51. This is a particular case of the general problem; to describe a circle passing through a given point and touching two straight lines given in position.

Let A be the given point between the two given lines which when produced meet in the point B. Bisect the angle at B by BD and through A draw AD perpendicular to BD and produce it to meet the two given lines in C, E. Take DF equal to DA, and on CB take CG such that the rectangle contained by CF, CA is equal to the square of CG. The circle described through the points F, A, G, will be the circle required. Deduce the particular case when the given lines are at right angles to one another, and the given point in the line which bisects the angle at B.

If the lines are parallel, when is the solution possible?

52. Let A, B, be the centres of the given circles, which touch externally in E; and let C be the given point in that whose centre is B. Make CD equal to AE and draw AD; make the angle DAG equal to the angle ADG: then G is the centre of the circle required, and GC its radius.

53. Let C be the given point in the given line AB, and D the centre of the given circle. Through C draw a line CE perpendicular to AB, on the other side of AB, take CE equal to the radius of the given circle. Join ED, and at D make the angle EDF equal to the angle DEC, and produce EC to meet DF in F.

54. If the three points be such as when joined by straight lines a triangle is formed; the points at which the inscribed circle touches the sides of the triangle, are the points at which the three circles touch one another. Euc. IV. 4. There are different cases which arise from the relative position of the three points.

55. Bisect the sides AB, BC, CA, in D, E, F, and join DE, EF, FD; then the circles described about the triangles ADF, BDE, CEF shall pass through the angles A, B, C of the equilateral triangle and shall touch each other.

56. Suppose the triangle constructed, then it may be shewn that the difference between the hypothenuse and the sum of the two sides is equal to the diameter of the inscribed circle.

57. With the given radius of the circumscribed circle, describe a circle. Draw BC cutting off the segment BAC containing an angle equal to the given vertical angle. Bisect BC in D, and draw the diameter EDF: join FB, and with centre F and radius FB describe a circle: this will be the locus of the centres of the inscribed circle (see Theorem 27, p. 339). On DE take DG equal to the given radius of the inscribed circle, and through G draw GH parallel to BC, and meeting the locus of the centres in G. G is the centre of the inscribed circle.

58. This may readily be effected in almost a similar way as the preceding Problem.

59. With the given radius describe a circle, then by Euc. III. 34.

60. Let AD make with a diameter AB (2 R) an angle DAB equal to one-third of a right angle, and let the required circle DHG whose centre is F, and radius $\frac{3}{4}$. R, touch the straight line AD in D and the circle in H. Join FC intersecting the circles at H, and FD, also draw FE parallel to DA, and AE to DF. Then since the circle DHG touches the given line AD (for the angle BAD being given, AD is given), and since its radius FD is given, the locus of the centre is the line EF at the given distance AE from AD. Also since HF and HC are given, CF is given, and C being given, the locus of F is the circle described about C with a radius equal to the sum of the radii HF, HC.

61. Let ABC be a triangle on the given base BC and having its vertical angle A equal to the given angle. Then since the angle at A is constant, A is a point in the arc of a segment of a circle described on BC. Let D be the centre of the circle inscribed in the triangle ABC. Join DA, DB, DC: then the angles at B, C, A, are bisected. Euc. IV. 4. Also since the angles of each of the triangles ABC, DBC are equal to two right angles, it follows that the angle BDC is equal to the angle A and half the sum of the angles B and C. But the sum of the angles B and C can be found because A is given. Hence the angle BDC is known and therefore D is the locus of the vertex of a triangle described on the base BC and having its vertical angle at D double of the angle at A.

62. Let the two circles touch each other in D and the given line in A, B, and let C, C′ be the centres of the circles. Join CA, C′B, C′C, and draw CE parallel to AB meeting AC in E. Hence the construction.

63. Divide the circle into three equal sectors, and draw tangents to the middle points of the arcs, the problem is then reduced to the inscription of a circle in a triangle.

64. The general case of this problem is when the given circles do not *touch* or *intersect* one another. Let A, B, C be the centres of the given circles. With centre B describe a circle with a radius equal to the difference or sum (as the case may require) of the radii of the circles whose centres are A and B: with centre C describe another circle with a radius equal to the difference or sum of the radii of the circles whose centres are A and C. Then the circle described touching these two circles and passing through the point A (Prob. 59, p. 351), will have its centre coincident with the centre of the required circle. Give the analysis of the problem.

65. This is the general case of Problem 63 supra.

66. The problem is the same as to find how many equal circles may be placed round a circle of the same radius, touching this circle and each other. The number is six.

67. This may be effected by Euc. IV. 10.

68. Apply Euc. IV. 10; I. 23; IV. 11.

69. See Problem 27, p. 298.

70. If one of the diagonals be drawn, this line with three sides of the pentagon forms a quadrilateral figure of which three consecutive sides are equal. The problem is reduced to the inscription of a quadrilateral in a square.

71. This may be deduced from Euc. IV. 11.

72. The line AC or BD (fig. Euc. IV. 10) is the side of a regular decagon inscribed in the circle. See the note, Euc. II. 11, p. 72.

73. Each side of the inscribed equilateral triangle subtends an arc equal to one third of the circumference. Hence the method is at once obvious.

74. Let the areas of the inscribed and circumscribed hexagons be expressed in terms of the radius of the circle.

75. The alternate sides of the hexagon will fall upon the sides of the triangle, and each side will be found to be equal to one third of the side of the equilateral triangle.

76. This construction may be effected in two different ways.

(1) When four sides of the hexagon fall upon the sides of the square. If AC be a diagonal of the square ABCD; a line EF may be drawn parallel to AC, by problem 43, p. 298, such that AE, EF, FC shall be equal to one another.

(2) When only two of the sides of the hexagon fall on the sides of the square. Bisect the opposite sides AB, CD in E, F; the problem is then reduced to that of drawing two lines from E, F to meet BC in G, H, such that EG, GH, HF shall be equal to one another.

77. A regular duodecagon may be inscribed in a circle by means of the equilateral triangle and square, or by means of the hexagon. If r be the radius of the circle, the area of the duodecagon is $3r^2$, which is the square of the side of an equilateral triangle inscribed in the same circle. Theorem 1, p. 332.

78. By Euc. I. 47, the perpendicular distance from the centre of the circle upon the side of the inscribed hexagon may be found. The comparison of the areas of the two figures may be made by comparing the sums of the areas of the triangles by which they are respectively formed, by drawing lines from the angular points of the figures to the centre of the circle which circumscribes the figures.

79. If the pentagon be equilateral and equiangular, the problem is impossible; it is however possible to inscribe a regular hexagon in an irregular pentagon, when the pentagon admits of an inscribed circle, and the points of contact are five of the points of the inscribed hexagon.

80. Each of the interior angles of a regular octagon may be shewn to be equal to three-fourths of two right angles, and the exterior angles made by producing the sides, are each equal to one-fourth of two right angles, or one half of a right angle.

81. This is found from the inscribed square.

82. If the alternate sides of the octagon be produced to meet one another, the figure thus formed is a square, and the area of the octagon may be shewn to be the difference between the area of the square, and twice the square of the side of the octagon.

83. Let the areas of the inscribed hexagon and the circumscribed octagon be expressed in terms of the radius of the circle.

84. The pentagon may be transformed into a square, and then the problem is to describe a regular octagon equal in area to a given square.

85. The value of the interior angle of any regular figure may be found by means of the note on Euc. IV. 16, p. 125.

86. If the least angle be denoted by θ, the other angles are $\theta + 10$, $\theta + 20$, &c. degrees; by applying the expression for the sum of an arithmetical series, and note p. 125, θ will be found to be 99°.

87. Let n be the number of sides; then the sum of the interior angles of the figure may be found by finding the sum of an arithmetic progression of n terms, whose first term is 120°, and common difference 5°: and by note on Euc. IV. 16, p. 125, the number of sides will be found to be 16 and 9. Construct the two figures, and shew that one of them contains re-entrant angles.

88. Proceed as in Problem 87.

89. Proceed as in Problem 87.

90. Proceed as in Problem 87.

91. The number of sides may be found by the note on Euc. IV. 16, p. 125.

92. By means of the note p. 125, the figure may be shewn to be a regular nonagon.

93. The same method as in Euc. IV. 14, may be employed for determining the centre of the circle which will circumscribe any regular polygon.

94. This may be effected by the same construction as Problem 27, p. 298.

95. Every regular polygon can be divided into equal isosceles triangles by drawing lines from the centre of the inscribed or circumscribed circle to the angular points of the figure, and the number of triangles will be equal to the number of sides of the polygon. If a perpendicular FG be let fall from F (figure Euc. IV. 14,) the centre on the base CD of FCD, one of these triangles, and if GF be produced to H till FH be equal to FG, and HC, HD be joined, an isosceles triangle is formed, such that the angle at H is half the angle at F. Bisect HC, HD in K, L, and join KL; then the triangle HKL may be placed round the vertex H, twice as many times as the triangle CFD round the vertex F.

96. Each of the vertical angles of the triangles so formed, may be proved to be equal to the difference between the exterior and interior angle of the heptagon.

97. See note on Euc. IV. 16, p. 125.

98. See Euc. I. 9, note p. 49; Problems 10, 11, p. 297; Euc. IV. 10; Euc. IV. 16, note p. 124; Problem 67, p. 336.

99. The equilateral triangle can be proved to be the least triangle which can be circumscribed about a circle.

100. Let ABC be the equilateral triangle, and let a, b, c be the centres of the squares described upon the sides opposite to the angles A, B, C respectively. The triangle formed by drawing ab, bc, ca may be proved to be an equilateral triangle, the sides of which are respectively equal to the line drawn from any angle of the given triangle to the centre of the square on the opposite side. If the numerical value of the side of the given triangle be given, the areas of the two triangles may be expressed in terms of the given side.

101. The area of the triangle formed by joining the centres, may be shewn to be four times the area of the triangle formed by joining the points of contact of the circles.

102. If the radius of the given circle be unity, the radius of each of the four equal circles which touch it externally and each other, may be shewn to be numerically equal to $1+\sqrt{2}$.

103. Take half of the side of the square inscribed in the given circle, this will be equal to a side of the required octagon. At the extremities on the same side of this line make two angles each equal to three-fourths of two right angles, bisect these angles by two straight lines, the point at which they meet will be the centre of the circle which circumscribes the octagon, and either of the bisecting lines is the radius of the circle.

104. (1) When $n=2$, the figure is a square. (2) When $n=4$, the figure is *any* triangle; and we have only to bisect the sides, which will be the points at which the inscribed figure has its angles situated. In all cases except the triangle, the given figure must be equilateral and equiangular.

HINTS, &c. TO THE THEOREMS. BOOK IV.

2. See Euc. IV. 4, 5. The centres of the two circles may be proved to coincide, and the diameter of the circumscribed circle may be shewn to be double of the diameter of the inscribed circle.

3. Let the figure be constructed, the sum of the sides of one triangle may be proved to be double the sum of the sides of the other: and the area of one, four times the area of the other. The parallelism of the lines is proved by Euc. III. 32; I. 29.

4. The line joining the points of bisection, is parallel to the base of the triangle and therefore cuts off an equilateral triangle from the given triangle. By Euc. III. 21; I. 6, the truth of the theorem may be shewn.

5. See Theorem 22, p. 301.

6. Prove aBc, cAb, bCa to be straight lines, and the angles at a, b, c equal.

7. Let three equilateral triangles be described upon AB, AC, BC, the sides of any triangle, and let D, E, F be the centres of the circles inscribed in the equilateral triangles on AB, AC, BC respectively. Let DE, EF, FD be drawn; then EFD is an equilateral triangle. Join DA, DB, EA, EC, FB, FC. At E in AE make the angle AEG equal to FEC, and take EG equal to ED, and join GA. Then the angles of the triangles GDE, DEF may be proved to be respectively equal, and each equal to two-thirds of two right angles.

8. Let the line AD drawn from the vertex A of the equilateral triangle, cut the base BC, and meet the circumference of the circle in D. Let DB, DC be joined: AD is equal to DB and DC. If on DA, DE be taken equal to DB, and BE be joined; BDE may be proved to be an equilateral triangle, also the triangle ABE may be proved equal to the triangle CBD.

The other case is when the line does not cut the base.

9. Let the figure be described. Join DC, then DC is a diameter of the circle described about the quadrilateral figure CFDE. Bisect DC in G, and join FG. If FS can be proved to be perpendicular to FG, then FS will be a tangent to the circle at F, Euc. III. 18.

10. Let ABC be an equilateral triangle inscribed in a circle, and let AB'C' be an isosceles triangle inscribed in the same circle, having the same vertex A. Draw the diameter AD intersecting BC in E, and B'C' in E', and let B'C' fall below BC. Then AB, BE, and AB', B'E', are respectively the semi-perimeters of the triangles. Draw B'F perpendicular to BC, and cut off AH equal to AB, and join BH. If BF can be proved to be greater than B'H, the perimeter of ABC is greater than the perimeter of AB'C'. Next let B'C' fall above BC.

11. Let the equilateral triangle ABC whose altitude is AD, be turned round its centre O till it assume the position *abc*, and let the base *bc* of the new position cut BC in E. Produce *ad* to meet BC in F. Then from the right-angled triangles ODF, *d*EF, the angle between the two positions of the altitude is proved to be equal to the angle between the bases BC, *bc*.

12. Let a diameter be drawn from any angle of an equilateral triangle inscribed in a circle, to meet the circumference. It may be proved that the radius is bisected by the opposite side of the triangle.

13. Let a circle be described upon the base of the equilateral triangle, and let an equilateral triangle be inscribed in the circle. Draw a diameter from one of the vertices of the inscribed triangle, and join the other extremity of the diameter with one of the other extremities of the sides of the inscribed triangle. The side of the inscribed triangle may then be proved equal to the perpendicular in the other triangle.

14. Let the angle BAC be a right angle, fig. Euc. IV. 4. Join AD. Then Euc. III. 17, note p. 108.

15. By the preceding theorem, the excess of the two sides containing the right angle above the hypothenuse is equal to the diameter of the inscribed circle. In this theorem the hypothenuse is equal to the diameter of the circumscribed circle.

16. Let P, Q be the middle points of the arcs AB, AC, and let PQ be joined, cutting AB, AC in D, E; then AD is equal to AE. Find the centre O, and join OP, QO.

17. Let the figure be constructed; the proof depends on Theorem 3, p. 313.

18. Let ABC be any triangle inscribed in a circle, and let the perpendiculars AD, BE, CF intersect in G. Produce AD to meet the circumference in H, and join BH, CH. Then the triangle BHC may be shewn to be equal in all respects to the triangle BGC, and the circle which circumscribes one of the triangles will also circumscribe the other. Similarly may be shewn, by producing BE and CF, &c.

19. Let ABC be a triangle, F the centre of the circumscribed circle (figure Euc. IV. 5,) FD, FE, FG, the perpendiculars from F on AB, AC, BC respectively. Draw DE, DG, GE. Then each of the quadrilaterals ADFE, BDFG, GFEC may be circumscribed by a circle, Euc. III. 22. Then by Euc. VI. E, and observing that twice the area of the triangle ABC is equal to the sum of the rectangles contained by the perpendiculars FD, FE, FG and the sides on which they respectively fall, and also to the rectangle by the sum of the sides and the radius of the inscribed circle, we may shew that the rectangle contained by the sum of the perpendiculars and the sum of the sides of the triangle, is equal to the rectangle contained by the sum of the sides, and the sum of the radii of the inscribed and circumscribed circles.

20. Let F, G, figure, Euc. IV. 5, be the centres of the circumscribed and inscribed circles; join GF, GA, then the angle GAF which is equal to the difference of the angles GAD, FAD, may be shewn equal to half the difference of the angles ABC and ACB.

21. Through C draw CH parallel to AB and join AH. Then HAC the differ-

ence of the angles at the base is equal to the angle HFC, Euc. III. 21, and HFC is bisected by FG.

22. In the figure, Euc. IV. 5. Let AF bisect the angle at A, and be produced to meet the circumference in G. Join GB, GC and find the centre H of the circle inscribed in the triangle ABC. The lines GH, GB, GC are equal to one another.

23. This property of the figure, Euc. IV. 4, exhibited by joining AD and producing it to meet the base, is shewn to follow from Euc. I. 32.

24. This is manifest from Euc. III. 21.

25. This is manifest from Euc. III. 11, 18.

26. The proof of this Theorem is contained in that of Problem 30, p. 334.

27. This Theorem may be stated more generally as follows:

Let AB be the base of a triangle, AEB the locus of the vertex; D the bisection of the remaining arc ADB of the circumscribing circle: then the locus of the centre of the inscribed circle is another circle whose centre is D and radius DB. For join CD: then P the centre of the inscribed circle is in CD. Join AP, PB; then these lines bisect the angles CAB, CBA, and DB, DP, DA may be proved to be equal to one another.

28. Let a circle inscribed in the triangle AED, (figure, Theorem 3, p. 113,) touch the base ED in H, and the sides AD, AE in K, L respectively. It may be easily shewn that EF is equal to DH, and BL or CK equal to ED.

29. The base BC is intersected by the perpendicular AD, and the side AC is intersected by the perpendicular BE. From Theorem 4, p. 314, the arc AF is proved equal to AE, or the arc FE is bisected in A. In the same manner may the arcs FD, DE be shewn to be bisected in B, C.

30. Let ABC be a triangle, and let D, E be the points where the inscribed circle touches the sides AB, AC. Draw BE, CD intersecting each other in O. Join AO, and produce it to meet BC in F. Then F is the point where the inscribed circle touches the third side BC. If F be not the point of contact, let some other point G be the point of contact. Through D draw DH parallel to AC, and DK parallel to BC. By the similar triangles, CG may be proved equal to CF, or G the point of contact coincides with F, the point where the line drawn from A through O meets BC.

31. The difference of the two squares is obviously the sum of the four triangles at the corners of the exterior square.

32. The areas of the successive inscribed squares will be found to be respectively, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., of the given square, and the sum of these squares may be found by finding the sum of the series $\frac{1}{2}$, $\frac{1}{4}$, &c.

33. Let the squares be inscribed in, and circumscribed about a circle, and let the diameters be drawn, the relation of the two squares is manifest.

34. The opposite angles of every quadrilateral about which a circle can be described are together equal to two right angles: and as the opposite angles of every parallelogram are equal to one another, it is obvious that no circle can be described about a parallelogram except it be rectangular.

35. Let one of the diagonals of the square be drawn, then the isosceles right-angled triangle which is half the square, may be proved greater than any other right-angled triangle upon the same hypothenuse.

36. This may be made to appear from Euc. III. 21, 27.

37. The four lines drawn from the centre of the circle to the angular points of the quadrilateral divide the circumference of the circle into four parts, the sum of each pair of opposite portions may be proved equal to half the circumference.

38. This will be manifest from the equality of the two tangents drawn to a circle from the same point.

39. This is the converse proposition of the preceding, and its truth may be proved indirectly, after having proved the direct form of the proposition.

40. Let AC, BD be joined in the figure, Euc. III. 14; the sides AC, BD are parallel. Join AD, BC. Then since the angles in equal segments of the same circle are equal, Euc. I. 27.

In the same way the opposite sides may be shewn to be parallel when the quadrilateral figure circumscribes the circle.

41. Let ABCD, A′B′C′D′ be two quadrilateral figures having their corresponding sides respectively equal, and let ABCD be circumscribed by a circle, and A′B′C′D′

not capable of being so circumscribed, the quadrilateral ABCD is greater than the quadrilateral A'B'C'D'. Draw the diameter AE, intersecting the side CD, and join CE, ED. On C'D' describe the triangle C'E'D' having its sides respectively equal to the sides of the triangle CED, and join A'E'. The proof depends on shewing that of the quadrilaterals ABCE, A'B'C'E', which have their sides respectively equal, the greater is that which has its fourth side equal to the diameter of the circumscribing circle.

42. This follows as a corollary from Theorem 45, infra.

43. Let the sides of the quadrilateral be produced, and the four circles described touching one side of the figure and the two adjacent sides produced. If two lines be drawn from each exterior angle to the centres of the circles adjacent, the figure so formed may be proved to be a quadrilateral having the sum of each pair of its opposite angles equal to two right angles.

44. Let the diagonal AD cut the arc in P, and let O be the centre of the inscribed circle. Draw OQ perpendicular to AB. Draw PE a tangent at P meeting AB produced in E: then BE is equal to PD. Join PQ, PB. Then AB may be proved equal to QE. Hence AQ is equal to BE or DP.

45. Let A, B, C, D be the angular points of the inscribed quadrilateral, and E, F, G, H those of the circumscribed one whose points of contact with the circle are at A, B, C, D: it is required to prove that the diagonals AC, BD, EF, GH intersect in one point.

If EF do not cut the diagonals AC, BD in the same point, let it cut AC in Q, and BD in Q'; and draw EK parallel to FC, and EL to FD. Then AE, EK, EB, EL may be shewn to be equal to one another. Again, since EF cuts AC in Q, and BD in Q', by Euc. VI. 4. KE : FC :: QE : QF; and LE : DF :: Q'E : Q'F; and since KE = LE, and FC = DF, therefore QE : QF :: Q'E : Q'F, which is impossible. Whence FE does not cut AC, BD in different points: that is, it passes through their intersection. In the same manner GH is shewn to pass through the intersection of AC, BD.

(2) If the point of intersection be *without* the circle, the same reasoning applies, with the single exception, that EAC and AKE are respectively the supplements of ACG instead of equal to it.

(3) It therefore follows that AB and CD will intersect in a point in GH, and that BC, AD will intersect in EF.

46. Let a circle inscribed in a square touch one of the sides in the point A, let ABC be an equilateral triangle inscribed in the circle, also let a circle inscribed in the triangle, touch BC, CA, AB in the points D, E, F respectively. AD being joined passes through O the common centre of the two circles. If any point P be taken in the circumference of the inner circle, and PA, PB, PC be drawn, then $PA^2 + PB^2 + PC^2 + OD^2 = 4 . AO^2$, or the area of the given square. By theorem 22, page 309, after joining PF, PC, PD: $PA^2 + PB^2 + PC^2 = PD^2 + PE^2 + PF^2 + 3 . BF^2$. If perpendiculars PL, PM, PN be drawn to the sides of the triangle BC, CA, AB; since the square of the chord PD drawn from any point P to the point of contact of the inscribed circle, is equal to the rectangle contained by the perpendicular PL and the diameter of the circle, $PD^2 + PE^2 + PF^2 = (PL + PM + PN) . 2DO$; also $PL + PM + PN = AD$ (Theorem 15, p. 308) $= 3 . OD$ (Euc. IV. 4). Whence may be deduced the required property.

Is the property true when the point P is not on the circumference, but any point within the circle?

47. This point will be found to be the intersection of the diagonals of the given parallelogram.

48. This theorem is the converse of part of Theorem 31, p. 340, but including also the rhombus as well as the square.

49. Let the inscribed circles whose centres are A, B touch each other in G, and the circle whose centre is C, in the points D, E; join A, D; A, E; at D, draw DF perpendicular to DA, and EF to EB, meeting in F. Let F, G be joined, and FG be proved to touch the two circles in G whose centres are A and B.

50. This is obvious from Euc. IV. 7, the side of a square circumscribing a circle being equal to the diameter of the circle.

51. Let a diameter be drawn from P through the centre Q, and join PA', PB',

PC′, also from A′, B′, C′ draw lines perpendicular to the diameter through P. Then PA′Q, PB′Q, PC′Q are three triangles, of which one will be obtuse-angled and the other two acute-angled. Then by Euc. II. 12, 13, $PA'^2 + PB'^2 + PC'^2$ may be found to be equal to $3.PQ^2 + 3.A'Q^2$. Again, by joining P′A, P′B, P′C, &c. in a similar way, $P'A^2 + P'B^2 + P'C^2$ may be found to be equal to the same quantity.

52. If BD be shewn to subtend an arc of the larger circle equal to one tenth of the whole circumference:—then BD is a side of the decagon in the larger circle. And if the triangle ABD can be shewn to be inscriptible in the smaller circle, BD will be the side of the inscribed pentagon.

53. It may be shewn that the angles ABF, BFD stand on two arcs, one of which is three times as large as the other.

54. It may be proved that the diagonals bisect the angles of the pentagon; and the fivesided figure formed by their intersection may be shewn to be both equiangular and equilateral.

55. The figure ABCDE is an irregular pentagon inscribed in a circle; it may be shewn that the five angles at the circumference stand upon arcs whose sum is equal to the whole circumference of the circle; Euc. III. 20.

56. Prove the five lines joining the points of intersection to be equal to one another, and the angles contained by every two lines which are adjacent to one another.

57. This is a modified form of stating one of the properties in Theorem 64, p. 341.

58. The figure may be proved to be both equilateral and equiangular by means of the isosceles triangles formed by producing the sides of the pentagon.

59. The angles at α, β, γ, δ, ϵ may be proved equal to one another, and each equal to four-fifths of a right angle.

60. If a side CD (figure, Euc. IV. 11) of a regular pentagon be produced to K, the exterior angle ADK of the inscribed quadrilateral figure ABCD is equal to the angle ABC one of the interior angles of the pentagon. From this a construction may be made for the method of folding the ribbon.

61. Let AP be drawn from A perpendicular on CD, figure, Euc. IV. 11. AP may be proved to pass through the centre of the circumscribing circle, as also the perpendiculars from the other angles of the figure. The relation of the angles may be found by means of Euc. I. 32. Cor.

62. The sides and diagonals of the two pentagons (see figure, Euc. IV. 11) may be shewn to have the same relation to each other which is proved in Euc. IV. 10.

63. By Euc. II. 11, $FO^2 = AF.FO$, FO is the side of the regular decagon inscribed in the circle, and OC is the side of the hexagon. Also, $CF^2 = CO^2 + OF^2$, and by Theorem 39, p. 355, CF is a side of the pentagon inscribed in the circle.

64. In the figure, Euc. IV. 10. Let DC be produced to meet the circumference in F, and join FB. Then FB is the side of a regular pentagon inscribed in the larger circle, D is the middle of the arc subtended by the adjacent side of the pentagon. Then the difference of FD and BD is equal to the radius AB. Next, it may be shewn, that FD is divided in the same manner in C as AB, and by Euc. II. 4, 11, the squares of FD and DB are three times the square of AB, and the rectangle of FD and DB is equal to the square of AB.

65. Each of the figures thus formed exterior to the hexagon is an equilateral triangle.

66. The angles contained in the two segments of the circle, may be shewn to be equal, then by joining the extremities of the arcs, the two remaining sides may be shewn to be parallel.

67. It may be shewn that four equal and equilateral triangles will form an equilateral triangle of the same perimeter as the hexagon, which is formed by six equal and equilateral triangles.

68. Let the alternate sides of the figure, Euc. IV. 15, be produced to meet; each of the triangles so formed exterior to the hexagon, may be proved equal in all respects to each of the six triangles into which the hexagon is divided by the diagonals.

69. Let the figure be constructed. By drawing the diagonals of the hexagon, the proof is obvious.

70. Let the figure be drawn, O being the centre and A, B any two opposite angles of the hexagon. If AP, BP′, OQ be the perpendiculars on the line, then the

sum of AP and BP′ may be proved equal to twice OQ, and in a similar way the sum of the pairs of perpendiculars from the two remaining pairs of opposite angles.

71. The circles will be the escribed circles of the six triangles formed by joining the centre of the circle and the angular points of the circumscribed hexagon.

72. This appears directly from Euc. I. 38; and IV. 15.

73. The square of the tangent, by Euc. III. 36, and the square of the side of the octagon may be shewn each to be equal to the same quantity, $(2-\sqrt{2})r^2$, where r is the radius of the circle.

74. By constructing the figures and drawing lines from the centre of the circle to the angles of the octagon, the areas of the eight triangles may be easily shewn to be equal to eight times the rectangle contained by the radius of the circle, and half the side of the inscribed square.

75. Let a regular polygon ABCDE be taken, O being the centre of the circumscribed circle, and E the bisection of the side DC opposite to the angle A. Join AO, OE, and prove that AO, OE are in the same straight line.

76. The first part may be proved by Euc. III. 21. The converse, when the number of sides is *odd*, follows, and may be proved ex absurdo. But when the number of sides is *even*, every pair of opposite angles may be equal, as in the case of the rectangle, which has all its angles equal, but not all its sides equal.

77. Let ABCDEF be a figure of six sides, having all its interior angles equal to one another, but not its opposite sides equal. Produce AB, DC to meet in G, and BC, ED to meet in H. Then AB may be shewn to be parallel to ED.

78. Let lines be drawn to the centre of the circle from the extremities of the lines and the points of contact, and the loci of the extremities of the lines may be shewn to be in the circumferences of two concentric circles, unless the parts of the lines on each side of the points of contact be equal.

79. Join the points of bisection of the equal lines and the centre of the given circle.

80. In the pentagon, hexagon, &c., by Euc. I. 32, the truth of the property is proved.

81. This may be readily shewn in the case of two polygons, one regular, and the other irregular, both having the same number of sides.

82. Let ABCDE (figure Euc. IV. 11,) be a regular polygon inscribed in a circle, and in the arc AB take any point M and join AM, CM. Then the triangle ABC may be shewn to be greater than the triangle AMC.

83. This Theorem is the general form of Theorem 41, p. 340, and may be proved in a similar manner for a polygon of five, six, &c., sides.

84. The proof of this property depends on the fact, that an isosceles triangle has a greater area than any scalene triangle of the same perimeter.

85. The sum of the arcs on which stand the 1st, 3rd, 5th, &c. angles, is equal to the sum of the arcs on which stand the 2nd, 4th, 6th, &c. angles.

86. Let two lines also be drawn from the centre of the circumscribed circle to the extremities of the same side.

87. Let ABCDEF be any irregular polygon of six sides inscribed in a circle, and let AC, CE, EA, BD, DF, FB be joined intersecting each other in a, b, c, d, e, f; then $abcdef$ is an irregular polygon of six sides; and if AC be intersected in a, b, then the sum of the interior angles at a, b, of the inner polygon, may be shewn to be equal to the sum of the interior angles at A, C of the exterior polygon. Euc. I. 32; III. 21. Suppose the sides of the exterior polygon to be equal, the sides of the interior polygon may readily be shewn to be also equal.

88. The opposite angles of the figure so constructed may be proved to be equal to two right angles. Euc. III. 22.

89. No Geometrical method is known whereby the circumference of a circle can be divided into *seven* or *eleven* equal parts.

90. From the given point draw lines to the angles of the polygon; then the area of the polygon is equal to the areas of the triangles thus formed, namely, the rectangle contained by the sum of the perpendiculars and half one of the equal sides. But the area of the polygon is also equal to n times the area of the equal triangles of which the polygon is composed, that is, n times the rectangle contained by the radius of the inscribed circle and half one side. Hence, &c.

91. This Theorem is the same as the preceding, and forms *Prop.* III. *of Stewart's General Theorems.*

92. Let ABCDEF be a regular hexagon, P, Q, R, S, T, V the points of contact of the inscribed circle, and from any point G let GH, GK, GL, GM, GN, GO be drawn perpendicular to the sides of the figure.

Draw PS, QT, RV intersecting each other in a the centre of the circle, and join GP, GQ, GR, GS, GT, GV. Draw GX, GY, GZ perpendicular to PS, QT, RV and join Ga. Then all the angles at a are equal. By Euc. I. 47, $GH^2+GX^2=GP^2$, &c.

$$\text{Hence } GH^2+GK^2+GL^2+GM^2+GN^2+GO^2+2\,(GX^2+GY^2+GZ^2)$$
$$=GP^2+GQ^2+GR^2+GS^2+GT^2+GV^2.$$

$$\text{Next shew that } GP^2+GQ^2+\&c.=6\,.\,Ga^2+6\,.\,Pa^2=6\,.\,d^2+6\,.\,r^2.$$

The points X, Y, Z, are in the circumference of a circle whose diameter is Ga, and the circumference is divided into equal parts in X, Y, Z, also b, the bisection of Ga, is the centre of this circle, and $GX^2+GY^2+GZ^2=3\,.\,Gb^2$. Whence may be shewn, that $GH^2+GK^2+\&c.=6\left(\frac{d^2}{2}+r^2\right)$.

Prove the property, when the figure is a regular pentagon.

This is *Prop.* V. *of Stewart's General Theorems.*

93. This proposition is more easily established by the method of co-ordinates than by pure Geometry. In this way it has been proved by Dr Wallace, in the Gentleman's Mathematical Companion, Vol. VI. p. 452.

The case of the triangle has been proved Geometrically by Mr Kay, in Leybourn's Mathematical Repository, (NS.) Vol. III. p. 35, and Trigonometrically also by Dr Wallace in the same place. A complete Geometrical demonstration, however, may be obtained by means of a general method of investigating certain classes of properties of the circle given by Lieut. Glenie, of the Royal Engineers, in Vol. VI. of the Edinburgh Transactions.

HINTS, &c. TO THE PROBLEMS. BOOK VI.

4. Let AB be the given perimeter of the required triangle. On AB describe a triangle ABC similar to the given triangle; bisect the angles at A and B by lines meeting in D; through D, draw DE, DF parallel to AC, BC, and meeting AB in E, F: then DEF is the triangle required.

5. On any side BC of the given triangle ABC, take BD equal to the given base; join AD, through C draw CE parallel to AD, meeting BA produced, if necessary, in E, join ED: then BDE is the triangle required.

6. (1) In every right-angled triangle when its three sides are in Arithmetical progression, they may be shewn to be as the numbers 5, 4, 3. On the given line AC describe a triangle having its sides AC, AD, DC in this proportion, bisect the angles at A, C by AE, CE meeting in E, and through E draw EF, EG parallel to AD, DC meeting in F and G.

(2) Let AC be the sum of the sides of the triangle, fig. Euc. VI. 13. Upon AC describe a triangle ADC whose sides shall be in continued proportion, (by Prob. 9, infra.) Bisect the angles at A and C by two lines meeting in E. From E draw EF, EG parallel to DA, DC respectively.

7. Describe a circle with any radius, and draw within it the straight line MN cutting off a segment containing an angle equal to the given angle, Euc. III. 34. Divide MN in P in the given ratio, and at P draw PA perpendicular to MN and meeting the circumference in A. Join AM, AN, and on AP or AP produced, take AD equal to the given perpendicular, and through D draw BC parallel to MN meeting AM, AN or these lines produced. Then ABC shall be the triangle required.

8. Let A, B be the two given points, and let P be a point in the locus so that PA, PB being joined, PA is to PB in the given ratio. Join AB and divide it in C in the given ratio, and join PC. Then PC bisects the angle APB. Euc. VI. 3.

Again, in AB produced, take AD to AB in the given ratio, join PD and produce AP to E, then PD bisects the angle BPE. Euc. VI. A. Whence CPD is a right angle, and the point P lies in the circumference of a circle whose diameter is CD.

9. By Euc. I. 47. $BC^2 - BE^2 = AC^2 - AE^2$, hence $BC^2 - AC^2 = BE^2 - AE^2$, or $(BC + AC).(BC - AC) = (BE + AE).(BE - AE)$, but $BE - AE = 2.DE$, also $BC - AC = DE$. $\therefore$ $BC + AC = 2.AB$, or the three sides AC, AB, BC are in Arithmetical progression.

10. Let PAQ be the given angle, bisect the angle A by AB, in AB find D the centre of the inscribed circle, and draw DC perpendicular to AP. In DB take DE such that the rectangle DE, DC is equal to the given rectangle. Describe a circle on DE as diameter meeting AP in F, G; and AQ in F′, G′. Join FG′ and AFG′ will be the triangle. Draw DH perpendicular to FG′, and join G′D. By Euc. VI. C, the rectangle FD, DG′ is equal to the rectangle ED, DK or CD, DE.

11. Let BC be the given base; draw BE perpendicular to BC and equal to the given altitude, and through E draw EM parallel to BC. At B make the angle CBF equal to the difference of the angles at the base. Divide the base in D so that BD may be to DC in the ratio of the sides, draw DG perpendicular to BF and produce it to meet EM in A. Join AB, AC; ABC is the triangle required.

12. Let AB be the given base, ACB the segment containing the vertical angle; draw the diameter AD of the circle, and divide it in E in the given ratio; on AE as a diameter, describe a circle AFE; and with centre B and a radius equal to the given line, describe a circle cutting AFE in F. Then AF being drawn and produced to meet the circumscribing circle in C, and CD being joined, ABC is the triangle required. For AF is to FC in the given ratio.

13. Let AC be the given base, and let DAC be the required triangle. Draw DB perpendicular to BC. Then from the hypothesis combined with Euc. VI. 8, it may be shewn that AB is equal to DC, and that AC is divided in B in extreme and mean ratio.

14. Let ABC be any triangle, and DEF the given triangle to which the inscribed triangle is required to be similar. Draw any line *de* terminated by AB, AC, and on *de* towards AC describe the triangle *def* similar to DEF, join B*f*, and produce it to meet AC in F′. Through F′ draw F′D′ parallel to *fd*, F′E′ parallel to *fe*, and join D′E′, then the triangle D′E′F′ is similar to DEF.

15. Employ Theorem 17, p. 353, and the construction becomes obvious.

16. Let ABC be the required triangle, CD the line bisecting the vertical angle, cutting AB in H, and meeting the circumscribed circle in D, DME a diameter drawn through D, and therefore bisecting the base AB in M. Let DE be bisected in O, then O is the centre of the circumscribing circle; also let P be the centre of the inscribed circle.

Then the line OP joining the centres, cuts the line bisecting the vertical angle in the centre of the inscribed circle. We have given, therefore, the base AB, the segment containing the vertical angle ACB, and the ratio CP to PH.

By means of the equiangular triangles DBC, DHB, the ratio of DB to DC may be shewn to be the same as the ratio of PC to PH, which is given. Hence the ratio of DB to DC, is given; also DB is given in magnitude, and therefore DC. Whence the construction.

17. This is similar in its construction to Prob. 12, supra, except that the point F, instead of being at a given distance from B, is in a semicircle on AB.

18. The line CD is not necessarily parallel to AB. Divide the base AB in C, so that AC is to CB in the ratio of the sides of the triangle.

Then if a point E in CD can be determined such that when AE, CE, EB, are joined, the angle AEB is bisected by CE, the problem is solved.

19. Let ABC be any triangle having the base BC. On the same base describe an isosceles triangle DBC equal to the given triangle. Bisect BC in E, and join DE, also upon BC describe an equilateral triangle. On FD, FB, take EG to EH as EF to EB: also take EK equal to EH and join GH, GK; then GHK is an equilateral triangle equal to the triangle ABC.

20. Let ABC be the required triangle, BC the hypothenuse, and FHKG the inscribed square; the side HK being on BC. Then BC may be proved to be divided in H and K, so that HK is a mean proportional between BH and KC.

21. Let AB be the given perimeter, and let CD be drawn parallel to AB at the distance of the given perpendicular; on AB describe a circle, and let F be the middle point of the semicircle on the opposite side of AB from CD; with centre F describe a circle through A, B, cutting CD in P or P′; make in this circle the arcs AG, BH, equal to BP, PA respectively; and draw PG, PH cutting AB in Q, R: then PQR is the triangle required.

Algebraically. Let x, y, be the sides, z the hypothenuse of the triangle, and p the perpendicular from the right angle on the hypothenuse: then x, y, z, may be found from the equations $x^2+y^2=z^2$, $xy=pz$, $x+y+z=a$, in terms of p the perpendicular, and a the perimeter.

22. Make an isosceles triangle, having its vertical angle equal to the given angle. Describe a triangle similar to this isosceles triangle, and having its perimeter equal to the given perimeter. Then the area of this triangle may be shewn to be greater than the area of any other triangle which has the same vertical angle and the same perimeter.

23. Let ABC be the given triangle. On BC take BD equal to one of the given lines, through A, draw AE parallel to BC. From B draw BE to meet AE in E, and such that BE is a fourth proportional to BC, BD, and the other given line. Join EC, produce BE to F, making BF equal to the other given line, and join FD: then FBD is the triangle required.

24. If a circle be described about the given triangle, and another circle upon the radius drawn from the vertex of the triangle to the centre of the circle, as a diameter, this circle will cut the base in two points, and give two solutions of the problem. Give the Analysis.

25. In the figure, Euc. VI. 13. If E be the middle point of AC; then AE or EC is the arithmetic mean, and DB is the geometric mean, between AB and BC. If DE be joined and BF be drawn perpendicular on DE; then DF may be proved to be the harmonic mean between AB and BC.

26. The two means and the two extremes form an arithmetic series of four lines whose successive differences are equal: the difference therefore between the first and the fourth, or the extremes, is treble the difference between the first and the second.

27. Let the two given lines meet when produced in A. At A draw AD perpendicular to AB, and AE to AC, and such that AD is to AE in the given ratio. Through D, E, draw DF, EF, respectively parallel to AB, AC and meeting each other in F. Join AF and produce it, and the perpendiculars drawn from any point of this line on the two given lines will always be in the given ratio.

28. This problem may be constructed in the same way under more general circumstances than those in which it is enunciated; namely, when A, B, G, are any points, and any line is substituted for BK, compatible with the construction following.

Bisect AB in V and join VG, produce VG to P, make GP = 2 . VG; on PG describe a circle, in which place the chords PQ, PQ′ equal to the given sum of the perpendiculars: then the line QG is that required, and Q′G is that upon which the difference (instead of the sum) of the perpendiculars shall be equal to PG. The proof depends on Prob. 3, p. 347.

29. Let the three given lines meet in A, B, C and form a triangle, and let the ratios of the three perpendiculars be as three lines m, n, p. On AC, BC, take AD, BE each equal to m, draw DF, EG each equal to m and parallel to AB. Join AF, BG and produce them to meet in O, the perpendiculars OP, OQ, OR from O drawn to AB, AC, BC, shall have the same ratios as m, n, p. From F draw FH perpendicular to AB, FK to AC, and FL parallel to AC. Then by the similar triangles.

30. Let AB be the given straight line, and let a perpendicular be drawn to AB from the point C. Divide AB in D, so that AD is to DB in the given ratio; then if from D a line DE be drawn to meet the perpendicular in E so that when AE, EB are joined, the angle AEB shall be bisected by ED, E will be the point required. Euc. VI. 3.

31. In BC produced take CE a third proportional to BC and AC; on CE describe a circle, the centre being O; draw the tangent EF at E equal to AC; draw FO cutting the circle in T and T′; and lastly, draw tangents at T, T′ meeting BC in P and P′. These points fulfil the conditions of the problem.

By combining the proportion in the construction with that from the similar triangles ABC, DBP, and Euc. III. 36, 37; it may be proved that CA . PD = CP^2.

The demonstration is similar for P′D′.

32. Let ABC be any triangle, and D the given point in the base BC. Divide DC in E so that CE may be to ED in the given ratio. Join AD, and from E draw EF to meet AC in F, and making the angle CFE less than CAD. Through F draw FG parallel to CB meeting AB in G, join DG and produce it to meet CA produced in H. Then DH is divided in G in the given ratio. What are the limits to the position of the point F?

33. Let P be the given point and AB the given straight line. Draw PQ perpendicular to AB and produce QP to R making QP to PR in the given ratio; through R draw CD parallel to AB: then any line drawn through P and terminated by the parallels will be divided at P in the given ratio.

34. Let AB be the given line from which it is required to cut off a part BC such that BC shall be a mean proportional between the remainder AC and another given line. Produce AB to D, making BD equal to the other given line. On AD describe a semicircle, at B draw BE perpendicular to AD. Bisect BD in O, and with centre O and radius OB describe a semicircle, join OE cutting the semicircle on BD in F, at F draw FC perpendicular to OE and meeting AB in C. C is the point of division, such that BC is a mean proportional between AC and BD.

35. Take any straight line AB, and find another AC, so that AC is to AB as $\sqrt{5} : 1$; and let AB, AC make any acute angle at A and join BC. On AC take AD equal to the given straight line, and through D, draw DE parallel to CB, and meeting AB in E, then AE is the line required.

36. Find two squares in the given ratio, and if BF be the given line (figure Euc. VI. 4), draw BE at right angles to BF, and take BC, CE respectively equal to the sides of the squares which are in the given ratio. Join EF, and draw CA parallel to EF: then BF is divided in A as required.

37. See Euc. VI. 13.

38. This may be effected in different ways, one of which is the following. At one extremity A of the given line AB draw AC making any acute angle with AB and join BC: at any point D in BC draw DEF parallel to AC cutting AB in E and such that EF is equal to ED, draw FC cutting AB in G. Then AB is harmonically divided in E, G.

39. In the fig. Euc. VI. 13. DB is the Geometric mean between AB and BC, and if AC be bisected in E, AE or EC is the Arithmetic mean.

The next the same as—to find the segments of the hypothenuse of a right-angled triangle made by a perpendicular from the right angle, having given the difference between half the hypothenuse and the perpendicular.

40. For "the base produced," read "*the part of the base produced.*" Let ABC be the given right-angled triangle, C the right angle, and BC the base.

At the vertex A in AB, make the angle BAD equal to the angle BAC, and let AD meet the base CB produced in D. Then D is the point required. Euc. VI. 3.

41. The construction is suggested by Euc. I. 47, and Euc. VI. 31.

43. Produce one side of the triangle through the vertex, and make the part produced equal to the other side. Bisect this line, and with the vertex of the triangle as centre and radius equal to half the sum of the sides, describe a circle cutting the base of the triangle.

44. This Problem is analogous to Problem 24, p. 349.

45. Suppose that ABCD the required square is constructed; and PA, PB, PC, the distances of the point P from the three angles of the square are given. Draw BQ perpendicular to PB and equal to it, and join QC. Then since ABC, PDQ are right angles, the angles ABP, CBQ are equal, and hence QC equal to AP is given, and P, Q are given points. Wherefore PQ being given in magnitude and position, and QC, CP in magnitude; the point C is given, and hence the side BC of the square.

Construction. Draw BQ perpendicular and equal to PB, the line which lies between AP, CP, and join PQ: on PQ as a base, and PA, PQ as sides, describe the triangle PCQ: then BC is the side of the square.

46. Let AB, AC, be the two given lines placed at right angles at A. Take AC

to AD in the given ratio, and join CD; with centre B and radius BP equal to the side of the given square, describe a circle cutting CD in P; draw PE parallel to AC, and EF parallel to CD: then AB, AC are divided in E and F as required.

47. If a triangle be constructed on AB so that the vertical angle is bisected by the line drawn to the point C. By Euc. VI. A. the point required may be determined.

48. Let AB, AC be the two given straight lines meeting at A, and P the given point between them. At A draw AD, AE so that AD is to AE in the given ratio, and containing the angle ADE equal to the given angle. Join DE, PA, through P draw PF parallel to AD meeting AB in F, and PG parallel to AE meeting AC in G. PF, PG, are the lines required.

49. Let the given line AB be divided in C, D. On AD describe a semicircle, and on CB describe another semicircle intersecting the former in P; draw PE perpendicular to AB; then E is the point required.

50. The line drawn through the given point and making equal angles with the two given lines is the line required. If a circle be described touching the two given lines at the points where the required line meets them, the rectangle contained by the segments of any other line drawn through the given point, is greater than the rectangle by the segments of that line which makes equal angles with the given line.

51. This problem is misplaced: it properly falls among the problems on Euc. XI.

Let ABCD be the given rectangle, whose sides AB, CD are each equal to $34a$, and whose other sides are each equal to $13a$. Take Ah, Ak, Be, Bf each equal to $12a$, join ef, hk, these will be the creases required. For draw Am, Bg perpendicular to hk, ef, and join gm; bisect CD in P; draw PR parallel to BC, and AD meeting gm in Q; join Pg, Pm; let the triangles Bfc, Ahk be turned about fe and hk till they take the respective positions fce, hdk, at right angles to the plane of the rectangle ABCD; and join Pc, Pd.

52. Let ABC be a right-angled triangle, having the right angle at B and the base BC greater than the perpendicular AB. Let P be the required point in BC (not AC) so that when AP is joined, and PD drawn perpendicular to AC, AP and PD shall be a minimum. Produce DP, to meet AB produced in E. Then ED or AP and PD may be proved to be less than the sum of two lines drawn to any other point of BC.

53. At any point D in BC draw DE perpendicular to BC meeting AB in E; in EA take EF to ED as 1 to 2, and join FD. Through A draw AG parallel to FD, and through G draw GP parallel to DE meeting AB in P. Then P is the point such that AP is half of PG.

54. Let these points be taken, one on each side, and straight lines be drawn to them: it may then be proved that these points severally bisect the sides of the triangle.

55. Let AB be equal to a side of the given square. On AB describe a semicircle; at A draw AC perpendicular to AB and equal to a fourth proportional to AB and the two sides of the given rectangle. Draw CD parallel to AB meeting the circumference in D. Join AD, BD, which are the required lines.

56. Describe a circle about the triangle, and draw the diameter through the vertex A, draw a line touching the circle at A, and meeting the base BC produced in D. Then AD will be a mean proportional between DC and DB. Euc. III. 36.

57. Let A, B be the two given points, and C a point in the circumference of the given circle. Let a circle be described through the points A, B, C and cutting the circle in another point D. Join CD, AB, and produce them to meet in E. Let EF be drawn touching the given circle in F, the circle described through the points A, B, F, will be the circle required. Joining AD and CB, by Euc. III. 21, the triangles CEB, AED are equiangular, and by Euc. VI. 4, 16, III. 36, 37, the given circle and the required circle each touch the line EF in the same point, and therefore touch one another. When does this solution fail?

Various cases will arise according to the relative position of the two points and the circle.

58. Let A be the given point, BC the given straight line, and D the centre of the given circle. Through D draw CD perpendicular to BC, meeting the circumference in E, F. Join AF, and take FG to the diameter FE, as FC is to FA. The circle de-

scribed passing through the two points A, G and touching the line BC in B is the circle required. Let H be the centre of this circle; join HB, and BF cutting the circumference of the given circle in K, and join EK. Then the triangles FBC, FKE being equiangular, by Euc. VI. 4, 16, and the construction, K is proved to be a point in the circumference of the circle passing through the points A, G, B. And if DK, KH be joined, DKH may be proved to be a straight line:—the straight line which joins the centres of two circles, and passes through a common point in their circumferences.

59. Let A be the given point, B, C the centres of the two given circles. Let a line drawn through B, C meet the circumferences of the circles in G, F; E, D, respectively. In GD produced, take the point H, so that BH is to CH as the radius of the circle whose centre is B, is to the radius of the circle whose centre is C. Join AH, and take KH to DH as GH to AH. Through A, K describe a circle ALK touching the circle whose centre is B, in L. Then M may be proved to be a point in the circumference of the circle whose centre is C. For by joining HL and producing it to meet the circumference of the circle whose centre is B in N; and joining BN, BL, and drawing CO parallel to BL, and CM parallel to BN, the line HN is proved to cut the circumference of the circle whose centre is B in M, O; and CO, CM are radii. By joining GL, DM, M may be proved to be a point in the circumference of the circle ALK. And by producing BL, CM to meet in P, P is proved to be the centre of ALK, and BP joining the centres of the two circles passes through L the point of contact. Hence also is shewn that PMC passes through M, the point where the circles whose centres are P and C touch each other.

Note. If the given point be in the circumference of one of the circles, the construction may be more simply effected thus:

Let A be in the circumference of the circle whose centre is B. Join BA, and in AB produced, if necessary, take AD equal to the radius of the circle whose centre is C, join DC, and at C make the angle DCE equal to the angle CDE, the point E determined by the intersection of DA produced and CE, is the centre of the circle.

60. Let the two given circles be without one another, and let A, B be their centres. Join AB cutting the circumferences in C, D; take CE, DF each equal to the radius of the required circle: the two circles described with centres A, B, and radii AE, BF respectively, will intersect one another, and the point of intersection will be the centre of the required circle. Distinguish the different cases.

61. Let the two given lines AB, BD meet in B, and let C be the centre of the given circle, and let the required circle touch the line AB, and have its centre in BD. Draw CFE perpendicular to HB intersecting the circumference of the given circle in F, and produce CE, making EF equal to the radius CF. Through G draw GK parallel to AB, and meeting DB in K. Join CK, and through B, draw BL parallel to KC, meeting the circumference of the circle whose centre is C in L; join CL and produce CL to meet BD in O. Then O is the centre of the circle required. Draw OM perpendicular to AB, and produce EC to meet BD in N. Then by the similar triangles, OL may be proved equal to OM.

62. Let AB be the given straight line, and C the centre of the given circle; through C draw the diameter DCE perpendicular to AB. Place in the circle a line FG which has to AB the given ratio; bisect FG in H, join CH, and on the diameter DCE, take CK, CL, each equal to CH; either of the lines drawn through K, L, and parallel to AB is the line required.

63. The locus of the intersections of the diagonals of all the rectangles inscribed in a scalene triangle, is a straight line drawn from the bisection of the base to the bisection of the shorter side of the triangle.

64. The tangent AC by Euc. III. 36, may be shewn to be equal to 3 . AB: the problem is reduced to finding a mean proportional between AB and 3 . AB.

65. Let A be the given point within the circle whose centre is C, and let BAD be the line required, so that BA is to AD in the given ratio. Join AC and produce it to meet the circumference in E, F. Then EF is a diameter. Draw BG, DH perpendicular on EF: then the triangles BGA, DHA are equiangular. Hence the construction.

66. This Problem only differs from the preceding in having the given point without the given circle.

67. Draw tangents EF, GH, common to the two circles whose centres are A, B to cut the line AB in C and D, (C being between A and B, and D in AB produced): then any line drawn through either of these points will fulfil the conditions of the Problem.

Take a line through D, the intercepted chords of which are KL and MN; and join AM, AN, AE, BK, BL, BF. Then AD : DB :: AE : BF :: AN : BL.

Whence the triangles ADN, BDL are similar, and AN is parallel to BL. Similarly AM is parallel to BK; and hence the isosceles triangles MAN, KBL have the angles at their vertices A, B, equal. The other angles therefore are equal, and the triangles are similar. Whence MN : KL :: AM : BK.

Hence to draw a pair of common tangents to two circles.

Draw any two parallel radii AM, BK, join MK, AB and produce them to intersect in D. Then D is the point from which a pair of common tangents may be drawn to the two circles. The points of contact may be found by describing circles on BD, DA as diameters.

68. Since the magnitude and position of the two circles are given, their radii and the distances of their centres are known. Let O, O′ be the centres of the larger and smaller circles, join OO′, O′C, from C draw CE to touch the larger circle in E. Join also OE, CO. Then Euc. III. 36. $CA . CB = CE^2$, and from the right-angled triangles CE is known, also the ratio of CA to CB is given. Hence the problem is reduced to this: to find two lines which have a given ratio to one another, and whose rectangle is equal to a given rectangle.

69. Let A be the given point in the circumference of the circle, C its centre. Draw the diameter ACB, and produce AB to D, taking AB to BD in the given ratio: from D draw a line to touch the circle in E, which is the point required. From A draw AF perpendicular to DE, and cutting the circle in G.

70. See Problem 8, p. 348.

71. Suppose the area of the grass-plot to be two-thirds of the rectangular field.

Let HM, LM be two lines drawn from H, L points in AB, BC and parallel to the sides, cut off the rectangle HMLB equal to two-thirds of ABCD, Problem 95, p. 352. Bisect LC, AH in P, Q, and through P, Q draw PR, QR parallel to the sides and meeting in R. On BA, BC, take BS, BT each equal to AQ, through S draw SX parallel to BC and meeting QR in X, and through T draw TYZ parallel to AB meeting SX in Y, and QR in Z. Then the rectangle RXYZ is equal to HMLB, and AQ is the width of the pathway round the grass-plot.

72. By means of Theorem 137, p. 362, the ratio of the diagonals AC to BD may be found to be as $AB . AD + BC . CD$ to $AB . BE + AD . DC$. figure, Euc. VI. D.

73. The two hexagons consist each of six equilateral triangles, and the ratio of the hexagons is the same as the ratio of their equilateral triangles.

74. Let ABC be any triangle and D be the given point in BC, from which lines are to be drawn which shall divide the triangle into any number (suppose five) equal parts. Divide BC into five equal parts in E, F, G, H, and draw AE, AF, AG, AH, AD, and through E, F, G, H draw EL, FM, GN, HO parallel to AD, and join DL, DM, DN, DO; these lines divide the triangle into five equal parts.

By a similar process, a triangle may be divided into any number of parts which have a given ratio to one another.

75. Let ABC be the larger, *abc* the smaller triangle, it is required to draw a line DE parallel to AC cutting off the triangle DBE equal to the triangle *abc*. On BC take BG equal to *bc*, and on BG describe the triangle BGH equal to the triangle *abc*. Draw HK parallel to BC, join KG; then the triangle BGK is equal to the triangle *abc*. On BA, BC take BD to BE in the ratio of BA to BC, and such that the rectangle contained by BD, BE shall be equal to the rectangle contained by BK, BG. Join DE, then DE is parallel to AC, and the triangle BDE is equal to *abc*.

76. Let ABC be the given triangle which is to be divided into two parts having a given ratio, by a line parallel to BC. Describe a semicircle on AB and divide AB in D in the given ratio, at D draw DE perpendicular to AB and meeting the circumference in E, with centre A and radius AE describe a circle cutting AB in F; the line drawn through F parallel to BC is the line required.

In the same manner a triangle may be divided into three or more parts having any given ratio to one another by lines drawn parallel to one of the sides of the triangle.

77. This is a particular case of Prob. 1, p. 254. It will be found that the side of the square is equal to one-third of the hypothenuse.

78. The square inscribed in a right-angled triangle which has one of its sides coinciding with the hypothenuse, may be shewn to be less than that which has two of its sides coinciding with the base and perpendicular.

79. Find M and N the sides of squares equal to the given triangle and the given area of the rectangle respectively; from A draw AD perpendicular on BC the base of the triangle; on AD describe a semicircle, bisect the arc in E and draw ED; next find a fourth proportional to M, N, and DE. In DB make DF equal to this line, and draw FGG′ parallel to AD and cutting the semicircle in G, G′: then one side of the required rectangle passes through G.

A second rectangle fulfilling the conditions is found by means of G′.

The limit of possibility is when FGG′ becomes a tangent to the semicircle; and the inscribed rectangle is then the greatest possible.

Give the Analysis of the Problem.

80. Take two points on the radii equidistant from the centre, and on the line joining these points, describe a square; the lines drawn from the centre through the opposite angles of the square to meet the circular arc, will determine two points of the square inscribed in the sector.

81. Since the area of the triangle is given, the side of the square which is double the area of the triangle is given. Find a third proportional to the given base, and the side of this square: this line will be equal to the altitude of the triangle.

82. The sides of the three squares inscribed in the triangle may be shewn to be inversely as the three sides of the triangle respectively assumed as the base. Hence that square is the greatest which is contiguous to the smallest of the three sides of the triangle.

83. Let ABCDE be the given regular pentagon. On AB, AE take equal distances AF, AG, join FG, and on FG describe a square FGKH. Join AH and produce it to meet a side of the pentagon in L. Draw LM parallel to FH meeting AE in M. Then LM is a side of the inscribed square.

84. Let ABC be the given triangle. Draw AD making with the base BC an angle equal to one of the given angles of the parallelogram. Draw AE parallel to BC and take AD to AE in the given ratio of the sides. Join BE cutting AC in F.

85. The greatest parallelogram which can be inscribed in a given triangle, is that which has one side parallel to the base of the triangle, and terminated in the points of bisection of the sides.

86. This is only a particular case of the general problem, when the rectilineal figures are a right-angled triangle and a square of given magnitudes.

87. Let AB be the base of the segment ABD fig. Euc. III. 30. Bisect AB in C, take any point E in AC and make CF equal to CE: upon EF describe a square EFGH: from C draw CG and produce it to meet the arc of the segment in K.

88. This parallelogram may be proved to be a square.

89. Analysis. Let ABCD be the given rectangle, and EFGH that required, having the point E on AB, F on BC, &c., and let LMKN be the parallelogram whose angular points, L, M, N, K are at the bisections of AB, BC, CD, DA.

Then $HA.AE+EB.BF=\text{fig. } ABCD-\text{fig. } EFGH=$ a given area.

Whence $2.HK.EL=\text{fig. } ABCD-\text{fig. } EFGH-2.AK.AL$.

The rectangle of the two distances HK, EL, from the middle of the sides is therefore given.

Also by the similar triangles HAE, EBF, may be deduced,

$$LE^2-HK^2=AL^2-AK^2.$$

Whence the rectangle, and the difference of the squares of LE, KH are given, to find the lines themselves.

90. In a straight line at any point A, make Ac to Ad in the given ratio. At A draw AB perpendicular to cAd, and equal to a side of the given square. On cd describe a semicircle cutting AB in b; and join bc, bd; from B draw BC parallel to bc, and BD parallel to bd: then AC, AD are the adjacent sides of the rectangle. For, CA is to AD as cA to Ad; Euc. VI. 2. and $CA.AD=AB^2$, CBD being a right-angled triangle.

91. Let D be the given point *within* the triangle ABC. In the base BC take BX to BC as the part to be cut off is to the whole triangle. Join BX, and describe

the parallelogram BEFG equal to the triangle ABX, having the sides BE, BG on BA, BC respectively, and the side EF passing through the point D. Draw DH parallel to AB, and on HG describe a semicircle, place HK in this semicircle, equal to HB, join GK, and with centre G and radius GK describe a circle cutting BC in L, M; the line drawn from L or M through D, cuts off the required part from the triangle ABC. Give the analysis.

92. A mean proportional between two homologous sides of the polygons, will be the corresponding side of the required polygon.

93. Analysis. Let ABCD be the given rectangle, and EFGH that to be constructed. Then the diagonals of EFGH are equal and bisect each other in P the centre of the given rectangle. About EPF describe a circle meeting BD in K, and join KE, KF. Then since the rectangle EFGH is given in species, the angle EPF formed by its diagonals is given; and hence also the opposite angle EKF of the inscribed quadrilateral PEKF is given. Also since KP bisects that angle, the angle PKE is given, and its supplement BKE is given. And in the same way, KF is parallel to another given line; and hence EF is parallel to a third given line. Again, the angle EPF of the isosceles triangle EPF is given; and hence the quadrilateral EPFK is given in species.

94. Reference may be made to Euc. VI. 31.

95. It is manifest that this is the general case of Prob. 3, p. 331.

If the rectangle to be cut off be two-thirds of the given rectangle ABCD.

Produce CB to E so that BE may be equal to a side of that square which is equal to the rectangle required to be cut off; in this case, equal to two thirds of the rectangle ABCD. On AB take AF equal to AD or BC; bisect FB in G, and with centre G and radius GE describe a semicircle meeting AB, and AB produced, in H and K. On CB take CL equal to AH and draw HM, LM parallel to the sides, and HBLM is two-thirds of the rectangle ABCD.

96. Draw any diameter AB, in AB take any point C, and through C draw DCE perpendicular to AB, making CE and CD each equal to half of AC. Join AD, AE and produce them to meet the circumference in F, G. Join FG.

97. Draw any diameter AB of the given circle; on AB take AC to AB as 4 to 5, and draw DCE perpendicular to AB. Join AE, EB, BD, BA, and the isosceles triangle AED is four times the isosceles triangle EBD.

98. In the figure Euc. III. 30; from C draw CE, CF making with CD, the angles DCE, DCF each equal to the angle CDA or CDB, and meeting the arc ADB in E and F. Join EF, the segment of the circle described upon EF and which passes through C will be similar to the segment ADB.

99. The side of the equilateral triangle is one-third, and the side of the regular pentagon is one-fifth of the given line. The *radii* of their inscribed circles may be expressed in terms of the sides of the triangle and pentagon respectively: and the numerical ratio of the radii will be found to be as $5\sqrt{5-\sqrt{5}}$ to $\sqrt{21}$.

100. By means of the similar triangles it may be proved that of any three of the consecutive circles, the sum of the radii of the first and second, is to their difference, as the sum of the radii of the second and third, is to their difference.

HINTS, &c. TO THE THEOREMS. BOOK VI.

3. This is the converse of the Cor. Euc. VI. 1, and may be proved indirectly.

4. See Note p. 5, Appendix.

5. See Note Euc. VI. A, p. 203.

6. The lines drawn making equal angles with homologous sides, divide the triangles into two corresponding pairs of equiangular triangles; by Euc. VI. 4, the proportions are evident.

7. Let DE be the position of the given line and B the given point through which the parallel is to pass. Stretch the string from B to D and take a continued length DA equal to DB. From A take any length to meet DE in any point E, and take a continued length EC equal to EA: the line joining B and C is parallel to DE.

8. All the parallelograms are manifestly similar and similarly situated with respect to each other: and every 4, 9, 16, 20, &c. of the smaller ones, form parallelograms similar to each other and the smaller one.

9. Join BC, DC. Draw AF intersecting DC in G, and let AF produced meet BC in H. Then DC is parallel to BC, and from the similar triangles FDG, FHC; FDE, FBC; ADE, ABC; FC may be shewn to be equal to m. FD: and DC = FC + DF.

10. Join ba cutting AE in F. Then by the similar triangles DbF, DFC; Dba, DBC; Aba, ABC; n. FE = $(n+1)$. DE, and DE = AE − AD. Also from the similar triangles Aba, ABC; n. FE = $(n-1)$. AE. Whence is shewn 2. AE = $(n+1)$. AD. By means of the Theorem 160, p. 364, and Euc. VI. 2, (for ba is parallel to BC) BC may be proved to be bisected in E.

11. By constructing the figure, the angles of the two triangles may easily be shewn to be respectively equal.

12. The area of a right-angled triangle being half of the rectangle contained by the base and perpendicular; the sum of the series of triangles will be found to depend upon the sum of the series $1 + \frac{1}{2} + \frac{1}{4} +$ &c. continued ad infinitum.

13. This may be shewn from Euc. I. 47; VI. 4, 16.

14. Let ABCD be a square and AC its diagonal. On AC take AE equal to the side BC or AB: join BE and at E draw EF perpendicular to AC and meeting BC in F. Then EC, the difference between the diagonal AC and the side AB of the square, is less than AB; and CE, EF, FB may be proved to be equal to one another: also CE, EF are the adjacent sides of a square whose diagonal is FC. On FC take FG equal to CE and join EG. Then as in the first square, the difference CG between the diagonal FC and the side EC or EF, is less than the side EC. Hence EC the difference between the diagonal and the side of the given square, is contained twice in the side BC with a remainder CG: and CG is the difference between the side CE and the diagonal CF of another square. By proceeding in a similar way, CG the difference between the diagonal CF and the side CE, is contained twice in the side CE with a remainder: and the same relations may be shewn to exist between the difference of the diagonal and the side of every square of the series which is so constructed. Hence, therefore, as the difference of the side and diagonal of every square of the series, is contained twice in the side with a remainder, it follows that there is no line which exactly measures the side and the diagonal of a square.

15. In the arc AB (fig. Euc. IV. 2) let any point K be taken, and from K let KL, KM, KN be drawn perpendicular to AB, AC, BC respectively, produced if necessary, also let LM, LN be joined, then MLN may be shewn to be a straight line. Draw AK, BK, CK, and by Euc. III. 31, 22, 21; Euc. I. 14.

16. Join AL, and produce it to meet the base BC in G. Join also DF intersecting AG in M. Then every line DF drawn parallel to the base BC is divided in the same proportion as the segments of the base made by a line drawn from the vertex of the triangle. Then conversely, if this proportion hold good, the line joining the points L, G must pass through A.

17. Divide the given base BC in D, so that BD may be to DC in the ratio of the sides. At B, D draw BB′, DD′ perpendicular to BC and equal to BD, DC respectively. Join B′D′ and produce it to meet BC produced in O. With centre O and radius OD describe a circle. From A any point in the circumference join AB, AC, AO. Prove that AB is to AC as BD to DC. Or thus. If ABC be one of the triangles. Divide the base BC in D so that BA is to AC as BD to DC. Produce BC and take DO to OC as BA to AC: then O is the centre of the circle.

18. A circle may be described about the four-sided figure ABDC. By Euc. I. 13; Euc. III. 21, 22. The triangles ABC, ACE may be shewn to be equiangular.

19. Let ABC be the given triangle, and let the line EGF cut the base BC in G. Join AG. Then by Euc. VI. 1, and Theo. 85, p. 358, it may be proved that AC is to AB as GE is to GF.

20. Since CE is equal to CA, the triangles CAB, CED are similar, Euc. VI. 6, and by Euc. I. 5, 32, the triangles CAB, DCB may be shewn to be similar.

21. Let ABC be the triangle, right-angled at C, and let AE on AB be equal to AC, also let the line bisecting the angle A, meet BC in D. Join DE. Then the triangles ACD, AED are equal, and the triangles ACB, DEB equiangular.

22. Let ABC be any triangle, let BD be drawn parallel to AC and equal to AB, and CE parallel to AB and equal to AC. Join DC, BE intersecting AB, AC in F, G respectively. Then by means of the similar triangles, two proportions may be found from which it may be proved that AF is equal to AG, and that either is a mean proportional between BF and CG.

23. See Theorem 16, p. 308.

24. If the property be assumed to be true; then by Euc. VI. 16, it follows that the difference of the squares of the sides of the triangle, is equal to the difference of the squares of the segments of the base; and therefore the difference of the squares of one side and the adjacent segment of the base, is equal to the difference of the squares of the other side and its adjacent segment of the base. Whence it follows, that the line drawn from the vertex of the triangle to the point of section of the base, is perpendicular to the base.

25. Let BCDE be the square on the side BC of the isosceles triangle ABC. Then by Euc. VI. 2, FG is proved parallel to ED or BC.

26. Let the n^{th} part of the given line be cut off by Euc. VI. 9, then by Euc. II. 1.

27. The case in which the two angles are equal is proved in Euc. VI. 15, the case where one angle is the supplement of the other offers no difficulty.

28. Produce EG, FG to meet the perpendiculars CE, BF, produced, if necessary; the demonstration is obvious.

29. Let the perpendiculars from B, C the angles at the base, meet the line bisecting the vertical angle A in E, F; and let the line bisecting the vertical angle, meet the base in D. Then twice the area of the triangle ABC is the sum of the rectangles contained by AD, BE and AD, CF. The triangles AFC, AEB are equiangular, as also the triangles CFD, BED.

30 This property is a particular case of Euc. VI. 2.

31. The lines joining the bisections of every two sides may be proved parallel to the remaining side of the triangle, and the equality of the triangles may be inferred from Euc. I. 38.

32. Draw AF, and the triangle AFC is equal to the triangle ABD; therefore the ratio of the triangle FCE to ABD is known. The numerical value of the ratio may be found from the note on Euc. II. 11, p. 72.

The second property is obvious from the similarity of the triangles.

33. This property may be deduced directly from Euc. VI. B, 3.

34. This property may be immediately deduced from Euc. VI. 8, Cor.

35. From D draw DE perpendicular to AB, then DE is equal to DC.

$$\text{And by Euc. VI. 3, 4, } DC : AC :: BE : BC.$$

$$\text{Whence may be shewn, } AC^2 : AD^2 :: BC^2 : BE^2 + BC^2:$$

$$\text{also } BE^2 = BD^2 - DE^2 = BD^2 - DC^2 = (BD + DC).(BD - DC) = BC.(BD - DC).$$

$$\text{Whence it follows that } AC^2 : AD^2 :: BC^2 : BC\,(BD - DC + BC) :: BC : 2.BD.$$

36. Let the line DF drawn from D the bisection of the base of the triangle ABC, meet AB in E, and CA produced in F. Also let AG drawn parallel to BC from the vertex A, meet DF in G. Then by means of the similar triangles; DF, FE, FG may be shewn to be in harmonic proportion.

37. See Euc. VI. A, note, p. 204.

38. Produce AC to G. Then the angle BCG is bisected by CF, Euc. VI. A, and the angle ACB by CE. Euc. VI. 3.

39. The angle at A the centre of the circle (fig. Euc. IV. 10) is one tenth of four right angles, the arc BD is therefore one tenth of the circumference, and the chord BD is the side of a regular decagon inscribed in the larger circle. Produce BC to meet the circumference in F and join BF, then BF is the side of the inscribed pentagon, and AB is the side of the inscribed hexagon. Join FA. Then FCA may be proved to be an isosceles triangle and FB is a line drawn from the vertex meeting the base produced. If a perpendicular be drawn from F on BC, the difference of the squares of FB, FC may be shewn equal to the rectangle AB, BC, (Euc. I. 47; II. 5, Cor.); or the square of AC, Euc. IV. 10.

40. The triangles formed by drawing the successive perpendiculars may be shewn to be equiangular, and each equiangular to the original triangle.

41. Draw the perpendicular CE from C on the base AB.

Then $CB^2 = CD^2 + BD^2 + 2 . BD . DE$. Euc. II. 12.

and $AC^2 = CD^2 + AD^2 - 2 . AD . DE$. Euc. II. 13.

Multiply the former by AD, and the latter by BD, and add the results,

$$\text{and } AD . BC^2 + BD . AC^2 = CD^2 (AD + BD) + AD . BD (BD + AD)$$
$$= CD^2 . AB + AD . BD . AB.$$

This result is Prop. II. of Matthew Stewart's General Theorems.

The Analytical Discussion of Matthew Stewart's General Theorems by T. S. Davies, Esq., F.R.S., will be found at p. 573, &c. Vol. XV. of the Transactions of the Royal Society of Edinburgh.

42. The segments cut off from the sides are to be measured from the right angle, and by similar triangles are proved to be equal; also by similar triangles, either of them is proved to be a mean proportional between the remaining segments of the two sides.

43. The triangles HCF, ABF may be shewn to be equiangular.

44. Draw FG perpendicular to BA, and FH perpendicular to CE. From the similar triangles AED, AGF, BFG, BCE; DE : CE :: BG : AG.

But $BG = AE - EG = AE - FH$, and $AG = AE + EG = AE + FH$. And from the similar triangles CFH, CEA, AC : CF :: AE : FH. Whence may be deduced the proportion DE : EC :: AC − CF : AC + CF.

45. This theorem is the same as theorem 19, p. 354, under a slightly modified form of expression.

46. Assuming the truth of Theorem 35, page 355, namely,

$$AC^2 : AD^2 :: BC : 2 . BD; \quad \therefore \ 2 . AC^2 : AD^2 :: BC : BD,$$

whence $2 . AC^2 - AD^2 : AD^2 :: BC - BD : BD$,

$$\text{and since } 2 . AC^2 - AD^2 = 2 . AC^2 - (AC^2 + DC^2) = AC^2 - CD^2,$$

the property is immediately deduced.

47. The enunciation of this theorem does not seem definite.

48. See Theorem 2, page 305.

49. Each of the lines may be proved to be divided at the point of intersection in the ratio of 2 to 1.

50. This theorem bears the same relation to Euc. VI. B as Euc. VI. A does to Euc. VI. 3. Describe a circle about the triangle ABC, produce EA to meet the circumference again in F and join FC. Then by the similar triangles BEA, FCA; the rectangle BA, AC is equal to the rectangle AE, AF. By Euc. III. 36, Cor. the rectangle FE, EA is equal to the rectangle BE, EC. And since FE is divided in A, Euc. II. 3, the rectangle FE, EA is equal to the rectangle EA, AF together with the square of AE. Hence, &c.

51. In the figure, Theorem 45, p. 302, draw PQ, PR, PS perpendiculars on AB, AD, AC respectively: then since the triangle PAC is equal to the two triangles PAB, PAD, it follows that the rectangle contained by PS, AC, is equal to the sum of the rectangles contained by PQ, AB, and by PR, AD. When is the rectangle by PS, AC, equal to the *difference* of the other two rectangles?

52. Suppose Bfh to intersect EF, EG, EH in f, g, h, and to meet EK in K. Through h draw hkl parallel to EB or AC and km parallel to EF. Then by means of the similar triangles may be proved, that Bf is to fh as fg is to $2mg$. Whence Bf is to Bh as fg is to gh, since $2mg$ is the difference between fg and gh.

Again, draw Hn parallel to Gg, and by a similar process is proved, Bg is to BK as gh to hK. The line Bfh might be drawn so as to meet any one of the equidistant points in the given line AC.

53. The angles made by the four lines at the point of their divergence remain constant. See Note on Euc. VI. A, p. 203.

54. The triangles AEC, CBE may be shewn to be equiangular. Then Euc. VI. 4.

55. This is an extension of theorem 49, page 356. If the base BC of the triangle ABC be produced to E, so that CE is equal to BC, and AC be bisected in D, then if BD be joined and produced to meet AE in F, then AF is one half of FE.

56. If the vertex of the triangle be in one of the sides, the inscribed square is

greatest when its altitude is greatest, or when the base of the triangle coincides with the base of the square. Every other inscribed triangle may be shewn to be less than this triangle.

57. Let ABCDEF be any hexagon inscribed in a circle, and let the opposite sides AB, DE; BC, FE; CD, FA when produced meet in P, Q, R respectively. Produce AB, DC to meet in a, and EF both ways to meet BAP in b and CDR in c. Then by Euc. III. 36, and considering abc as a triangle intersected by the transversals PED, QCB, RFA respectively: it may be proved that $aP.bQ.cR = Pb.Qc.Ra$, which is the condition fulfilled when a straight line intersects the three sides ab, ac, bc of the triangle produced, in the points P, Q, R. See Appendix, p. 20.

58. The hexagon is not necessarily regular. By joining the points of contact, the three diagonals may be proved to intersect in one point by means of poles and polars.

59. The square inscribed in the circle may be shewn to be equal to twice the square of the radius, and five times the square inscribed in the semicircle, to four times the square of the radius.

60. The three triangles formed by three sides of the square with segments of the sides of the given triangle, may be proved to be similar. Whence by Euc. VI. 4, the truth of the property.

61. By constructing the figure, it may be shewn that twice the square inscribed in the quadrant is equal to the square of the radius, and that five times the square inscribed in the semicircle is equal to four times the square of the radius. Whence it follows that, &c.

62. By Euc. I. 47, and Euc. VI. 4, it may be shewn that four times the square of the radius is equal to fifteen times the square of one of the equal sides of the triangle.

63. See Problem 30, page 334.

64. The triangle cut off may be proved to be one-fourth of the given triangle.

65. In the figure Euc. IV. 7. Draw AD, then AD is the side of the inscribed square. Join EF, meeting AD in L, and the circumference in M, and draw AM; AM is the side of the inscribed octagon. Then it may be shewn that $A_1 : A :: EL : EA$, and $A : A_2 :: EA : EF$; also since the triangles ELA, EFA are similar, $EL : EA :: EA : EF$. Whence it follows that, &c.

66. The intersection of the diagonals is the common vertex of two triangles which have the parallel sides of the trapezium for their bases.

67. From one of the given points two straight lines are to be drawn perpendicular, one to each of any two adjacent sides of the parallelogram; and from the other point, two lines perpendicular in the same manner to each of the two remaining sides. When these four lines are drawn to intersect one another, the figure so formed may be shewn to be equiangular to the given parallelogram.

68. Suppose the side AD greater than BC, and from B, C, draw BE, CE′ perpendiculars on AD. Then by the right-angled triangles, &c.

69. Let AK be any rectangle contained by two lines AB, BC, and CK, AE the squares upon BC, AE. Then by Euc. VI. 1.

70. This is only another form of stating the general property of three lines in harmonical proportion. It may be deduced from the note to Euc. VI. A, page 203.

71. There appears to be some inaccuracy in the enunciation: for four lines PA, PB, PC, PD in harmonical proportion cannot be drawn from a point P to meet a straight line in four points A, B, C, D taken in order.

72. See Appendix, p. 20.

73. Let the figure be drawn, and let HP, KQ, LR, when produced, meet in O: also let KH, PQ meet in M; HL, PR in N; KL, QR in S. Then, since the triangles HKO, KLO, HLO, are cut by the transversals, PQM, QRS, PRN respectively; therefore

$$\frac{KQ}{QO}\cdot\frac{OP}{PH}\cdot\frac{HM}{MK}=1;\quad \frac{KQ}{QO}\cdot\frac{OR}{RL}\cdot\frac{LS}{SK}=1;\quad \frac{LR}{RO}\cdot\frac{OP}{PH}\cdot\frac{HN}{NL}=1;$$

whence is deduced $\frac{HM}{MK}\cdot\frac{KS}{SL}\cdot\frac{LN}{NH}=1$; which is the condition fulfilled when the three sides of the triangle HKL are produced, and are cut by a transversal in the points M, N, S. Hence the points M, N, S are in a straight line.

74. Let a point m be taken in CD, and mpq be drawn parallel to AB, and

intersecting FE, HG, in p, q: through q, let rqs be drawn parallel to CD, and intersecting EF, AB in r, s; then $mp : pq :: sq : qr$, may be proved from the similar triangles. In the same manner, if through s, tsv be drawn parallel to EF, and intersecting GH, CD in t, v, then $qs : qr :: ts : sv$, and similarly if a line be drawn through v parallel to the line next in order, &c.

75. See note to Euc. VI. A, page 203.

76. Let ABC be the triangle, M the middle point of BC, and AD = 2 . DM; draw AK, PL parallel to the base, the former meeting EF produced in K, and the latter through D meeting AB and AC in P and L. Then PD = PL, and by the similar triangles KE . FD = KF . ED, or DF . (KD − DE) = (KD + DF) . DE; whence is deduced the equality required.

77. This is the same property as that enunciated in Theorem 53, page 356, in a slightly altered form of expression.

78. Let AB, CD intersect each other in E, and be terminated by two unlimited lines given in position: and let ab, cd be drawn parallel to AB, CD respectively, intersecting each other in e, and also terminated by the two given lines. Then by the similar triangles and the composition of the ratios.

79. Let PA, PB, PC, PD be four straight lines drawn from P, and let mpq be drawn parallel to PA and meeting PB, PC, PD in m, p, q, so as to be bisected by PC in p. Through p draw any line EFpG meeting the other three lines in E, F, G. Then EG is divided harmonically in F, p.

80. By converting the proportion by Euc. VI. 16, and observing that

DB = DC + CB, CB = AC + 2 . CE and AD = DC − AC.

81. Constructing the figure, the right-angled triangles SCT, ACB may be proved to have a certain ratio, and the triangles ACB, CPM in the same way, may be proved to have the same ratio.

82. Draw DG perpendicular on AE. The triangle CDB is isosceles and DF is drawn from the vertex perpendicular on the base: also the triangles DFB, DAB are equal in all respects. Hence CF, FB, AB, are equal to one another, and AB is half of BC. Similarly BE is half of AB. Then from the similar triangles AGD, FDB, the property may be deduced.

83. See Note Euc. VI. A, p. 203. The bases of the triangles CBD, ACD, ABC, CDE may be shewn to be respectively equal to DB, 2 . BD, 3 . BD, 4 . BD.

84. The triangles DOE, EOB are readily proved to have the same ratio as the triangles EOB, BOA by Euc. VI. 1.

85. This property follows as a corollary to Euc. VI. 23, for the two triangles are respectively the halves of the parallelograms, and are therefore in the ratio compounded of the ratios of the sides which contain the same or equal angles: and this ratio is the same as the ratio of the rectangles by the sides.

86. Let the figure be constructed, then from the three similar right-angled triangles, Euc. VI. 19.

87. Since the three rectangles are equal, AB . AG = CD . CH = EF . EK. Hence AB : CD :: CH : AG and CD : EF :: EK : CH. Then supposing EK − CH = CH − AG, there may be deduced AB − CD : CD − EF :: AB : EF. And conversely. See Note on Def. III. and Prop. A, p. 203.

88. Every triangle may be shewn to be four times the area of the triangle about which it is described.

89. Let C, C′ be the centres of the two circles, and let CC′ the line joining the centres intersect the common tangent PP′ in T. Let the line joining the centres cut the circles in Q, Q′, and let PQ, P′Q′ be joined: then PQ is parallel to P′Q′. Join CP, C′P′, and then the angle QPT may be proved to be equal to the alternate angle Q′P′T.

90. Let C, C′, C″ be the centres of the three circles; C is the centre of the largest, C″ of the smallest. Let the tangents to the circles whose centres are C, C′; C, C″; C′, C″ meet in A, B, C respectively. Join the points A, B, C; then AB shall be in the same straight line as BC. Join C, C′, C″ and produce CC′, CC″, C′C″ these lines meet the tangents in A, B, C respectively. Through C′ draw C′E parallel to AB, then BC may be proved also parallel to C′E.

91. Let the chord AB be bisected in E by the chord CD. Let the tangents at A, B meet in P, and the tangents at C, D meet in Q, join PQ, and PQ is parallel

to AB. Join PE and produce it to the centre O, also join OQ cutting CD in F. Draw the radii OC, OA. Then the triangles OFE, OPQ are equiangular and right-angled, also the right angle BEP is equal to the alternate angle EPQ.

92. Let two circles whose centres are C, C′ cut one another. Let any two points P, Q, be taken in the circumference of one, and tangents be drawn at P, Q. Take P′, Q′ points in the circumference of the other, such that the tangents at P′, Q′ may be parallel to the tangents at P, Q. Draw PP′ intersecting CC′ in D. Join QD, Q′D. If QDQ′ can be proved to be a straight line, then QQ′, and PP′ intersect CC′ in the same point.

Join PC, P′C′, and by the similar triangles PCD, P′C′D the other property is deduced.

93. Join AB, and divide it in C so that AC is to BC as AP to BP: and if in AB produced, ED be taken to BD, as AP to BP; the point D may be proved to be the centre of the circle which is the locus of the point of intersection of the two lines. If the lines be equal, the locus is a straight line. Give the analysis of both cases.

94. The arithmetical ratio of $r.r'$ to $\frac{1}{3}.a^2$ may be deduced from Theorem 137, p. 362, Euc. IV. 4, and note to Euc. II. 11, p. 72.

95. This Theorem is the same as Theorem 123, p. 361.

96. By means of Euc. IV. 4, and Theorem 137, p. 362, this Theorem may be shewn to be true.

97. This is the same as Theorem 103, p. 360, under a slightly varied form of expression.

98. Let a tangent be drawn to touch the circle at P, and let PM be drawn perpendicular to the diameter ACB, C being the centre of the circle. At A, C, B, draw lines perpendicular to the radius meeting the tangent at P in A′, C′, B′. Then AA′, MP, CC′, BB′ are proportional. Draw A′R, PQ parallel to AB and meeting PM, BB′ in R, Q respectively. Then by the similar triangles PA′R, B′PQ, the required proportion may be deduced, observing that A′A is equal to A′P; B′P to B′B, and CC′ an arithmetic mean between AA′ and BB′.

99. The lines so drawn may be proved by Euc. VI. 3, to be proportional to the segments of the base of the triangle SEL, Theorem 123, p. 361.

100. Let any tangent to the circle at E be terminated by AD, BC tangents at the extremity of the diameter AB. Take O the centre of the circle and join OC, OD, OE: then ODC is a right-angled triangle and OE is the perpendicular from the right angle upon the hypothenuse.

101. Let BA, AC be the bounding radii, and D a point in the arc of a quadrant. Bisect BAC by AE, and draw through D, the line HDGP perpendicular to AE at G, and meeting AB, AC, produced in H, P. From H draw HM to touch the circle of which BC is a quadrantal arc; produce AH, making HL equal to HM, also on HA, take HK equal to HM. Then K, L, are the points of contact of two circles through D which touch the bounding radii, AB, AC.

Join DA. Then, since BAC is a right angle, AK is equal to the radius of the circle which touches BA, BC in K, K′; and similarly, AL is the radius of the circle which touches them in L, L′. Also, HAP being an isosceles triangle, and AD is drawn to the base, AD^2 is shewn to be equal to AK . KL. Euc. III. 36; II. 5, Cor.

102. The radius of the second circle is half the radius of the first, the radius of the third half that of the second circle, and so on. The radii of the second, third, &c. circles form a Geometric series.

103. Let O, O′ be the centres of the inscribed and escribed circles. Join OD_1, OB, O′B, OD_2. Then the triangles OBD_1, $O'BD_2$ may be shewn to be similar; whence may be shewn $BD_1 . BD_2 = R.r$. And by joining OE, OC, $O'C_1$, $O'E_2$, in a similar way may be shewn, $R.r = CE_1 . CE_2$.

104. By the point C is to be understood that in which BD cuts the quadrant ACB. Complete the semicircle BAG, BG being the diameter and E the centre. Join AC, AG; then by means of the quadrilateral ACBG inscribed in the circle, DCA may be shewn to be half a right angle: also ADB a right angle subtended by AB. Hence the locus of D is a semicircle; and the ratio of AB to BG may be shewn to be as $1 : \sqrt{2}$.

105. Through E one extremity of the chord EF, let a line be drawn parallel to

one diameter, and intersecting the other. Then the three angles of the two triangles may be shewn to be respectively equal to one another.

106. This is a repetition of Theorem 99, p. 359, under a somewhat different form of expression.

107. Let the tangents at A and C, A and B, B and C, meet in the points G, H, K, respectively.

Since the transversals FC, DA, EA intersect the triangle HKG,

$$\frac{GF}{FH}\cdot\frac{HB}{BK}\cdot\frac{KC}{CG}=1;\quad \frac{HA}{AG}\cdot\frac{GD}{DK}\cdot\frac{KB}{BH}=1;\quad \frac{HE}{EK}\cdot\frac{KC}{CG}\cdot\frac{GA}{AH}=1;$$

and observing that HA = HB, KB = KC, GC = GA;

$$\text{hence } \frac{GF}{FH}\cdot\frac{HE}{EK}\cdot\frac{KD}{DG}=1,$$

which is the condition fulfilled when a transversal intersects the three sides AB, AC, CB produced of the triangle, in the points D, E. F. (See Appendix, p. 22.)

108. It may be proved that the vertices of the two triangles which are similar in the same segment of a circle, are in the extremities of a chord parallel to the chord of the given segment.

109. This is the converse to Euc. VI. D, and may be proved indirectly.

110. Since the lines joining B, C, D, are equal, the consequence is obvious from the Proposition.

111. Perhaps the simplest mode of shewing the truth of this property is by means of transversals. The triangles CBP, BAQ are cut by the transversals ADQ, CDP, respectively.

Whence BA . PD . CQ = AP . DC . QB, and BC . QD . PA = CQ . DA . PB: from which the required proportion may be deduced.

112. Let the line AD drawn from the vertex A, meet the base of the isosceles triangle ABC in D, and let AD produced meet the circumference of the circumscribed circle in E. Then by Euc. II. 3; III. 35, and Theorem 27, p. 309.

113. This follows at once from Euc. III. 36, 37.

114. Correct the enunciation thus:—Let AD, FC, meet in P, and AE, BK in Q: then the points P, Q, B, C, E, D are in the circumference of one circle. For let FC meet AB in R. Then it has been proved, Euc. I. 47, that the angle BFR is equal to RAP. Also the angles FRB, ARP are equal; wherefore the angles FBR, APR are equal, and hence APR is a right angle. Whence, again, DPC is a right angle, and equal to DBC. It is hence in the same semicircle on DC; that is, in the circle BCED. In the same manner Q may be shewn to be in the same circle BCDE.

115. Let ABC be a triangle, and let the line AD bisecting the vertical angle A be divided in E, so that BC : BA + AC :: AE : ED. By Euc. VI. 3 may be deduced BC : BA + AC :: AC : AD. Whence may be proved that CE bisects the angle ACD, and Euc. IV. 4, that E is the centre of the inscribed circle.

116. For "bisected," read "divided into parts, one of which is double the other, the smaller segment being estimated from the centre of the circle."

Let ABC be the triangle; Q the centre of the circumscribing circle; P the intersection of the perpendiculars BG, CH: D, E the middle points of BA, CA; divide PQ in R, so that PR = 2 . QR; and join BR, RE: also draw DQ, QE, ED. Then the triangles BPC, EQD may be shewn to be equiangular, and hence BP = 2 . QE.

Again, PQ meeting the parallels QE, BP, the angles RQE, RPB are equal; and by hypothesis RP = 2 . RQ: whence the sides about the equal angles are proportional, that is, EQ : QR :: BP : PR, and the angles QRE, PRB are equal. The points B, R, E are therefore in one line. The same triangles give PR : RQ :: BR : RE, and hence BR = 2 . RE; or the point R is distant from B, two-thirds of the line BE drawn to the middle of the opposite side AC.

117. If the extremities of the diameters of the two circles be joined by two straight lines, these lines may be proved to intersect at the point of contact of the two circles; and the two right-angled triangles thus formed may be shewn to be similar by Euc. III. 34.

118. From the centres of the two circles let straight lines be drawn to the extre-

mities of the sides which are opposite to the right angles in each triangle, and to the points where the circles touch these sides. Then by similar triangles.

119. Let DB, DE, DCA be the three straight lines, fig. Euc. III. 37; let the points of contact B, E be joined by the straight line BC cutting DA in G. Then BDE is an isosceles triangle, and DG is a line from the vertex to a point G in the base. And two values of the square of BD may be found, one from Theo. 27, p. 309; Euc. III. 35; II. 2; and another from Euc. III. 36; II. 1. From these may be deduced, that the rectangle DC, GA is equal to the rectangle AD, CG. Whence the, &c.

120. Let the arc AE be double the arc AB of the circle whose centre is C. Let CD, CF, be the perpendiculars on the chords of the arcs AB, AE. Produce CF to meet the circumference in B and G, join GA and draw CH perpendicular to GA. The proportion is deduced from the similar triangles CBD, GFA.

121. Draw FG to bisect the angle DFE, and draw DK, EH perpendicular on FG; and let FK meet AB in G.

$$\text{Then } 2.GB : BF :: 2.HE : FE, \text{ and } 2.AG : AF :: 2.DK : FD;$$

by similar triangles: and by compounding these proportions, observing that AF = FB, AG = GB, and 4.HE.DK = 4.DC.CE, there results

$$AB^2 : AF^2 :: 4.EC.CD : FD.FE; \text{ similarly } BC^2 : CE^2 :: 4.FA.AD : FE.ED;$$

$$\text{whence } AB^2 : BC^2 :: DE.FA : EC.DF.$$

122. This is manifest from Euc. III. 36, 37.

123. Join OE. Then OE is equal to OA, and Euc. VI. 6, the triangles OES, OLE are equiangular. Whence it may be shewn that the angle SEL is bisected by EA.

124. Let BD touch the inner semicircle in E, and let O be its centre. Join OE. The triangles DAB, EOB are equiangular.

125. Let ABC be the triangle, and F a point in its base BC;

let the circles AFB, AFC be described, and their diameters AD, AE, be drawn;

$$\text{then } DA : AE :: BA : AC.$$

For join DB, DF, EF, EC, the triangles DAB, EAC may be proved to be similar.

126. In the enunciation, for "two circles" read "two equal circles, whose centres lie on opposite sides of the line ABCD."

The proof offers no difficulty. In every other case the theorem does not hold good.

127. Let the figure be constructed, and the similarity of the two triangles will be at once obvious from Euc. III. 32; Euc. I. 29.

128. Let the figure be drawn, and BC, CD, BD be joined. Then ABCD is a quadrilateral figure inscribed in a circle, and BD, AC are the diagonals. By Euc. VI. D, 17, the first proportion is deduced; and the other in a similar way.

129. Let the figure be drawn, and join HI. Then EF, HI are parallel to KN, a side of the triangle BKN. Euc. III. 37; VI. 2.

130. Let O be the centre of the inscribed circle DEF, and P that of the escribed circle HIK; these are in the line bisecting the angle C. Join MB, LA cutting COP in N and R; draw the several radii to the points of contact; and join OA, OB, PA, PB. Then prove that FK is equal to the difference of the sides AC, CB; and therefore to AM. Next, the lines BM, AL are perpendicular to CP, which bisects the common vertical angle, and CNB, CRL are right angles, as are also the angles made by OF, AB. Describe semicircles about ONFB and OFRA, and join NF, RF. Then the angle AFR = AOR = BOF = BNF; and the alternate angles FAR, FBN are equal. The triangles AFR, BNF are therefore equiangular, and AR : AF :: FB : BN; also 4.AF.FB = 4.AR.BN = AL.BN.

131. By Euc. IV. 4, twice the area of the triangle is equal to the rectangle contained by the sum of the sides and the radius of the inscribed circle. By Theorem 137, p. 362, the area is expressed in terms of the sides and the radius of the circumscribed circle. Whence the property required may be deduced, observing that one of the sides of the triangle is half the sum of the other two sides.

132. Since the line mnp is a transversal to the triangle ABC; $An.Cp.Bm = nC.pB.mA$; and by Euc. III. 36, the values of t_p^2, t_m^2, t_n^2 may be expressed in terms of An, nC, &c.: whence the property may be deduced.

133. Let A be the centre of the circle whose circumference passes though B the centre of the other circle, and CD the line which joins the intersections of the two

circles; draw BF cutting the first circle in F and the second in E, and draw EG perpendicular to CD: then FE : EG is a given ratio.

For, join BC, CF, FD, CE, and draw EP perpendicular to CF. The angle FCE may be shewn to be equal to the angle DCE; or EC bisects the angle FCD, and hence EG is equal to EP. But the angle FEG being given, and EPF being a right angle, the ratio FE to EP is given; that is, the ratio FE to EG is given.

134. See Theorems 153, 154, infra.

135. Let the sides of the triangle ABC be divided in the ratio of n to 1 in the points D, E, F. Join DE, EF, FD. Then the ratios of each of the triangles ADF, BDE, CEF to the triangle ABC may be found by Theorem 85, p. 358, in terms of n, whence also the ratio of the triangle DEF to the triangle ABC in terms of n.

136. Let the chords AB, CD intersect each other in E, so that AE is to EB as CE to ED. Then it may be shewn that the lines joining DB, AC are parallel, and that the line bisecting the angle at E bisects these parallels.

137. By Euc. VI. E. $BA.AC = EA.AD$. Multiply these equals by BC, and interpret the result.

138. For let the circle be described about the triangle EAC, then by the converse to Euc. III. 32, the truth of the proposition is manifest.

139. The triangles ABC, ADB may be shewn to be equiangular.

140. Let the figure be constructed as in Theorem 3, p. 313, the triangle EAD being right-angled at A, and let the circle inscribed in the triangle ADE touch AD, AE, DE in the points K, L, M respectively. Then AK is equal to AL, each being equal to the radius of the inscribed circle. Also AB is equal to GC, and AB is half the perimeter of the triangle AED.

Also if GA be joined, the triangle ADE is obviously equal to the difference between the figure AGDE and the triangle GDE, and this difference may be proved equal to the rectangle contained by the radii of the two circles.

141. Let the figure be constructed, then from the isosceles triangles, ED is shewn to be equal to EA, and EG to EB. Then Euc. VI. 13.

142. Let FG join the intersections of the circles, and cut AE in C.

Then, $AC.CD = FC.CG = BC.CE$, or, $AC : CE :: BC : CD$; whence,

$$AC + CE : BC + CD :: AC : CB, \text{ and } AC + CE : BC + CD :: CE : CD;$$

compounding these proportions and putting for $AC + CE$ and $BC + CD$, their equals, we have $AE^2 : BD^2 :: AC.CE : BC.CD$.

143. Let ABC be any triangle, and let D, E be the centres of the circumscribed and inscribed circles respectively. Join AD, and through D draw the diameter FDG and join AE; AE produced meets the diameter in F. Draw EH perpendicular to AC and join DE, EC, FC, CG. Then FC is equal to FE, and by Theorem 27, p. 309, $DE^2 = DA^2 - AE.EF = DA^2 - AE.FC$; also the triangles AEH, GFC being similar, $AE.FC = GF.EH$. Whence the truth of the theorem may be shewn.

144. This property follows directly from Euc. VI. C.

145. Join DC, then in the triangles ADB, ADC, the angle ACD is equal to the angle ADB, both standing upon equal arcs of the same circle, also the angle DAB is common to the two triangles. Hence the triangles are equiangular, and by Euc. VI. 4, the property is manifest.

146. Let ACB be the common diameter of the two circles which touch each other in the point A, and through C the centre of the smaller circle, let PP′ be drawn perpendicular to AB, and meeting the inner circle in Q, Q′: also let the tangents from P, P′ touch the inner circle in T, T′. Join CT, CT′. Then PT, P′T′, may be proved each equal to CT, CT′.

147. Let the circles cut each other in A, B, join AB, and on AB as a diameter describe a circle cutting the two given circles, and from A draw a straight line ACDE meeting the circumference in C, D, E. From B the other extremity of the diameter, draw BF, BG perpendicular to AB and meeting the circumferences of the two given circles in F, G. Then CD is to DE as BG to BF. The triangles CDB, ABG are similar, as also the triangles BED, ACF.

148. Half the difference between the sums of the opposite sides is equal to the distance between the points where the two circles touch one of the sides of the figure. This distance may be proved to be a mean proportional between the diameters of the circles.

149. Let the diagram be constructed according to the enunciation; and let PP′ be the common tangent; draw PS parallel to OO′, (O, O′ being the centres) meeting OP in S; produce PO, P′O′ to meet DE, D′E′ in Q, Q′. Then the triangles ABC, ABC′ are equiangular, whence BC : BA :: BA : BC′. Again join DD′ and bisect DD′ in M, and make Mm = half the difference of the sides AD, AD′. Then DE.D′E′, and $P'P^2$, may be each shewn to be equal to $4r^2-(R-r)^2$, R, r being the radii of the two circles.

150. Let AB be a line given in position, and P the given point. At P make the angle APD equal to the given angle, taking the point D, so that AP is to PD in the given ratio.

Draw a line through the point D making with PD an angle equal to the angle PAB. Then this line is the locus of the extremity D of the line PD. This may be proved by taking another point A′ in the given line, and making the angle A′PD′ equal to the angle APD. Then the triangles A′PA, D′PD may be shewn to be similar.

151. By constructing the figure and joining AC and AD, by Euc. III. 27, it may be proved that the line BD falls upon BC.

152. Join AG, AF, AD. Then, since the circles are equal, the segments AFB, AGB contain supplementary angles. Whence the angles AFG, AGF are equal, and AF is equal to AG. Again, AEB is a right angle; and hence AE being perpendicular to the base GF of the isosceles triangle, bisects FG. See Prob. 23, p. 316.

In the enunciation, "a third circle drawn with centre A," &c. appears to be superfluous.

153 and 154 may be taken together, with a few other properties, some of which, however, have been noticed in other places.

(1) Construct according to the enunciation, and complete the diameter AB through A; since
OS : OA :: OA : OL, we have OA − OS : OS + OA :: OL − OA : OL + OA, or since OA = OB, this becomes AS : SB :: AL : LB, which expresses that LB is harmonically divided in S and A.

(2) Join PO, QO; since PO = OA, therefore OS : OP :: OP : OL; and the triangles POS, LOP having the angle at O common and the sides about that angle proportionals, they are similar. Whence OS : OP or OA :: SP : PL; or the ratio of SP : PL is constant. In like manner the ratio SQ : QL is also constant, and the same as SP : PL. This is Theorem 154, (1).

(3) Join AP, BP; then since OS : OA :: OA : OL, hence

OS : OA :: OA − OS : OL − OA :: SA : AL,

OS : OA :: OA + OS : OL + OA :: BS : BL.

Whence SA : AL :: SP : PL, and SB : BL :: SP : PL; and the lines AP, BP bisect the interior and exterior angles SPL, SPQ of the triangle SPL at P.

Similarly, QA, QB bisect the exterior and interior angles of the triangle SQL at Q.

(4) Let the perpendicular at S produced meet the circumference in C, D: and join OC, CL: since $OS.OL = OA^2 = OC^2$, and OSC is a right angle, it follows that OCL is also a right angle, and that LC is a tangent to the circle at C.

In the same way it may be shewn that LD is a tangent at D. Whence the tangents at C and D meet the diameter AB produced in the same point.

(5) By the right-angled triangles OCL, CLS; $OL.LS = CL^2 = PL.LQ$; or the four points P, Q, O, S are in the circumference of a circle. Whence the exterior angle PSA of the quadrilateral is equal to the opposite angle OQP. But by the similar triangles OQS, OLQ, the angle OSQ = OQL. Whence QSP = ASP; and QSC = PSC.

(6) Produce QS to meet the circle in F: then ASF = QSB = QSA; and hence SP = SF; wherefore $PS.SQ = SF.SQ = SC^2$, a constant magnitude. This is 154, (2).

(7) Since SE bisects the interior angle PSQ, and SL the exterior angle PSF, of the triangle PSQ, PE : EQ :: PS : SQ :: PL : LQ; or LQ is harmonically divided in P, E. This is Theorem 153.

(8) Produce AP, BQ to meet in G, and let AQ, BP meet in H; then G, H will be in the line CD. For in the triangle PSQ, the three lines QA, BP, SC have been shewn to bisect the angles; wherefore these lines meet in a point.

Also, SC bisects PSQ, and AP, BQ bisect the exterior angles at P and Q; therefore they also meet in a point. Whence G, H are in the line CD.

(9) The lines BG, AG, are bisected in K and I by the circle which passes through the points P, S, O, Q.

(10) Let the circle PSOQ cut SG in M; and draw MO, MQ, MP.

Then since SM bisects the angle QSP, it bisects the circumference QMP on which QSP stands; and hence MQ = MP. Also, since QO = OP, it follows that MO is perpendicular to QK, and is a diameter of the circle PSOQ. Whence OQM, OPM are right angles. But OQ, OP are radii of the circle BQPA, and hence QM, PM are tangents at Q and P; and they meet at M in SC produced. Wherefore tangents at P and Q always meet in the line SC produced. This is 154, (3).

(11) If any chord QSF be drawn through the pole S, and QL, FL be drawn; then the angle SLF = SLQ.

For, join SP: then QS . SP = AS . SB = QS . SF: whence SF = SP, and the triangles PLS, FLS are equal in all respects, and hence the angle SLF = SLQ.

(12) Conversely. If QF be drawn through S, and lines be drawn from Q, F to make equal angles with LV drawn through L; the line which bisects the angle FLQ is a diameter passing through S.

Note. The line SG is called the *polar*, and L the *pole;* as are also the line LV and the point S, so called respectively. Taken together, either point and its respective line are called *reciprocal polars;*—as for instance SG and L.

The characteristic property of the *pole* and *polar* to which it is most convenient to refer, is, that if the diameter of a circle AB be produced and be harmonically divided in S and L; then a perpendicular to AB through S is the polar of L, and a perpendicular to AB through L is the polar of S.

155. Let AB be that diameter of the given circle which when produced is perpendicular to the given line CD, and let it meet that line in C; and let P be the given point: it is required to find D in CD, so that DB may be equal to the tangent DF.

Make BC : CQ :: CQ : CA, and join PQ; bisect PQ in E, and draw ED perpendicular to PQ meeting CD in D: then D is the point required. Let O be the centre of the circle, draw the tangent DF; and join OF, OD, QD, PD. Then QD may be shewn to be equal to DF and to DP.

When P coincides with Q, determined as in the construction, *any* point D in CD fulfils the conditions of the problem: that is, there are innumerable solutions.

156. Apply the method of transversals. See Appendix, p. 15, &c.

157. By assuming Euc. VI. 1, it may be shewn that

$$\frac{\text{figure ABPCA}}{\text{triangle AMN}} = \frac{\text{MP}}{\text{PN}} + \frac{\text{PN}}{\text{MP}}.$$

Also by theorem 85, p. 358, that $\dfrac{\text{triangle ABC}}{\text{triangle AMN}} = 2 + \dfrac{\text{MP}}{\text{PN}} + \dfrac{\text{PN}}{\text{MP}}$. Whence it follows that the triangle BPC is twice the triangle AMN. What is the fixed point?

158. It may be shewn by the theory of transversals, that when the three lines drawn from the angles to the opposite sides of a triangle, pass through any point P, $\text{A}b . \text{B}c . \text{C}a = a\text{B} . b\text{C} . c\text{A}$. By means of Euc. III. 36; it may be shewn, that

$$\frac{\text{A}b'}{b'\text{C}} \cdot \frac{\text{C}a'}{a'\text{B}} \cdot \frac{\text{B}c'}{c'\text{A}} = \frac{\text{A}b}{b\text{C}} \cdot \frac{\text{C}a}{a\text{B}} \cdot \frac{\text{B}c}{c\text{A}}.$$

The second part is to be shewn indirectly, by supposing one of the lines from the angle C, to intersect the opposite side in some other point c' different from c, and then shewing that c, c' coincide.

159. First. Join FA, AH: and prove that F, A, H, are in one line. Secondly. Join BG, HC; and prove that BG, HC are parallel, and GH, BC equal. Thirdly. Produce GH, BC to meet in M; and shew that the triangles BMG and CMH are isosceles. Lastly. Prove that FK bisects their bases, GB, HC at right-angles; and hence coincides with the perpendicular from M to the base in both triangles.

160. See Appendix, p. 18.

161. See Appendix, p. 16.

162. Let DG, EH, FK, be drawn perpendicular to the middle of the sides BC,

CA, AB of the triangle ABC, and each equal to half the side from the middle of which it is drawn; join GH, HK, KG, AK, KB, BG, GC, CH, and HA; produce GC, and draw HQ perpendicular to it, and AP perpendicular to BC. Then the two right-angled triangles APC, CQH, are similar; whence $AP^2 = 2 . CQ^2$; also $GC^2 = 2 . CD^2$: and Euc. II. 12, $GH^2 = GC^2 + CH^2 + 2 . GC . CQ$, which may be shewn to be equal to $\frac{1}{2} . CB^2 + \frac{1}{2} . CA^2 + 2 .$(triangle ABC). Similarly for HK^2, KG^2. Whence $GH^2 + KG^2 + HK^2$ is found.

163. Let E, F, G be the centres of the circles inscribed in the triangles ABC, ADB, ACD. Draw EH, FK, GL perpendiculars on BC, BA, AC respectively, and join CE, EB; BF, FA; CG, GA. Then the relation between R, r, r', or EH, FK, GL may be found from the similar triangles, and the property of right-angled triangles.

164. Let ABC be any triangle, and from A, B let the perpendiculars AD, BE on the opposite sides intersect in P: and let AF, BG drawn to F, G the bisections of the opposite sides, intersect in Q. Also let FR, GR be drawn perpendicular to BC, AC and meet in R: then R is the centre of the circumscribed circle. Join PQ, QR, then PQ is in the same straight line as QR.

Join FG, and by the equiangular triangles GRF, APB, AP is proved double of FR. And AQ is double of QF, and the alternate angles PAQ, QFR are equal. Hence the triangles APQ, RFQ are equiangular, therefore, &c.

165. Let ABC be a triangle, D the centre of the inscribed circle, draw DE perpendicular to the base BC, E is the point where the inscribed circle touches the base. Join CE and bisect it in F, bisect also BC in G. Then the points G, D, F may be proved to be in a straight line.

Draw GH perpendicular to BC, and DK perpendicular to GH. Join CK and produce it to meet BC in L, join also GD, DF. Then LG is equal to GE, and by the similar triangles CE may be proved to be bisected in F. Hence G, D, F must be in the same straight line.

166. This question is rather Algebraical than Geometrical. If the expression for the area of a triangle in terms of the sides may be assumed, the equation which connects the radii of the three circles may be deduced from the expression for the area of a triangle in terms of the sides and the radius of the inscribed circle, and theorem 137, p. 362.

167. From Euc. IV. 6, and Theo. 1, p. 332, the arithmetical ratio of the sides may be found: also the ratio of their areas.

168. Let a perpendicular be drawn from the centre C on ab, AB two sides (supposed parallel) of the inscribed and circumscribed polygons meeting them in d, D. Join Da: then Da is a side of a regular inscribed polygon of double the number of sides. The areas of the three polygons are the same multiples of the triangles adC, aDC, ADC, which may easily be shewn to have the proportion stated.

169. The first property may be perhaps more clearly seen by first shewing it to be true in a figure of three sides, then in one of four sides, &c.; and lastly in any polygon whatever. As all the figures described upon the parts of one side are similar to the given figure, any two sides of the given figure are proportional to the homologous sides of the smaller figures, and hence the property may be shewn respecting the sides. Also the second property may be proved by Euc. VI. 20.

170. Each pentagon can be divided into the same number of triangles, and by Euc. VI. 19.

171. The triangles which form the two regular figures are the same in number, and are similar. By the similarity of triangles the proportion is obvious.

172. The area of the inscribed equilateral triangle may be proved to be equal to half the inscribed hexagon, and the circumscribed triangle equal to four times the inscribed triangle.

173. See Theorem 168, supra.

HINTS, &c. TO THE PROBLEMS. BOOK XI.

1. SEE Géométrie par A. M. Legendre, (dixième édition), p. 156.

2. Describe a circle passing through the three given points, and from the centre draw a line perpendicular to its plane. Then every point in this perpendicular fulfils the conditions required.

3. This is an indeterminate Problem. If however, the circle be in that plane which passes through the given point, and be perpendicular to the two given planes, the problem is reduced to that of describing a circle which shall pass through a given point, and touch two given straight lines.

4. Let ACB, ADB, be the triangles, CD being perpendicular to the plane of ACB. Then the angles DAC, DBC, and the line DC being given, the lines DA, DB, can be constructed. Draw ab through C perpendicular to CD, and make the angles C′Da, C′Db, the complements of the given angles; then Ca, Cb will be of the same length as CA, CB. Whence the angle which they are to form being given, the triangle may be constructed, and its base is the line required.

5. About the given line let a plane be made to revolve, till it passes through the given point. The perpendicular drawn in this plane from the given point upon the given line is the distance required.

6. Let A, B, be the given points, and GH the given straight line; draw AC, BD perpendicular on GH, and in the plane AGH produced, draw DB′ perpendicular to GH, and equal to DB: join AB′, meeting GH in E, and draw EB. Then AE + EB is the minimum. For the triangles EDB, EB′D are equal, being right-angled at D, and having one side common, and the others equal. Whence the angle BEH is equal to GEA, each being equal to B′EH. The conclusion follows from the demonstration of Theorem 17, p. 369.

7. Through any point in the first line draw a line parallel to the second; the plane through these is parallel to the second line. Through the second line draw a plane perpendicular to the forenamed plane cutting the first line in a point.

Through this point draw a perpendicular in the second plane to the first plane, and it will be perpendicular to both lines.

8. Through any point draw perpendiculars to both planes; the plane passing through these two lines will fulfil the conditions required.

9. Let ABCD be the given triangular pyramid, of which the vertex is A, and base, the triangle BCD. Bisect BC in E, and join AE, BE: then AEB is an isosceles triangle, having its base AB equal to the side of the equilateral triangle, and its sides equal to the perpendicular from the vertex on the base. From A draw AF perpendicular to BE: AF is the perpendicular required.

Or thus. Let ABC be the equilateral triangle which forms the base of the pyramid, and D its centre. On AC describe a semicircle, and make AE equal to AD. Join AE, then AE is the perpendicular altitude.

From D draw the perpendicular DF equal to CE, and join FC, FA, FB. Then the triangle AFD is equal in all respects to the triangle ACE, and hence AF is equal to AC. In like manner the sides FC, FB are each proved equal to a side of the equilateral triangle ACB.

10. Through each line draw a plane parallel to the other; these planes will be parallel, and obviously form two of the faces of the parallelopiped. Through each line and one extremity of the other, draw a plane; and a second plane parallel to it through the remaining extremity. This will complete the figure; but there will be four varieties of cases according as the extremities are situated.

11. Every possible combination of the lines taken three at a time will form the pyramids, since the respective faces may be so formed. These according to the ordinary method are fifteen in number.

12. Bisect the base by a line drawn in the given direction, whether parallel to a given line, or tending to a given point. The plane drawn through the bisecting line and the vertex of the pyramid, gives the solution of the problem.

13. The section will be in all cases a parallelogram, but not necessarily rectangular. Any plane drawn perpendicular to a face through a line equal to the edge of the cube, will fulfil the condition.

HINTS, &c. TO THE THEOREMS. BOOK XI.

3. Let AD, BE be two parallel straight lines, and let two planes ADFC, BEFC pass through AD, BE, and let CF be their common intersection, fig. Euc. xi. 10. Then CF may be proved parallel to BE and AD.

4. See the figure Euc. xi. 10. If the lines be unequal, the figure may be shewn not to agree with the definition of a prism. Euc. xi. def. 13.

5. This theorem is analogous to Euc. xi. 8. Let two parallel lines AC, BD meet a plane in the points A, B. Take AC equal to BD and draw CE, DF, perpendiculars on the plane, and join AE, BF. Then the angles CAE, DBF, are the inclinations of AC, BD to the plane, Euc. xi. def. 5, and these angles may be proved to be equal.

6. Let lines be drawn in each plane through the points where the lines cut the planes, then by Euc. i. 29.

7. AB, BC, CD may be shewn to be three consecutive edges of a rectangular parallelopiped, of which AD is the diagonal.

8. If the intersecting plane be perpendicular to the three straight lines, by joining the points of their intersection with the plane, the figure formed will be an equilateral triangle. If the plane be not perpendicular, the triangle will be isosceles.

9. Let AB, CD be parallel straight lines, and let perpendiculars be drawn from the extremities of AB, CD on any plane, and meet it in the points A′, B′, C′, D′. Draw A′B′, C′D′, these are the projections of AB, CD on the plane, and may be proved to be parallel.

10. Let AE meet the straight lines BE, DE in the plane BED, fig. Euc. xi. 6, and let the angle AEB measure the inclination of AE to the plane BDE; then the angle AEB is less than the angle AED. Draw AB perpendicular to the plane, make ED equal to EB and join BD, AD. Euc. i. 18, 19.

11. Let the three parallel straight lines AD, BE, CF be cut by the parallel planes ABC, DEF, and A, B, C, the points of intersection of the lines, be joined, as also D, E, F: then the figure ABC may be proved to be equal and similar to the figure DEF.

12. Let AB be at right angles to the plane BCED, and let the perpendiculars from AB intersect the plane GHKL in the line MN, and let HNK be the common intersection of the planes CBDE, GHKL. Join AM, BN, and prove MN to be a straight line perpendicular to HK.

13. This may be readily proved by Euc. xi. 17.

14. Let AB, A′B′ be any portions of the two straight lines. At B′ draw B′C′ parallel to AB, and B′C perpendicular to the plane passing through A′B′C′. Let the plane passing through A′B′C intersect the line AB in the point A. In the plane A′B′C from A draw AA′ perpendicular to A′B′, and AC perpendicular to AA′. Then the plane CAB passing through the line AB may be shewn to be parallel to the plane A′B′C′ passing through the line A′B′, and that no other parallel planes can be drawn through AB, A′B′. Also AA′ is the perpendicular distance between the two planes, and that AA′ is less than any other line which can be drawn between the two planes.

15. The figure formed by lines drawn from a point above the plane of a circle to every point in its circumference is a cone. If the point be in the perpendicular to the plane drawn from the centre of the circle, the cone is a right cone, and all lines from the point to the circumference are equal; if the point be not in the perpendicular, the cone is an oblique cone and has only the two lines equal, as may readily be shewn.

16. Let BC be the common intersection of the two planes ABCD, EFGH which are inclined to each other at any angle. From K at any point in the plane ABCD, let KL be drawn perpendicular to the plane EFGH, and KM perpendicular to BC, the line of intersection of the two planes. Join LM, and prove that the plane which passes through KL, KM is perpendicular to the line BC.

17. Let GH be the edge of the wall, A, B the two points, and let the line joining A, B meet the edge of the wall GH in E. If the points AE, BE make equal

angles with GH, then AE, EB may be proved to be less than any other two lines drawn from A, B, to meet GH in any other point E′.

18. Let AB, AC drawn from the point A, and A′B′, A′C′ drawn from the point A′, in two parallel planes, make equal angles with a plane EF passing through AA′, and perpendicular to the planes BAC, B′A′C′. Let AB in the plane ABC be parallel to A′B′ in the plane A′B′C′; then AC may be proved to be parallel to A′C′.

19. Let HM be the common section of the two planes MN, MQ; and let AB be drawn from a point A in HM perpendicular to the plane MN: then, if planes be drawn through AB to cut the planes MN, MQ in lines which make the angles CAD, EAF with each other, and that the plane BACD is perpendicular both to MN and MQ, the angle CAD will be greater than EAF. Shew that the angle BAD is less than the angle BAF, and it follows that CAD is greater than EAF.

20. Let the depth be taken as the fixed unit, and let the breadth be double the depth, and the length double the breadth. Let the parallelopiped and the cube be constructed. Then the cube may be shewn to consist of the same number of cubic units as the parallelopiped. See note on def. A, p. 253.

Similarly, if the breadth be treble, or any other multiple of the depth, and the length treble, or the same multiple of the breadth, the equality of the two figures may be shewn to exist.

21. This theorem is analogous to the corresponding theorem respecting a rectangular parallelogram.

The axis of a parallelopiped must not be confounded with its diagonal.

22. This theorem is analogous to Euc. II. 4.

23. There is some inaccuracy in the enunciation of this theorem.

24. Let O be one of the solid angles of a cube whose three adjacent edges are OA, OB, OC; OBEC being the base of the cube. On OA, OB, OC, let any three points A′, B′, C′ be taken, and join A′B′, B′C′, C′A′. Then the square of the area of the base A′B′C′ of the solid OA′B′C′ is equal to the sum of the squares of the areas of the faces, OA′B′, OA′C′, OC′B′.

Join OE intersecting B′C′ in E′, and join A′E′. Then A′E′ may be shewn to be perpendicular to the base C′B′ of the triangle A′B′C′, and by Euc. I. 47, and note page 68, the truth of the property is shewn.

25. Let the figure be described, then in a similar manner to Theorem 2, page 367, by employing Euc. II. 12, 13, instead of Euc. I. 47, the truth of the theorem may be proved.

26. This is to shew that the square of the diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three edges.

27. Let a rectangular parallelogram ABCD be formed by four squares, each equal to a face of the given cube, and let EF, GH, KL, be the lines of division of the four squares. Let BD the diagonal of ABCD, cut EF in M; the square of BM to the square of AB is as 17 to 16. Let BG the diagonal of ABHG cut EF in N; the square of BN is to the square of AB, as 20 is to 16: hence there is some square between that of BM and BN which bears to the square of AB, the ratio of 18 to 16, or of 9 to 8.

28. If any point A in a sheet of paper be taken as the vertex of any pyramid (suppose a triangular pyramid), the three plane angles which can be formed at A, are equal to four right angles, and therefore greater than the sum of the three plane angles with which it is possible to form a solid angle.

29. Let BCD be the base of the pyramid. Take C′D′ equal to CD in the same line, and join AC′, BC′, AD′, BD′. Then the triangular base BC′D′ is equal to BCD, Euc. I. 38. And since A is a fixed point, the altitude of the pyramids ABCD, ABC′D′ is the same, and pyramids of the same altitude on equal bases are equal.

30. See Euc. VI. def. 1. From the vertex A draw a line to any point B in the base of the pyramid, and meeting the given section in B′. From the angular points of the base draw lines to the point B; also from the angular points of the given section to the point B′. Then any triangle in the section, may be shewn to be similar to the corresponding triangle in the base. Euc. VI. 20.

HINTS, &c. TO THE PROBLEMS. BOOK XII.

1. THERE is no method by which a square can be described, by plane geometry, exactly equal to the area of a circle. The tract of Archimedes on the mensuration of the circle, consists of the three following propositions. 1. Every circle is equal to the right-angled triangle whose base and perpendicular are equal to the radius and circumference of the circle. 2. The area of a circle is to the square described on its diameter as 11 to 14 nearly. 3. The circumference of a circle is equal to three times the diameter and a part of the diameter which is less than $\frac{10}{70}$ of the diameter, but greater than $\frac{10}{71}$ of the diameter.

2. The radius of a circle whose area is double that of another, is equal to the side of a square whose area is double that of the square of the radius of the given circle.

3. A similar remark applies here as to the preceding problem.

4. First, to bisect a circle by a concentric circle. Let C be its centre, AC any radius. On AC describe a semicircle, bisect AC in B, draw BD perpendicular to AC, and meeting the semicircle in D; join CD, and with centre C, and radius CD, describe a circle; its circumference shall bisect the given circle. Join AD. Then by Euc. VI. 20, Cor. 2, the square on AC is to the square on CD as AC is to CB: and Euc. XII. 2. In the same way, if the radius AC be trisected, and perpendiculars be drawn from the points of trisection to meet the semicircle in D, E, the two circles described from C with radii CD, CE shall trisect the circle. And generally, a circle may be divided into any number of equal parts.

NOTE. By a similar process a circle may be divided into any number of parts which shall have to each other any given ratios.

5. To divide the circle into two equal parts. Let any diameter ACB be drawn, and two semicircles be described, one on each side of the two radii AC, CB: these semicircles divide the circle into two equal parts which have their perimeters equal. In a similar way a circle may be divided into three parts, by dividing the diameter into three equal parts, AB, BC, CD; and describing semicircles upon AB, AC on one side of the diameter, and then semicircles upon DC, DB on the other side of the diameter.

6. By Euc. XII. 2. The squares of the radii of the two circles may be shewn to be in the ratio of 3 to 1.

7. The area of the circle of which the quadrant is given, is to the area of the circle which touches the three circles, as 36 is to 1. And the quadrant is one-fourth of the area of the circle. Hence the area of the quadrant is to the area of the circle as 9 to 1.

8. The meaning of the enunciation of this problem is not very clear.

9. By reference to Theorem 2, p. 371, and Euc. XII. 2, the parts of the diameter may be proved to bear to each other the ratio of 1 to 2.

10. If planes be drawn through the bisections of three of the edges of the tetrahedron at right angles to the edges, the point of their mutual intersection, is the centre of the sphere which circumscribes the tetrahedron.

11. Take a point A on the spherical surface of the fragment as a centre, and with any radius AB describe a circle upon it. Take two other points C, D in the circumference of this circle, and describe a plane triangle A′B′C′ having its sides equal to the distances AB, BC, CA, respectively. Describe a circle about the triangle A′B′C′, and draw the diameter A′D′; with centres A′, D′ and radius equal to AB, describe circles intersecting each other in E′, and through the points A′, D′, E′ describe a circle; the diameter of this circle will be equal to that of the sphere of which the fragment is given.

12. By Euc. I. 47, expressions for the squares of the sides of the triangle may be found, from which it will appear that the three sides of the triangle are mean proportionals between every two of the three diameters of the spheres.

13. (1) The regular tetrahedron. Each of the angles of an equilateral triangle is one third of two right angles; a solid angle may therefore be formed by three angles

of three equal and equilateral triangles, and the figure formed by the three bases of the triangles is manifestly an equilateral triangle equal in magnitude to each of the three given equilateral triangles. The angles of inclination of every two of the four faces are also equal.

(2) The regular Octahedron. Through any point O draw three straight lines perpendicular to each other, take OA, O*a*, OB, O*b*, OC, O*c* equal to one another, and join the extremities of these lines. The faces ABC, A*b*C, &c. are equilateral triangles equal to one another and eight in number: also the inclinations of every two contiguous faces are equal.

(3) The regular Icosahedron. A solid angle may be formed with five angles, each equal to the angle of an equilateral triangle. At the point A of any equilateral triangle ABC, let a solid angle be formed with it and four other equal and equilateral triangles ABD, ADE, AEF, AFC, each equal to the triangle ABC. Next at the point B, let another solid angle be formed with the triangle ABC and four others BCH, BHK, BKD, BDA, each equal to it. The solid angle at B is equal to the solid angle at A, and the inclinations of every two contiguous faces are equal; also the two solid angles have two faces ABC, ABD common. Next let a third solid angle be formed at C, by placing the two triangles CFG, CGH contiguous to the three CAB, CFA, CHB. The solid angle at C is equal to that at A or B, and the inclinations of the contiguous faces make equal angles. Thus two equal and equilateral triangles are placed contiguous one to another, forming three solid angles at A, B, C, and having every two contiguous faces equally inclined: also the solid angles formed at D, E, F, G, H, K, have alternately *three* and *two* angles of the equilateral triangles. In the same manner let another figure equal to this be formed with ten equal and equilateral triangles, each equal to the triangle ABC.

If these two figures be connected together, so that the points at which there are *two* angles of one figure, may coincide with the points which contain *three* angles of the other, there will be formed at the points D, E, F, G, H, K, six equal solid angles, each contained by five angles of the equilateral triangles, and every two contiguous faces will have the same inclination.

Hence a figure of twenty faces is formed each equal to the equilateral triangle ABC, and having the inclinations of every two contiguous faces equal.

(4) The regular Hexahedron. Since three right-angles may form a solid angle, it is therefore obvious that the solid angle formed by three equal squares, has every two of the faces equally inclined to one another; and with three other squares, each equal to the former, a figure is formed, bounded by six equal squares, and having every two contiguous faces at right-angles to one another.

(5) The regular Dodecahedron. Since three angles each equal to the angle of a regular pentagon may form a solid angle: let ABCDE be a regular pentagon, and with two others each equal to this, let a solid angle at A be formed; the inclinations of every two contiguous faces will be equal. At the points B, C, D, E successively, let solid angles be formed by pentagons equal to ABCDE. The solid angles at B, C, D, E are each equal to the solid angle at A, and the inclination of every two contiguous faces is the same. Thus is formed a figure with six equal and regular pentagons, having the inclination of every two contiguous faces equal, and the angles at the linear boundary of the figure alternately consisting of an angle of a pentagon and of two angles of two pentagons equally inclined to each other.

Next, let another figure equal to this be constructed with six pentagons, each equal to the pentagon ABCDE.

If these two figures be so placed that the angular points of the plane angles in the linear boundary of one, may coincide with the points at which there are two angles in the other figure; at each of these points will be formed ten solid angles, each equal to the angle at A, and having the inclination of every two contiguous faces equal to one another. Hence a regular figure is formed having twelve equal faces, and the inclinations of every two contiguous faces equal to one another.

14. This is repeated by mistake. It is the same as Problem 11, page 368.

15. Let A be the vertex, and BCD the triangle forming the base of the tetrahedron. Bisect each of the dihedral angles at the base by three planes which mutually intersect each other in the point F, which will be the centre. Then a perpendicular from F, drawn upon any one of the faces will be the radius of the inscribed sphere.

16. For the inclination of any two contiguous faces of the octahedron, see Theorem 34, page 376.

The octahedron is divisible into two pyramids whose bases are squares, and the four slant sides equilateral triangles.

17. Every oblique pyramid may be proved to be equal to a right pyramid of the same base and altitude.

Every right pyramid whose base is not triangular may be divided into triangular pyramids of the same altitude.

Every pyramid on a triangular base may be proved equal to one-third of a prism of the same base and altitude.

Hence, any pyramid may be proved to be one-third of a prism of the same base and altitude.

18. If the centres of the upper sphere, and the three upon which it rests, be joined, the figure is a tetrahedron: and the same remark may be made with respect to each of the three, and the spheres upon which they severally rest.

HINTS, &c. TO THE THEOREMS. BOOK XII.

5. Let a regular polygon be inscribed in the circle. The straight lines drawn from the centre to all the angles of the polygon, will divide the polygon into as many equal isosceles triangles, as there are sides of the polygon, and perpendiculars drawn from the centre on each side, will be equal to the common altitude of all the triangles. Each of these triangles is equal to half the rectangle contained by the base and altitude of the triangle. Hence the area of the polygon is equal to half the rectangle contained by the common altitude and the sum of the sides of the polygon. Now if the sides of the regular polygon be diminished in magnitude and their number increased, the perimeter of the polygon may be made continually to approach to the perimeter of the circle, and at length be made to differ from it by a magnitude less than can be assigned. In that case also the perpendiculars on the sides of the polygons differ from the radius of the circle by a length less than can be assigned. Hence the area of the polygon and the area of the circle differ from each other by a quantity less than can be assigned, and therefore the area of the circle is equal to half of the rectangle contained by two straight lines which are equal to the radius and the circumference of the circle: or the area of the circle is equal to the rectangle contained by the radius and a straight line equal to half the circumference of the circle.

6. The angle in a segment which is one-fourth of the circumference of a circle, is equal to one of the interior angles of a regular octagon. The ratio of the two angles will be found to be as 3 to 2.

7. Let AB, A′B′ be arcs of concentric circles whose centre is C and radii CA, CA′, and such that the sector ACB is equal to the sector A′CB′. Assuming that the area of a sector is equal to half the rectangle contained by the radius and the included arc: the arc AB is to the arc A′B′ as the radius A′C is to the radius AC. Let the radii AC, BC be cut by the interior circle in A′, D. Then the arc A′D is to the arc AB, as A′C is to AC; because the sectors A′CD, ACB are similar: and the arc AB′ is to the arc AD, as the angle ACB′ is to the angle ACD, or the angle ACB. Euc. VI. 33.

From these proportions may be deduced the proportion:—as the angle ACB is to the angle A′CB′, so is the square of the radius A′C to the square of the radius AC.

And by Euc. XII. 2, the property is manifest.

8. Let AB, A′B′ be arcs of two concentric circles, whose centre is C. ACB, A′CB′ two sectors such that the angle ACB is to the angle A′CB′, as $A'C^2$ is to AC^2.

Let AC, BC be cut by the interior circle in A′, D;

Then the arc A′B′ is to the arc A′D, as the angle A′CB′ is to the angle A′CD, or the angle ACB. Euc. VI. 33.

And the arc A′D is to the arc AB, as the radius A′C is to the radius AC, by similar sectors.

By means of these two proportions and the given proportion, the rectangle con-

tained by the arc AB and radius AC, may be proved equal to the rectangle contained by the arc A'B' and the radius A'C.

9. The sum of the squares of the segments of the diagonals, is equal to the sum of the squares of each pair of opposite sides of the quadrilateral figure. Hence by Euc. XII. 2; I. 47; V. 18, the property is proved.

10. The radii of the circles may be proved to be proportional to the two sides of the original triangle. Then by Euc. XII. 2; VI. 19.

11. Let the two lines intersect each other in A, and let C be the centre of the last circle. Join CA, and draw CB perpendicular to AB one of the lines.

Then CA and CB are given. Let C', C'', &c., be the centres of the circles which successively touch one another. Draw C'B', C''B'', &c., perpendicular to AB and C'D, C''D' parallel to AB meeting CB in D, C'B' in D', &c. Then by means of the similar triangles, the radii C'B', C''B'', &c., may be expressed in terms of CB and CA. Hence the sum of the series of the circles may be expressed in terms of the area of the last circle.

12. The squares of the four segments, are together equal to the square of the diameter. Theorem 4, p. 314; and Theorem 129, p. 325. Then by Euc. XII. 2; V. 18, the truth of the Theorem is manifest.

13. This is shewn by Euc. I. 47; XII. 2; V. 18.

14. The demonstration of this property is contained in that of Problem 5, p. 374.

15. Let Cc be the line joining the centres of the two circles whose planes are parallel, and let ACB, acb be parallel diameters drawn in each. Join AC, Bb, then ABba is a quadrilateral figure having two of its sides AB, ab parallel. It is then required to shew that the lines joining Ab, Ba intersect at the same point D in the line Cc. If ab be equal to AB, the figure ABba is a rectangular parallelogram.

16. Let A be any point above the plane of the circle whose centre is C and diameter BCD. Join CA, and let a plane pass through any point c in AC or AC produced. Through c in this plane draw bcd parallel to BCD. Join BA, DA and produce them to meet bcd in d and b. Then b, d may be proved to be two points in the circumference of the circle whose diameter is bcd, and by means of the similar triangles Acd, ACD, the areas of the two circles may be shewn to be proportional to the squares of AC and Ac.

17. Let the arc of a semicircle on the diameter AB be trisected in the points D, E; C being the centre of the circle. Let AD, AE, CD, CE be joined, then the difference of the segments on AD and AE, may be proved to be equal to the sector ACD or DCE.

18. The proof of this property depends on the demonstration of Theorem 2, p. 371, and the relation between the area of two circles described upon two lines as diameters, one of which is double of the other.

19. Let the figure be described, and the demonstration will be obvious from the consideration of the parts.

20. The triangles ABD, ABC have the angle ABD common, and ACB may be proved equal to BAD, by Euc. I. 29; III. 32. And therefore the angle CAB is equal to the angle ADB. Also the triangles CAB, CEA may be shewn to be equiangular, and AD equal to AE. Then by Euc. VI. 4.

21. Let the diameter AB be divided into five equal parts, in C, D, E, then C, D are the second and third points of division. The semicircles AEC, AFD are described on one side of the diameter, and BGC, BHD on the other. Then since the semicircumferences of circles are proportional to their diameters, the perimeter of the figure AECGBHDF is shewn to be equal to the perimeter of the original circle.

By Euc. XII. 2, the area of the figure AECGBHD may be shewn to be one fifth part of the area of the circle.

The general case, when the diameter is divided into n equal parts is proved in the same way.

22. This is shewn from Euc. XII. 2; I. 47; V. 18.

23. Assuming that the area of a sector of a circle is equal to half the rectangle contained by the radius and the arc, the sector AOC is shewn to be equal to the triangle AOB.

24. By Euc. XII. 2, the area of the quadrant ADBEA is equal to the area of the semicircle ABCA.

25. Let POQ be any quadrant, O being the centre of the circle, and let BG, DH be drawn perpendicular to the radius PO, and OB, OD be joined. The triangle GBO is equal to DHO.

26. The segments on BC, BA, AC may be shewn to be similar. And similar segments of circles may be proved to be proportional to the squares of their radii, Euc. XII. 2, and to the squares of the chords on which they stand, Euc. VI. 6.

If Euc. VI. 31 be extended to *any similar figures*, the equality follows directly.

27. The triangles CEA, CEB are equal, and the difference of the two segments may be shewn to be equal to the difference of the parts of the semicircle made by CE. The difference of the same parts may also be shewn to be equal to double the sector DEC.

28. Let AB be the hypothenuse of the right-angled triangle ABC, and let the semicircles described upon the sides AC, BC, intersect the hypothenuse in D. Join AD. AD is perpendicular to AB. The segments on AC, AD, and on one side of CD are similar; and the segments on AC may be shewn to be equal to the segments on AD, CD. Also the segment on BC may be shewn to be equal to the segments on BD and the other side of CD.

If however Euc. VI. 31 be true for *all similar figures*, the conclusions above stated follow at once from the right-angled triangles.

29. (*a*) Join BD, CD, DA, Euc. III. 31; I. 14. (*b*) Produce CD to meet the arc of the quadrant in E. Then the sector ACE is half of the quadrant: also the semicircle CDA may be shewn to be equal to half the quadrant. (*c*) The segments on CD and DA are similar and equal, if the figure bounded by DA, AC, and the arc CD be added to each, the remaining part of the semicircle on AC is equal to the triangle ACD which is a right-angled isosceles triangle.

30. This theorem is analogous to Euc. III. 14.

31. Let D be the given point, and from D let DA be drawn through the centre E, and meeting the surface in C, A. Let DB be a line from D touching the sphere at B. Join BE. Then the triangle DBE (fig. Euc. III. 36) is in a plane passing through D, and E the centre of the sphere, and the distances DE, EB are always the same. Hence it follows that BD is always of the same length. Euc. I. 47.

The sphere which touches the six edges of any tetrahedron, has four circular sections touching the sides of the four triangles which form the surface of the tetrahedron.

32. Let the circle ADB cut the circle AEB in the diameter AB at any angle, C being their common centre. Next let the plane perpendicular to AB cut the circumference of the circle ADB in D, F, and the circumference of AEB in E, G. Then E, D, G, F may be proved to be in the circumference of a circle.

33. See the Géométrie par M. Vincent, p. 450.

34. Let ABCD be a regular tetrahedron. From A in the plane ABC draw AE perpendicular to BC, and join DE in the plane BCD, also from A draw AG perpendicular to the line DE. Then the angle AEG is the inclination of the two faces ABC, DBC of the tetrahedron, and the base EG is one-third of the hypothenuse AE in the right-angled triangle AGE.

Let $abcdef$ be a regular octahedron whose faces are equal to those of the tetrahedron. Join a, f two opposite vertices. Draw ag in the plane abc perpendicular to bc, and ge perpendicular to af. Draw fg in the plane fbc, and from f draw fh perpendicular to ag produced.

Then agf is the inclination of two faces of the octahedron. Also in the right-angled triangle fhg, gh may be proved to be one-third of fg, and fg is equal to AE. Hence the triangles fgh, AEF are equal in all respects. Therefore the angle fgh is equal to the angle AEB. Hence the angle AEF is the supplement of the angle agf, or the inclination of two contiguous faces of a tetrahedron, is the supplement of the inclination of two contiguous faces of an octahedron.

ADDENDUM.

PROBLEM 16, page 333.

Inscribe in a circle a triangle whose sides or sides produced, shall pass through three given points in the same plane.

Lemma I. Let there be given two points A, B, and a circle DCE whose centre is S: from the point A, draw ADC to cut the circle; through B, C, D describe a circle cutting the line AB in K: then K is a fixed point, however the line *ADC* may be drawn; and if KD be drawn to meet the circle in H, the line *HE, drawn to* the intersection E of BC with the circle, will be parallel to AB.

For, first, $BA \cdot AK = CD \cdot DA =$ a given magnitude, namely, the square of the tangent from A to the circle. Also BA is given, and hence AK is also given, and K is a fixed point, however ADC be drawn from A to cut the circle.

And, secondly, since CDHE is a quadrilateral inscribed in the circle CDE, the exterior angle made by producing EH, is equal to the interior opposite angle DCE. In the same way the angle DKA is equal to DCB. Whence these angles are equal, and HE is parallel to AB.

Lemma II. The same conditions as before being given, draw the diameter TV through K and S; and find the point F such that $SF \cdot SK = ST^2$: then joining FD, the angle EDF will be equal to the difference between BKS and a right angle, however the point C be taken in the circle DCE.

For, draw HX parallel to KS, and from X draw the diameter XSG, and join HG, GD; also draw KP perpendicular to AB.

Then since XHG is an angle in a semicircle, it is a right angle; and since HX is parallel to KS, HG is perpendicular to KS; and the angle EHG is equal to PKS.

Moreover, since HG is perpendicular to KS, the line DG always passes through F, and hence the line FG makes with DE the constant angle PKS, however C may be taken in the circle DCE.

Having premised these two Lemmas, we may proceed to the construction of the Problem, as follows:

Let A, B, Q be the three given points. Find the points K, F as in the Lemmas, together with the angle PKS. Join QF, and on it describe a segment to contain the angle PKS; and let it cut the circle DCE in D. Then D is one of the angular points of the triangle. Join DA meeting the circle in C, and CB meeting the circle in E, and draw ED. It will pass through Q by the reasoning of the Lemmas.

The same principle may be applied to the solution of Problem 58, p. 324.

Any two points, A, B, being given within a circle CDE, it is required to find a point D, so that the difference of the angles BDE, ADC may equal a given angle.

The points A, B may be taken any where either within or without the circle; and the construction will be the same.

Analysis. Let the point D be supposed to be found, such that the angle EDB − DCA = the given angle. Make ECL equal to that angle, and join LC. Then, obviously, if BD meet the circle in H, and DA meet it in G, the chord GH will be parallel to LC. Find F and K, by the Lemmas, then the angle FHG is given. Wherefore through F draw FQ parallel to LC, and make the angle HFG equal to the given angle. Draw BH meeting the circle in D, and join DA: then these are the lines required.

NOTE.—The second Lemma is only a variation of the last Porism of Euclid's third Book on that subject.

The 57th proposition in Dr Simson's Restoration of the Porisms, leads directly to the construction in the manner here given.

The former Problem, though not mentioned directly by Pappus, nor found in any ancient author, was without doubt considered by the Greek geometers.

It has been regarded by modern geometers as an extension of the 117th Proposition of the Seventh Book of the Collections of Pappus, namely:—when the three points are not in the same straight line.

The Problem itself, as well as Proposition 117 of Pappus, has engaged the attention of several distinguished modern geometers. Bonnycastle, in p. 348 of his Geometry, has given a concise account of the several solutions by mathematicians on the continent; as also Dr Traill in p. 95 of his life of Simson. In p. 97, he has given Simson's solution of the Problem, which, from a note attached, appears to have been completed in 1731. Simson's "Opera Reliqua" was published by the munificence of Earl Stanhope in 1776.

In 1742, M. Cramer proposed the problem to M. de Castillon, but it was not till 1776 that Castillon published a geometrical solution in the Berlin Memoirs of that year. In the same volume is a solution by La Grange, by means of trigonometrical formulæ. Carnot, in p. 383 of his *Géométrie de Position*, has given a modified form of La Grange's Solution. In the Petersburgh Acts for 1780, are solutions of the same Problem by Euler, Lexell and Fuss. In the Memoirs of the Italian Society (Tom. IV. 1788) are two papers respecting this problem; one by Ottajano, in which is given a geometrical solution of the problem, and an extension to the case of a polygon of any number of sides, which he inscribes in a given circle, so that the sides respectively shall pass through the same number of points. Ottajano also gives a sketch of the history of the problem. The other paper is by Malfatti, and contains a solution of the general problem of the polygon last mentioned.

In the Berlin Memoirs for 1796 is a paper by Lhuilier, containing an algebraical solution of the most general case of the polygon. He also extends the problem to the conic sections, and adds a similar one respecting the sphere. An extension of this Problem to the Conic Sections has also been effected by Poncelet in his *Traité des Propriétés Projectives*. M. Brianchon has considered the Problem in the case where a conic section is substituted for the circle, and where the three points are in one line. His solution will be found in the Journal de l'Ecole Polytechnique. (Tom. IV.)

Dr Wallace and Mr Lowry applied the 57th Porism to this problem in Vol. II. of the Old Series of Leybourn's Repository. In Vol. I. of the New Series of that work, a new and very elegant analysis of the Porism was given by Mr Noble, which has been the main guide in demonstrating the two Lemmas here used. The same method, slightly modified, applies to any inscribed polygon. The most elegant system of investigation, however, that has ever been published, is that of Mr Swale, in the second number of his Apollonius. Mr Lowry has also given the solution of the problem in the case where the ellipse is substituted for the circle, and where the polygon has any number of sides. See Leybourn's Repository, Vol. II., New Series, p. 189.